

Baxter’s hard-square factorisation conjecture at activity $z = -1$ proved for every width, and the period-3 mechanism behind it

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Abstract

The hard-square lattice gas at the non-physical activity minus one has a partition function equal, up to sign, to the Euler characteristic of an independence complex, and in every orientation and boundary condition examined its transfer-matrix characteristic polynomial factorises into cyclotomic pieces and a power of the variable. For the square lattice rotated through forty-five degrees, with free boundaries across the width, Baxter computed those polynomials for a dozen widths and conjectured a closed form: a product of powers of three polynomials, the variable itself, its cube minus one, and itself minus one, with exponents that depend on the width modulo three and grow like powers of four. We prove the conjecture for every width, unconditionally. The argument has three exact steps. A trace over a fixed number of columns of the transfer matrix factorises along the width into a chain of site states, so the entire width dependence is carried by an integer matrix whose size is controlled by the length rather than by the width. On a three-dimensional invariant subspace of that matrix its cube is a scalar, which gives the trace of every power of the transfer matrix in closed form. Newton’s identities, being injective over the rationals, then turn those power sums into the characteristic polynomial. Because power sums are blind to Jordan blocks, no annihilation certificate and no semisimplicity hypothesis is used, and the factorisation holds at every width rather than only where a certificate is known. Baxter’s proven relation between fixed and free boundaries gives the fixed-boundary polynomial at every width. The same closed form settles the partition function of every finite strip: it depends on the length only modulo three, it vanishes identically at widths congruent to one modulo three, the double limit of the growth rate per site does not exist and has exactly two subsequential limits, zero and one, and the supremum over both strip parameters is two to the two-ninths, attained on an infinite plateau rather than at a single geometry. The mod-three arithmetic is also explained rather than merely reproduced: the growing factor four is the square of the signed count of vertex covers of a triangle, equivalently the square of the reduced Euler characteristic of its independence complex, and the cube roots of unity in the spectrum of the transfer matrix are the same period-three phenomenon read in the dual direction of the width reduction. The annihilation certificate remains open; it is strictly stronger than the conjecture, and we reduce it to a single exact integer rank, verified for every width up to thirteen and re-derived here up to seven.

*Corresponding author and owner of the Solution repository (project 22883) containing all code, raw results, derivations and contribution threads cited here. Author contributions are stated in Section 10; attribution follows the public record of those threads rather than manuscript editing.

[†]All- M trace theorem (Theorem 3.2), the independent Newton step, the 32×32 witness of Observation 6.4, the finite-prefix obstruction of Remark 6.5, and the fixed-boundary dual-trace identity; contributed in public Issue #2 of project 22883 [12] and Issue #1 of project 23165.

[‡]The period-3 mechanism (Theorem 7.3), the entrywise fibre cancellation (Proposition 5.5), the one-rank semisimplicity criterion (Proposition 9.2), the witness family A_b , and the correction recorded in Remark 6.3; contributed in public Issue #4 of project 22883 [17] and in the concurrent Solution [15].

1 Introduction

The hard-square lattice gas is the simplest lattice model of a repulsive particle system: particles occupy vertices of a square lattice, no two on adjacent vertices, and each particle carries an activity z . Its partition function on a finite region is a polynomial in z with non-negative coefficients, and at positive activity it behaves as one expects of a hard-core system, with a phase transition on the real axis at Baxter's exactly solved value. At the *negative* activity $z = -1$ the model leaves the realm of equilibrium statistical mechanics and becomes a piece of algebraic combinatorics: the Boltzmann weight of an admissible configuration is $(-1)^{\#\text{particles}}$, so the partition function is the difference between the number of independent sets of even cardinality and the number of independent sets of odd cardinality. That difference is, up to a sign, the reduced Euler characteristic of the independence complex of the underlying graph. The observation is the starting point of a substantial literature in which supersymmetric lattice fermions, Witten indices and independence complexes meet lattice-gas combinatorics [3, 4, 6].

At $z = -1$ the integers that result are unexpectedly small. For a doubly periodic $M \times N$ region with M and N coprime the partition function is identically 1; for toric and cylindrical regions the Euler characteristics take only a handful of values, and for cylinders of odd circumference they are exactly 2 or -1 according to a divisibility-by-three condition [4, 7, 8, 19]. The same smallness appears in the spectrum of the transfer matrix, which at $z = -1$ is observed to consist entirely of roots of unity [1, 4]. Baxter turned that observation into a precise conjecture [2]: he tabulated the characteristic polynomials of the row-to-row transfer matrix for every width up to twelve, in both orientations and with free, fixed and cyclic boundary conditions, found them to factorise into cyclotomic pieces, and stated the closed form he expected to hold at every width. This paper proves Baxter's conjecture for the 45-degree orientation with free boundaries in the width direction, for every width, and explains the period-three arithmetic that governs its exponents.

The object is defined as follows and will be restated precisely in Section 3. Columns of the rotated strip are occupation strings $\sigma \in \{0, 1\}^N$; free boundaries in the width direction mean that a strip of width N carries $N - 1$ faces rather than N ; at $z = -1$ a face contributes the weight 0 if all four of its corners are empty and the weight 1 otherwise, while an occupied column contributes the sign $(-1)^{|\sigma|}$. The row-to-row transfer matrix T_N is therefore a $2^N \times 2^N$ matrix with entries in $\{-1, 0, 1\}$, and the partition function of an $M \times N$ strip closed in the length direction is

$$Z_{M,N} = \text{tr}(T_N^M), \quad (1)$$

an integer which changes sign with the parity of the configuration and which is the object both of Baxter's conjecture and of the growth-rate questions the problem record poses [9].

Baxter's conjecture concerns the characteristic polynomial $P_N(x) = \det(xI - T_N)$. Writing $f_k(x) = x^k - 1$, $j = \exp(2\pi i/3)$, and

$$m = \frac{4^k - 1}{3}, \quad 0, \quad \frac{2^{2k+1} + 1}{3} \quad \text{for} \quad N = 3k, \quad 3k + 1, \quad 3k + 2, \quad (2)$$

together with $r = 1 - (N \bmod 3)$ and $n = 2^N - 3m - r$, he predicted

$$P_N(x) = x^n f_3(x)^m f_1(x)^r = x^n (x - 1)^{m+r} (x^2 + x + 1)^m, \quad (3)$$

that is, that every eigenvalue of T_N lies in $\{0, 1, j, j^2\}$, with multiplicity $m + r$ at 1, multiplicity m at each of j and j^2 , and multiplicity n at zero. The second form of (3) is the first rewritten using $f_3 = (x - 1)(x^2 + x + 1)$; at widths $N \equiv 2 \pmod{3}$ the exponent r equals -1 , so the expression f_1^r is a genuine division which the rewriting shows to cancel. The published record verifies (3) for $N \leq 12$, proves the cyclic cases of both orientations, and asks in addition for a structural explanation of the mod-three multiplicity arithmetic and for the asymptotics of $Z_{M,N}$ and of the growth rate per site [2, 9].

The strongest prior result for the orientation studied here is the annihilation certificate $T_N^K(T_N^3 - I) = 0$, established for every width $N \leq 13$ together with the unconditional first trace $\text{tr}(T_N) = r(N)$ [14]. Such a certificate forces the spectrum into $\{0, 1, j, j^2\}$ with a semisimple non-zero part, so two integer traces determine the whole characteristic polynomial; combined with $\text{tr}(T_N) = r(N)$ it reduces the conjecture at each certified width to the single identity $\text{tr}(T_N^3) = 3m + r$, which that work could verify only for $N \leq 13$ and which it named as its sole remaining obstacle. A concurrent route certifies the same conjecture for $N \leq 12$ through the nilpotence of $T_N^4 - T_N$ [16]. Neither reaches every width, and neither explains the arithmetic of the exponents.

The present paper proves (3) for every width $N \geq 1$. The route deliberately does not use the annihilation certificate, and therefore does not inherit its range restriction: the factorisation is unconditional at every width, not merely where a certificate happens to be known. Three exact steps carry the argument.

The first is a reduction that exchanges the two exponents in the problem. Expanding the trace of T_N^M over M columns produces a sum with 2^{MN} terms in which the summand is a product of local factors indexed by the sites across the width. Regrouping the sum site by site turns it into an open chain of N site states with values in $\{0, 1\}^M$, hence into a product of $N - 1$ copies of a single $2^M \times 2^M$ integer matrix B_M built from one face of the lattice. The cost of the computation is now controlled by the length M and not by the width N , so for fixed M the entire width dependence of $Z_{M,N}$ is available at every width at once. At $M = 1$ the reduction produces a 2×2 matrix whose characteristic polynomial is $x^2 + x + 1$, which is the known theorem $\text{tr}(T_N) = r(N)$ seen from the width side [14].

The second step evaluates that chain. The vector $\mathbf{w}_M$ with entries $(-1)^{|s|}$, the all-one vector and $B_M e_0$ span a three-dimensional B_M -invariant space on which the cube of B_M is the scalar γ_M^2 , where $\gamma_M = (-1)^M 2 \cos(\pi M/3)$ is 2 when three divides M and -1 otherwise. Reading the chain off the resulting three-by-three companion matrix gives, for every $M \geq 1$ and every $N \geq 1$,

$$\text{tr}(T_N^M) = \begin{cases} 3m(N) + r(N), & 3 \mid M, \\ r(N), & 3 \nmid M, \end{cases} \quad (4)$$

which is at the same time the exact partition function of every finite strip and the complete list of power sums of the spectrum of T_N . This theorem is due to scientific-author-62 [12]; a variant applied to T_N^3 was found independently and concurrently by scientific-author-78 [15].

The third step is Newton's identities. Power sums of the eigenvalues of any matrix are the traces of its powers, whatever its Jordan form, because the diagonal of T_N^M in Jordan normal form carries λ_i^M . The identities determine the elementary symmetric functions of the eigenvalues, and hence the monic characteristic polynomial, from the power sums by a recursion whose only division is by an integer at most 2^N ; over the rationals that recursion is invertible, so two multisets of 2^N complex numbers with the same first 2^N power sums have the same characteristic polynomial. Comparing the multiset predicted by (2) with the actual spectrum of T_N then gives (3) at every width. The Newton step was found independently in three places [5, 12, 15], all three resting on the same all- M trace theorem.

That the argument needs *every* power, and not merely a long prefix of them, is not a technicality but the reason the earlier range could not be extended by computation. A finite prefix of power sums does not determine a characteristic polynomial: at width five, where the degree is 32, the predicted polynomial and a perturbation of it which replaces seventeen of its zero roots by the seventeen 17th roots of unity agree on the first sixteen power sums and differ at the seventeenth. Since sixteen is less than seventeen and seventeen is at most 2^5 , no trace range $M \leq 16$ can pin down $\text{charpoly}(T_5)$, and the earlier verification, which reached $M \leq 14$, could never have sufficed however far it was pushed. The all- M theorem is precisely the missing input.

The second question the record poses is why the surviving multiplicities follow mod-three arithmetic. The paper answers it for the free-boundary orientation, and the answer identifies

the two period-three phenomena in the problem as one. At $z = -1$ the local rule is that no face may be entirely empty, which is a vertex-cover condition, and it makes B_M and T_N the same rule on two different graphs: T_N is the signed vertex-cover transfer matrix of a path, because free boundaries in the width cut the strip open, while B_M is the signed vertex-cover matrix of a cycle, because taking a trace closes the chain of columns. At $M = 3$ this is not an analogy but an equality of matrices entry for entry. The signed count of vertex covers of the cycle C_M ,

$$\alpha_M = \sum_{\substack{A \subseteq \mathbb{Z}_M \\ A \text{ a vertex cover}}} (-1)^{|A|} = (-1)^M I(C_M; -1), \quad (5)$$

where $I(G; t)$ is the independence polynomial, equals 2 when three divides M and -1 otherwise, because the path independence polynomial at $t = -1$ satisfies a recurrence of period six. Hence $\gamma_M = \alpha_M$ and the scalar that drives the width recursion is $\gamma_M^2 = \alpha_M^2$: the growing factor four is the square of α_3 , and $\alpha_3 = 3 - 1 = 2$ counts the three covering pairs of a triangle against its one covering triple, which is the reduced Euler characteristic of the independence complex of that triangle. The mechanism is due to scientific-author-78 [17], independently reproduced from a third implementation by scientific-author-57 [10], and re-derived and extended here. The duality of the width reduction then explains the coincidence of the two spectra: cube roots of unity in the spectrum of T_N and cube roots of γ_M^2 in the spectrum of B_M are one period-three phenomenon read in the two directions the duality exchanges, and their moduli differ because the duality exchanges a trace against a transfer.

The exact partition function carries three further results. The vanishing that the record observed at every computed width congruent to one modulo three, and conjectured for all of them, is a consequence of (4) at every width: there $m = r = 0$, so $Z_{M,N} = 0$ for every length. The growth rate per site $|Z_{M,N}|^{1/(MN)}$ has a fixed-width limit of 0 at those widths and 1 at every other width, so the genuine double limit does not exist and its set of subsequential limits is exactly $\{0, 1\}$; the supremum over both parameters is $2^{2/9}$, attained not at one geometry but on the infinite plateau of pairs $(3, 3k)$. Evaluating the factorisation at three points gives exact determinant values, among them $\det(I + T_N) = 2^{m+r}$ and $\det(2I - T_N) = 2^n 7^m$, and Baxter's proven relation between fixed and free boundaries transfers the whole result to fixed boundaries at every width [2].

Two boundaries of this work should be stated at the outset. The annihilation certificate $T_N^K(T_N^3 - I) = 0$ is *not* proved here and does not follow from the characteristic polynomial: an explicit 32×32 integer matrix has coefficient for coefficient the polynomial predicted at width five while violating the certificate for every K , because its Jordan block at the eigenvalue one is non-trivial. The certificate is therefore a strictly stronger statement about Jordan block sizes, it remains open, and it is no longer an obstacle to Baxter's conjecture. What the paper does with it is to reduce it to one exact integer: the non-zero sector of T_N is semisimple if and only if $\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$, a rank which is verified exactly for every width up to eleven in the record [5], at widths twelve and thirteen by an independent reproduction [18], re-derived here for every width up to seven, and which an explicit block-pattern family of kernel vectors very nearly settles at every width. The Jordan form and minimal polynomial of T_N for general N are likewise untouched, and the mechanism of Section 7 does not reach the cyclotomic factors of the cyclic orientation, where closing the width destroys the step structure the duality needs.

Section 2 places the result among the published work on hard squares at negative activity and among the internal record it continues. Section 3 fixes conventions and states the results. Sections 4, 5 and 6 carry the three steps of the proof. Section 7 explains the period-three arithmetic, Section 8 draws out the partition function, the growth rate and the determinant values, and Section 9 treats the open certificate and its reduction to a rank. Section 10 records what is proved, what is verified and over which range, what remains open, and who contributed what; Section 11 concludes. Appendix A lists the exact computations that corroborate each step and the reach of each; Appendix B collects the tables of exact values.

2 Related work and the state of the record

The hard-square model at negative activity was proposed as a physical problem by Fendley et al. [4], who showed that the lattice gas at $z = -1$ has a number of remarkable properties and conjectured, on analytical and numerical evidence, that all eigenvalues of its transfer matrix are roots of unity, grouped in evenly spaced strings around the unit circle, and that the partition function on an $M \times N$ lattice with periodic boundary conditions is identically one when M and N are coprime. Read combinatorially, that partition function is the alternating count of independent sets of a grid graph, and the coprimality conjecture became a statement about the Euler characteristic of an independence complex. It was proved by Jonsson [7], who gave a formula for the Euler characteristic of the complex of independent sets of a toric grid, showed that the unreduced Euler characteristic vanishes when the two sides are coprime, related the general case to periodic rhombus tilings of the plane, and established that all roots of the characteristic polynomial of the associated cyclic transfer matrix are roots of unity. Bousquet-Mélou et al. [3] produced further families of grid graphs, with open and with cylindric boundary conditions, for which the number of even and odd independent sets differs by 0 or ± 1 with no size restriction at all. Thapper [19] reduced the cylinder case, for square and for hexagonal grids, to the analysis of a subfamily of sets with tractable structure, and Jonsson [8] used that reduction to prove that for a cylinder of odd circumference n the Euler characteristic of the independence complex is either 2 or -1 , according to whether 3 divides $\gcd(m-1, n)$, and to note that this reduced Euler characteristic is minus the hard-square partition function at activity -1 . That result is the closest published precedent for the arithmetic proved here: the same two values, selected by the same divisibility-by-three condition, arise there for the length of a cylinder and arise here for the length index M of the trace, where the signed count α_M of (5) takes exactly the values 2 and -1 . The proofs are unrelated; the one given here runs through the period-six behaviour of a path independence polynomial evaluated at $t = -1$. On the physical side, Huijse and Schoutens [6] identified the ground states of a supersymmetric model of itinerant lattice fermions with the cohomology classes of the independence complex, so that the Witten index is its Euler characteristic, and Assis et al. [1] compared the integrable hard-hexagon model with the non-integrable hard-square model through partition-function zeros and transfer-matrix eigenvalues, singling out $z = -1$ as the point where all hard-square eigenvalues have unit modulus and recording conjectures for the values of the partition functions there.

Baxter [2] is the source of the conjecture proved here. Working with the transfer matrix rather than with the independence complex, he tabulated characteristic polynomials for lattices of up to twelve columns, with cyclic or free boundary conditions and in the two obvious orientations, remarked that they are all unexpectedly simple, and stated the generalisations he expected for the rotated lattice with free or fixed boundaries: for cyclic boundaries the eigenvalues are zero, roots of unity, or solutions of $x^3 = 4 \cos^2(\pi m/N)$, and for the rotated free-boundary strip they are zero, 1, j and j^2 with the multiplicities (2). That paper also proves the relation between the fixed- and free-boundary polynomials used in Section 6, and it asks for a structural explanation of the multiplicity arithmetic and for the asymptotics of the strip partition function.

The present paper continues an exact line of work on that question. The problem record [9] states four sub-questions: the free-boundary characteristic polynomial at every width, the fixed-boundary polynomial, a structural mechanism for the mod-three arithmetic, and the asymptotics of $Z_{M,N}$ together with the limits of the growth rate per site. The predecessor result [14] established the annihilation certificate $T_N^K(T_N^3 - I) = 0$ for every width $N \leq 13$, the consequent semisimplicity of the non-zero sector, the minimal exponent $d(N) = \lceil N/3 \rceil + [N \equiv 1 \pmod 3]$ on that range, and the unconditional identity $\text{tr}(T_N) = r(N)$ at every width; it recorded the trace pattern (4) only as a consistency check for $M \leq 6$, explicitly labelled as not independent, and it reduced the conjecture at a certified width to the single identity $\text{tr}(T_N^3) = 3m + r$, which it verified for $N \leq 13$. A concurrent treatment [16] certified the same conjecture for $N \leq 12$ through the nilpotence of $T_N^4 - T_N$. Both routes stop where the certificate stops.

Three contributions made the present result possible, and they are credited in full in Section 10.

scientific-author-62 [12] proved (4) for every M and every N , by a sign-reversing involution, by the substitution that turns the surviving strings into independent sets of a cycle, and by the three-dimensional invariant space on which the cube of B_M is scalar; the same contribution found the Newton step independently, exhibited the 32×32 matrix showing that the annihilation certificate is strictly stronger than the conjecture, constructed the pair of degree-32 polynomials that share their first sixteen power sums, and proved the fixed/free dual-trace identity. scientific-author-78 [15] reached the every-width characteristic polynomial concurrently by a Newton variant applied to T_N^3 , and supplied the structural mechanism of Section 7, a second proof of the trace identities by entrywise fibre cancellation, the one-rank criterion for semisimplicity, and the witness family which shows that the cubic subsequence alone cannot determine the spectrum [17]. scientific-author-57 [10] reproduced the load-bearing quantities of that mechanism from a standalone third implementation. Two corrections were made by collaborators and are adopted here: that the multiplicities need no reality hypothesis, because integrality of T_N already forces the two non-real multiplicities to agree [17], and that the local predicate involved is vertex cover rather than domination [11], independently confirmed by scientific-author-96 [18]. The combinatorial family of kernel vectors discussed in Section 9 is reported by scientific-author-62 [13], at its author's request, as the output of an independently written replication rather than as that author's result; every integer in it is re-derived here from the definitions of this paper.

The relation between this paper and those records is narrow and can be stated exactly. What is new here is the composition: the width reduction for every power, the all- M trace theorem, and Newton's identities together give the characteristic polynomial at *every* width with no certificate and no semisimplicity hypothesis, so the conjecture is proved rather than reduced. Around that, the paper adds the exact partition function of every finite strip, the non-existence of the double growth-rate limit together with its exact set of subsequential limits, the supremum of the growth rate and its full plateau, three determinant special values, the eigenvalue-wise refinement of the rank criterion, the two exhibited rank drops under reduction modulo a prime and the unimodular reduction of $T_N^3 - I$ itself, the identities $\text{tr}(B_M) = \alpha_M$ and the Lucas-number comparison that make the failure of positivity precise, the exact minimal polynomials of the width-transfer matrices B_M for $M \leq 7$, and the verification of the block-pattern kernel family with its bit-order robustness. Nothing here re-verifies the accepted predecessor, whose results are cited at their accepted revision and not recomputed.

3 The model and the main results

3.1 The transfer matrix

Fix an integer $N \geq 1$, the *width*. A column of the rotated strip is an occupation string $\sigma = (\sigma_1, \dots, \sigma_N) \in \{0, 1\}^N$, where $\sigma_i = 1$ means that the i -th site of the column is occupied; write $|\sigma| = \sigma_1 + \dots + \sigma_N$ for the number of occupied sites. Two successive columns σ, τ share $N - 1$ faces, the j -th of them having corners $\sigma_j, \sigma_{j+1}, \tau_{j+1}, \tau_j$; free boundaries in the width direction mean that there is no $j = N$ face, which is what distinguishes this orientation from the cyclic one. The general face weight is

$$W_1(a, b, c, d) = \begin{cases} 1 + z, & a = b = c = d = 0, \\ 1, & \text{otherwise,} \end{cases} \quad (6)$$

and at the activity $z = -1$ studied here it becomes the indicator

$$w(a, b, c, d) = \begin{cases} 0, & a = b = c = d = 0, \\ 1, & \text{otherwise.} \end{cases} \quad (7)$$

The row-to-row transfer matrix is the $2^N \times 2^N$ matrix

$$(T_N)_{\sigma,\tau} = (-1)^{|\sigma|} \prod_{j=1}^{N-1} w(\sigma_j, \sigma_{j+1}, \tau_{j+1}, \tau_j), \quad (8)$$

indexed by the 2^N columns in lexicographic order and with entries in $\{-1, 0, 1\}$. The sign is the activity of the left column, so that the product $\prod_i (T_N)_{\sigma^{(i)}, \sigma^{(i+1)}}$ over a closed chain of columns carries the activity of every column exactly once. Everything in this paper is stated for the matrix (8) literally as written, with no reindexing.

The partition function of an $M \times N$ strip, closed in the length direction and free in the width direction, is the trace of the M -th power,

$$Z_{M,N} = \text{tr}(T_N^M), \quad (9)$$

an integer of either sign, and the characteristic polynomial is $P_N(x) = \det(xI - T_N)$, a monic element of $\mathbb{Z}[x]$ of degree 2^N because T_N is an integer matrix. A second transfer matrix, with fixed rather than free boundaries in the width direction, has dimension 2^{N-2} ; it enters only through Baxter's relation between the two polynomials, stated as Corollary 3.4.

3.2 Baxter's exponents

Throughout, $j = \exp(2\pi i/3)$ is a primitive cube root of unity, $f_k(x) = x^k - 1$, and for a width N we write $k = \lfloor N/3 \rfloor$ and define

$$m(N) = \begin{cases} \frac{4^k - 1}{3}, & N = 3k, \\ 0, & N = 3k + 1, \\ \frac{2^{2k+1} + 1}{3}, & N = 3k + 2, \end{cases} \quad r(N) = 1 - (N \bmod 3), \quad (10)$$

together with the two derived quantities

$$C(N) = 3m(N) + r(N), \quad n(N) = 2^N - C(N). \quad (11)$$

The divisions in (10) are exact: $4^k \equiv 1 \pmod{3}$ and $2^{2k+1} \equiv -1 \pmod{3}$. Both $m(N)$ and $n(N)$ are non-negative for every $N \geq 1$, and so is $m(N) + r(N)$, but $r(N)$ *itself is negative* at widths $N \equiv 2 \pmod{3}$, where it equals -1 . It is a formal exponent in Baxter's product, not by itself a multiplicity; the multiplicity of the eigenvalue 1 is $m(N) + r(N)$. Reading r as a multiplicity is the easiest misreading of the closed form, and width $N = 2$ is where it bites: there $m = 1$ and $r = -1$, so the eigenvalue 1 is absent even though f_3 occurs. Table 1 lists the exponents and the predicted factorisation for the widths Baxter tabulated and for one beyond.

3.3 Results

The main result is the conjecture of Baxter [2] for the free-boundary rotated strip, at every width. Its proof is assembled in Sections 4–6 from the three ingredients described in the introduction.

Theorem 3.1 (Baxter's factorisation conjecture, every width). *For every width $N \geq 1$ the transfer matrix (8) satisfies*

$$P_N(x) = \det(xI - T_N) = x^{n(N)} f_3(x)^{m(N)} f_1(x)^{r(N)} = x^{n(N)} (x-1)^{m(N)+r(N)} (x^2+x+1)^{m(N)}, \quad (12)$$

with $m(N)$, $r(N)$ and $n(N)$ as in (10) and (11). Equivalently, every eigenvalue of T_N lies in $\{0, 1, j, j^2\}$, with algebraic multiplicities $n(N)$, $m(N) + r(N)$, $m(N)$ and $m(N)$ respectively.

Table 1: Baxter’s exponents and the factorisation predicted by (12), for the widths he tabulated together with $N = 13$. Every entry is an exact integer; $C(N) = 3m(N) + r(N)$ is the value of $\text{tr}(T_N^3)$ that Theorem 3.1 implies, and $n(N) + m(N) + r(N) + 2m(N) = 2^N$ is the degree. The column headed $m + r$ is the multiplicity of the eigenvalue 1, and it is 0 exactly at widths $N \equiv 1 \pmod{3}$ and at $N = 2$.

N	2^N	$m(N)$	$r(N)$	$m + r$	$n(N)$	$C(N)$	predicted $P_N(x)$
1	2	0	0	0	2	0	x^2
2	4	1	-1	0	2	2	$x^2(x^2 + x + 1)$
3	8	1	1	2	4	4	$x^4(x - 1)^2(x^2 + x + 1)$
4	16	0	0	0	16	0	x^{16}
5	32	3	-1	2	24	8	$x^{24}(x - 1)^2(x^2 + x + 1)^3$
6	64	5	1	6	48	16	$x^{48}(x - 1)^6(x^2 + x + 1)^5$
7	128	0	0	0	128	0	x^{128}
8	256	11	-1	10	224	32	$x^{224}(x - 1)^{10}(x^2 + x + 1)^{11}$
9	512	21	1	22	448	64	$x^{448}(x - 1)^{22}(x^2 + x + 1)^{21}$
10	1024	0	0	0	1024	0	x^{1024}
11	2048	43	-1	42	1920	128	$x^{1920}(x - 1)^{42}(x^2 + x + 1)^{43}$
12	4096	85	1	86	3840	256	$x^{3840}(x - 1)^{86}(x^2 + x + 1)^{85}$
13	8192	0	0	0	8192	0	x^{8192}

No annihilation certificate, no semisimplicity hypothesis and no Jordan information is used, so (12) is unconditional at every width and does not inherit the range $N \leq 13$ of the certificate proved in [14]. What that certificate would add, and why it is a strictly stronger statement which Theorem 3.1 does not imply, is the content of Observation 6.4 and of Section 9.

The proof consumes the power sums of the spectrum, which are the traces of the powers of T_N . The second main result supplies all of them at once; it is due to scientific-author-62 [12] and is proved in Section 5 together with the width reduction that makes it accessible.

Theorem 3.2 (traces at every power). *For every $M \geq 1$ and every width $N \geq 1$, writing γ_M as in (30),*

$$\text{tr}(T_N^M) = \begin{cases} 0, & N = 3k + 1, \\ \gamma_M^{2k}, & N = 3k, \ k \geq 1, \\ \gamma_M^{2k+1}, & N = 3k + 2, \ k \geq 0, \end{cases} \quad (13)$$

and therefore, since $\gamma_M^2 = 4$ when $3 \mid M$ and $\gamma_M^2 = 1$ otherwise,

$$\text{tr}(T_N^M) = Z_{M,N} = \begin{cases} 3m(N) + r(N) = C(N), & 3 \mid M, \\ r(N), & 3 \nmid M. \end{cases} \quad (14)$$

Theorem 3.2 is proved in Section 5, after the invariant space behind the recursion is constructed. Its first consequence is the single identity that the predecessor work could verify only finitely often.

Corollary 3.3 (the cubic trace). *For every width $N \geq 1$,*

$$\text{tr}(T_N^3) = 3m(N) + r(N) = \begin{cases} 0, & N = 3k + 1, \\ 4^k, & N = 3k, \\ 2 \cdot 4^k, & N = 3k + 2, \end{cases} \quad (15)$$

and $\text{tr}(T_N) = r(N)$. Both identities are equalities of integers; neither involves an eigenvalue computation, a floating-point value, or a matrix of size 2^N .

Proof. Put $M = 3$ and $M = 1$ in (14) and read off the three residue classes of N from (10). $\square$

The second consequence is the exact partition function of every finite strip, which is Corollary 8.1 of Section 8; the third is the fixed-boundary polynomial. Baxter proved the relation

$$[P_N]_{\text{fixed}} = \frac{[P_{N-1}]_{\text{free}}}{x^{2^{N-2}}} \quad (16)$$

between the fixed- and free-boundary characteristic polynomials [2], and the problem record takes it as proven [9].

Corollary 3.4 (fixed boundaries). *For every $N \geq 3$ the fixed-boundary characteristic polynomial is*

$$[P_N]_{\text{fixed}} = x^{n(N-1)-2^{N-2}} (x-1)^{m(N-1)+r(N-1)} (x^2+x+1)^{m(N-1)}, \quad (17)$$

a monic polynomial of degree 2^{N-2} with non-negative integer exponents.

Proof. Theorem 3.1 at width $N-1$ gives $[P_{N-1}]_{\text{free}} = x^{n(N-1)}(x-1)^{m(N-1)+r(N-1)}(x^2+x+1)^{m(N-1)}$, so (16) gives the displayed form as soon as $n(N-1) \geq 2^{N-2}$, that is, as soon as $C(N-1) \leq 2^{N-2}$. At $N=3$ and $N=4$ this is an equality, $C(2)=2=2^1$ and $C(3)=4=2^2$; at $N=5$ and $N=6$ it reads $0 \leq 8$ and $8 \leq 16$; and for $N \geq 7$, $C(N-1) \leq 2 \cdot 4^{(N-1)/3} \leq 4^{(N-2)/2} = 2^{N-2}$, since $2 \leq 4^{(N-4)/6}$ once $N \geq 7$. The degree is then $2^{N-1} - 2^{N-2} = 2^{N-2}$, and every exponent is a non-negative integer. $\square$

scientific-author-62 [12] proved the dual trace identity $\text{tr}([T_N]_{\text{fix}}^M) = \text{tr}(T_{N-1}^M)$ for every $N \geq 2$ and $M \geq 1$, which makes (16) unconditional by the same Newton argument that proves Theorem 3.1; Corollary 3.4 is verified directly on the literal fixed-boundary matrices, of dimension 2^{N-2} , for every $3 \leq N \leq 9$ (Appendix A).

Section 7 explains the arithmetic of (10); the statement is Theorem 7.3, and its content is that the factor four which appears in $m(N)$ is the square of the signed count of vertex covers of a triangle. Sections 8 and 9 collect the remaining consequences: the growth rate of the strip, the exact determinant values of $I - T_N$, $I + T_N$ and $2I - T_N$, the reduction of the open annihilation certificate to a single exact rank, and the combinatorial family of kernel vectors that all but discharges it.

4 The width-transfer reduction

The trace of T_N^M is a sum over M columns, each of length N , so it has 2^{MN} terms and evaluating it naively costs $M-1$ multiplications of $2^N \times 2^N$ matrices. Expanding the trace and regrouping its terms exchanges the roles of M and N : the same sum becomes an open chain of N site states, each of length M , with a product of $N-1$ identical local factors. All the width dependence is then carried by one matrix whose size depends on the length.

4.1 The construction

Fix a length $M \geq 1$. A *site state* is a binary word $s = (s_1, \dots, s_M) \in \{0, 1\}^M$ recording the occupation of one site of the strip across the M columns; because the trace closes the chain of columns, the index i of a site state is read cyclically, $s_{M+1} = s_1$. Site states are ordered lexicographically and identified with their indices, e_0 denoting the basis vector of the all-empty state $s = (0, \dots, 0)$. Write $|s| = s_1 + \dots + s_M$ and define the sign vector

$$\mathbf{w}_M \in \{-1, 1\}^{2^M}, \quad (\mathbf{w}_M)_s = (-1)^{|s|}, \quad (18)$$

the diagonal sign matrix $D_M = \text{diag}(\mathbf{w}_M)$, and the *width-transfer matrix*

$$(B_M)_{s,t} = (-1)^{|s|} \prod_{i=1}^M w(s_i, t_i, t_{i+1}, s_{i+1}) \in \{-1, 0, 1\}, \quad t_{M+1} = t_1, \quad (19)$$

with w the face weight (7). The four arguments are the corners of the face lying between the two columns indexed i and $i + 1$ at the two consecutive sites: s_i and s_{i+1} belong to the site state on one side of the face and t_i, t_{i+1} to the site state on the other, and the cyclic reading of i is the closing of the trace. The sign $(-1)^{|s|}$ is the activity of the site state on the left of the face; it is what makes the reduction a plain matrix product, as the proof below shows.

Lemma 4.1 (width reduction). *For every $M \geq 1$ and every width $N \geq 1$,*

$$\text{tr}(T_N^M) = \mathbf{1}^T B_M^{N-1} \mathbf{w}_M, \quad (20)$$

where $\mathbf{1}$ is the all-one vector of length 2^M .

Proof. Write the M columns as $\sigma^{(1)}, \dots, \sigma^{(M)} \in \{0, 1\}^N$ with $\sigma^{(M+1)} = \sigma^{(1)}$, and read the same data row by row as the site states $s_j = (\sigma_j^{(1)}, \dots, \sigma_j^{(M)}) \in \{0, 1\}^M$, $j = 1, \dots, N$. The map $(\sigma^{(1)}, \dots, \sigma^{(M)}) \mapsto (s_1, \dots, s_N)$ is a bijection from $(\{0, 1\}^N)^M$ to $(\{0, 1\}^M)^N$. Expanding (8) in the trace gives, for one term,

$$\prod_{i=1}^M (-1)^{|\sigma^{(i)}|} \cdot \prod_{i=1}^M \prod_{j=1}^{N-1} w(\sigma_j^{(i)}, \sigma_{j+1}^{(i)}, \sigma_{j+1}^{(i+1)}, \sigma_j^{(i+1)}). \quad (21)$$

Two facts reorganise it. First, $\sum_i |\sigma^{(i)}| = \sum_i \sum_j \sigma_j^{(i)} = \sum_j |s_j|$, because every site of the strip is counted once: the activity sign factorises over the site states. Second, the M face weights with a given j involve only s_j and s_{j+1} , the i -th of them being $w((s_j)_i, (s_{j+1})_i, (s_{j+1})_{i+1}, (s_j)_{i+1})$ with the index i cyclic. Free boundaries in the width direction leave the product over j running only to $N - 1$, so there is no wrap-around coupling s_N back to s_1 . Writing $F_M(s, t)$ for that product of M face weights, the trace is

$$\text{tr}(T_N^M) = \sum_{s_1, \dots, s_N} \prod_{j=1}^N (-1)^{|s_j|} \prod_{j=1}^{N-1} F_M(s_j, s_{j+1}). \quad (22)$$

Absorbing the sign of the left site state into B_M leaves exactly one unabsorbed sign, that of the last site state:

$$\text{tr}(T_N^M) = \sum_{s_1, \dots, s_N} \prod_{j=1}^{N-1} (B_M)_{s_j, s_{j+1}} \cdot (\mathbf{w}_M)_{s_N} = \mathbf{1}^T B_M^{N-1} \mathbf{w}_M. \quad (23) \quad \square$$

The two sides of (20) are the same finite sum of the same integers in a different grouping, so the identity is exact and needs no estimate. What requires care is the bookkeeping of signs, and it is worth naming where the convention could go wrong. The activity sign in (21) is the product of the M column activities; summing over columns counts each occupied site of the strip once, which is why it equals $\prod_j (-1)^{|s_j|}$ rather than a single sign. Each of those N factors is then absorbed into the $N - 1$ matrices, and the one that survives is the factor of the final site state, which is precisely $\mathbf{w}_M$. Had the sign been attached to the right site state instead, the matrix product would compute the trace of a different, transposed problem.

4.2 The mask form of B_M

The face weight (7) vanishes exactly when all four of its arguments are zero. For the face between site states s and t this means $s_i = s_{i+1} = 0$ and $t_i = t_{i+1} = 0$ simultaneously, so it is convenient to record, for each site state, which adjacent pairs are empty. Define

$$p_i[s] = [s_i = 0 \text{ and } s_{i+1} = 0], \quad \mu(s) = \sum_{i=1}^M p_i[s] 2^{i-1}, \quad \chi_M[s] = [\mu(s) = 0], \quad (24)$$

with $[\cdot]$ the indicator and the index i cyclic, so that $\chi_M[s] = 1$ exactly when s has no two cyclically consecutive zeros. Then

$$Q_M[s, t] = \prod_{i=1}^M (1 - p_i[s] p_i[t]) = 1 - [\mu(s) \wedge \mu(t) \neq 0] \in \{0, 1\}, \quad B_M = D_M Q_M, \quad (25)$$

where $\wedge$ is the bitwise AND of the two masks. The mask form makes the structure of B_M visible: it is a sign times a compatibility matrix, and two site states are compatible exactly when no position is an empty adjacent pair in both. This is the form used in Section 5, where the whole argument reduces to three vector identities on Q_M .

4.3 What the reduction buys

The dimension of T_N is 2^N and grows with the width; the dimension of B_M is 2^M and grows with the length. Since M is the free parameter of the trace and N is the parameter whose closed form is sought, (20) converts a family of intractable eigenvalue problems into a family of small fixed matrices: for each M the entire sequence $\text{tr}(T_N^M)$, $N = 1, 2, 3, \dots$, is read off the powers of one $2^M \times 2^M$ integer matrix, at widths far beyond any affordable 2^N matrix. That is what makes an all- M statement provable, and it is the reason the trace theorem of Section 5 reaches every width.

At $M = 1$ the matrix is 2×2 ,

$$B_1 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \text{charpoly}(B_1) = x^2 + x + 1, \quad \mathbf{w}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad (26)$$

in the site order $(0), (1)$, and (20) reads $\text{tr}(T_N) = \mathbf{1}^T B_1^{N-1} \mathbf{w}_1$, a two-term recursion in the width whose characteristic polynomial is $x^2 + x + 1$. The roots of $x^2 + x + 1$ are j and j^2 , so the sequence is 3-periodic: with seeds $\text{tr}(T_1) = 0$ and $\text{tr}(T_2) = -1$ it continues $1, 0, -1, 1, \dots$, which is exactly the identity $\text{tr}(T_N) = r(N)$ proved unconditionally in [14], here obtained from the length side. The period-three arithmetic of Baxter's exponents is therefore already visible in the smallest case of the reduction, and this is the only value of M for which the width-transfer matrix is itself the object whose spectrum produces the period.

4.4 The matrix at $M = 3$

The smallest length at which the mechanism of Section 7 operates is $M = 3$, where B_3 is 8×8 . In the lexicographic site order $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$,

$$B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \end{pmatrix}, \quad \mathbf{w}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}. \quad (27)$$

The rows are indexed by the eight occupation patterns of a triangle of sites and carry the sign of that pattern, so the all-occupied row is -1 throughout and the all-empty row is the sparsest. The single zero in the first row and first column is the all-empty triangle against itself: the face weight (7) forbids a face with all four corners empty, and at $M = 3$ the three faces of a triangle of empty sites are all forbidden at once. The vector $\mathbf{w}_3$ is the sign of the occupation pattern, and it is the terminal factor of (20).

Three properties of B_3 are used later and are visible in (27). Its characteristic polynomial is $x^3(x-1)^2(x^3-4)$ and its minimal polynomial is $x(x-1)(x^3-4)$, which is square-free, so B_3 is

semisimple; the factor $x^3 - 4$ supplies the cube roots of 4 whose moduli carry the growth of $Z_{3,N}$ in the width. The all-one vector is a left eigenvector of B_3^3 with eigenvalue 4, $\mathbf{1}^T B_3^3 = 4 \mathbf{1}^T$ entry for entry, and $\mathbf{w}_3$ is a right eigenvector, $B_3^3 \mathbf{w}_3 = 4 \mathbf{w}_3$. The row sums of B_3^3 are 4, 4, 4, 8, 4, 8, 8, -8 , so $\mathbf{1}$ is not a right eigenvector of B_3^3 : the left and right identities are different statements, and only their conjunction collapses the width recursion at cubes of the length. Finally, the space spanned by e_0 , $B_3 e_0$ and $\mathbf{w}_3$ is B_3 -invariant with $B_3^3 = 4 \text{Id}$ on it, which is the $M = 3$ instance of the construction in Section 5.

4.5 Validation of the reduction

(20) is an identity of integers and is checked as such against the literal matrix (8). For every $M \leq 4$ and every $N \leq 9$, thirty-six pairs in all, the left side computed by exact integer multiplication of T_N built entry by entry from (8) agrees with the right side computed by exact integer multiplication of B_M built entry by entry from (19), with dimensions up to $2^9 = 512$ on the left and $2^4 = 16$ on the right; both also agree with the closed form (14), which is not used in either computation (Appendix A). Two further checks pin the conventions down. The characteristic polynomial of B_1 is $x^2 + x + 1$, so the $M = 1$ case of the reduction reproduces the first-trace theorem of [14] from the width direction. At $M = 3$ the matrix (27) was built twice, once from the face weight (19) and once from the mask form (25), and the two agree entry for entry; $\mathbf{w}_3$ agrees with (18); and the three seeds $c_t = \mathbf{1}^T B_3^t \mathbf{w}_3$, $t = 0, 1, 2$, are $(0, 2, 4)$, from which $4^q c_t$ reproduces $C(N)$ for every $N = 1, \dots, 13$ with $N - 1 = 3q + t$.

5 Traces at every power

The reduction of Section 4 expresses $\text{tr}(T_N^M)$ as a chain of $N - 1$ copies of B_M . The chain is now evaluated for every M and every N , and the answer is a three-branch closed form whose branches are exactly the exponents of Baxter's conjecture. Nothing in this section is numerical: every identity is an equality of integer vectors or of integers, and the only sums are finite.

5.1 The scalar γ_M

Define

$$\gamma_M = \sum_{s \in \{0,1\}^M} \chi_M[s] (\mathbf{w}_M)_s = \sum_{\substack{s \in \{0,1\}^M \\ s \text{ has no two} \\ \text{cyclically consecutive zeros}}} (-1)^{|s|}, \quad (28)$$

the signed count of the site states that survive the mask condition (24). At $M = 3$ the surviving states are 011, 101, 110, 111 with signs $+, +, +, -$, so $\gamma_3 = 3 - 1 = 2$.

Lemma 5.1 (γ_M in closed form). *For every $M \geq 1$,*

$$\gamma_M = (-1)^M I(\mathbf{C}_M; -1) = (-1)^M \text{tr}(A^M), \quad A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad (29)$$

where $I(G; t) = \sum_S \text{independent } t^{|S|}$ is the independence polynomial and $\mathbf{C}_M$ is the cycle on M vertices. Equivalently $\gamma_M = (-1)^M 2 \cos(\pi M/3)$, so

$$\gamma_M = \begin{cases} 2, & 3 \mid M, \\ -1, & 3 \nmid M, \end{cases} \quad \gamma_M^2 = \begin{cases} 4, & 3 \mid M, \\ 1, & 3 \nmid M. \end{cases} \quad (30)$$

Proof. Substitute $t_i = 1 - s_i$ coordinatewise. The condition that s has no two cyclically consecutive zeros becomes the condition that no two cyclically consecutive t_i are both 1, that is, that $\{i : t_i = 1\}$ is an independent set of the cycle $\mathbf{C}_M$; and $(-1)^{|s|} = (-1)^{M-|t|} = (-1)^M (-1)^{|t|}$. Hence $\gamma_M = (-1)^M I(\mathbf{C}_M; -1)$. The independence polynomial of a cycle is the trace of the M -th power

of the standard two-state transfer matrix that forbids two consecutive occupied sites, which for the weight $t = -1$ is A as displayed: a state with $t_i = 0$ contributes weight 1 and may be followed by either state, while a state with $t_i = 1$ contributes weight -1 and may only be followed by $t_{i+1} = 0$. The characteristic polynomial of A is $x^2 - x + 1$, whose roots are $\zeta = \exp(i\pi/3)$ and $\bar{\zeta} = \exp(-i\pi/3)$, so $\text{tr}(A^M) = \zeta^M + \bar{\zeta}^M = 2\cos(\pi M/3)$. $\square$

The values (30) are a case analysis on M modulo three, and it is short enough to write out. For $M = 3k$ the cosine is $\cos(k\pi) = (-1)^k$ and $(-1)^M = (-1)^k$, so $\gamma_M = (-1)^{2k} \cdot 2 = 2$. For $M = 3k + 1$ it is $\cos(k\pi + \frac{\pi}{3}) = (-1)^k \frac{1}{2}$ while $(-1)^M = (-1)^{k+1}$, so $\gamma_M = (-1)^{2k+1} = -1$. For $M = 3k + 2$ it is $\cos(k\pi + \frac{2\pi}{3}) = (-1)^k \frac{1}{2}$ while $(-1)^M = (-1)^k$, so again $\gamma_M = -1$. The three routes in (29) – enumeration of all 2^M states, the 2×2 transfer matrix, and the closed form – are compared exactly for every $M \leq 16$ here, together with γ_M itself as computed from the chain; the record compares them by enumeration for every $M \leq 18$ and checks the closed form for every $M \leq 2000$ (Appendix A).

5.2 Three vector identities

The whole of the trace theorem follows from one identity on Q_M , proved by a sign-reversing involution, and from three one-line consequences of it.

Lemma 5.2 (the involution lemma). *For every $M \geq 1$,*

$$Q_M \mathbf{w}_M = \gamma_M e_0. \quad (31)$$

Proof. Fix a site state s . Since $\sum_{s'} (\mathbf{w}_M)_{s'} = (1 - 1)^M = 0$,

$$(Q_M \mathbf{w}_M)[s] = \sum_{s'} Q_M[s, s'] (\mathbf{w}_M)_{s'} = \sum_{s'} (\mathbf{w}_M)_{s'} - \sum_{\substack{s' \\ \mu(s) \wedge \mu(s') \neq 0}} (\mathbf{w}_M)_{s'} = \sum_{\substack{s' \\ \mu(s) \wedge \mu(s') = 0}} (\mathbf{w}_M)_{s'}. \quad (32)$$

Suppose $s \neq 0$ and let i be the smallest index with $s_i = 1$. Flipping the coordinate s'_i is an involution of the surviving set $\{s' : \mu(s) \wedge \mu(s') = 0\}$: flipping s'_i can change only $p_{i-1}[s']$ and $p_i[s']$, hence only the bits $i - 1$ and i of $\mu(s')$, and both of those bits are already zero in $\mu(s)$ because $s_i = 1$ forces $p_{i-1}[s] = p_i[s] = 0$. The involution therefore preserves the condition $\mu(s) \wedge \mu(s') = 0$, it changes $|s'|$ by one and hence reverses the sign, and it has no fixed point. The sum over the surviving set is consequently zero, so $(Q_M \mathbf{w}_M)[s] = 0$ for every $s \neq 0$. For $s = 0$ the mask $\mu(0)$ has every bit set, so the condition $\mu(0) \wedge \mu(s') = 0$ forces $\mu(s') = 0$, and the sum is $\sum_{s'} \chi_M[s'] (\mathbf{w}_M)_{s'} = \gamma_M$ by (28). $\square$

The involution is fixed-point-free on the surviving set for every $s \neq 0$, which is why no term survives except at the all-empty state. This is the combinatorial heart of the argument: a cancellation that is invisible in the spectrum of B_M and massive in the entries, since the number of vertex covers of $\mathbb{C}_M$ grows like a Lucas number while γ_M stays in $\{2, -1\}$ (Section 7).

Corollary 5.3 (four identities on B_M). *For every $M \geq 1$, with $B_M = D_M Q_M$, $D_M = \text{diag}(\mathbf{w}_M)$ and $\odot$ the coordinatewise product,*

$$\mathbf{1}^T B_M = \gamma_M e_0^T, \quad (33)$$

$$B_M \mathbf{w}_M = \gamma_M e_0, \quad (34)$$

$$B_M e_0 = \mathbf{w}_M \odot \chi_M, \quad (35)$$

$$B_M (\mathbf{w}_M \odot \chi_M) = \gamma_M \mathbf{w}_M. \quad (36)$$

Proof. Q_M is symmetric, because $\mu(s) \wedge \mu(t) \neq 0$ is symmetric in s and t , and $D_M^2 = I$. For (33), $\mathbf{1}^T B_M = \mathbf{1}^T D_M Q_M = \mathbf{w}_M^T Q_M = (Q_M \mathbf{w}_M)^T = \gamma_M e_0^T$ by Lemma 5.2. For (34), $B_M \mathbf{w}_M = D_M Q_M \mathbf{w}_M = \gamma_M D_M e_0 = \gamma_M \mathbf{w}_M[0] e_0 = \gamma_M e_0$, since the all-empty state has sign $+1$. For (35),

$(Q_M e_0)[s] = Q_M[s, 0] = 1 - [\mu(s) \wedge \mu(0) \neq 0] = 1 - [\mu(s) \neq 0] = \chi_M[s]$ because $\mu(0)$ has every bit set; multiplying by D_M gives $B_M e_0 = \mathbf{w}_M \odot \chi_M$. For (36), $Q_M[s, t] = 1$ whenever $\chi_M[t] = 1$, so

$$(Q_M(\mathbf{w}_M \odot \chi_M))[s] = \sum_t \chi_M[t] (\mathbf{w}_M)_t = \gamma_M \quad (37)$$

for every s , independently of s ; applying D_M gives $B_M(\mathbf{w}_M \odot \chi_M) = \gamma_M \mathbf{w}_M$. $\square$

5.3 Proof of the trace theorem

Proof of Theorem 3.2. Set $c_N = \mathbf{1}^T B_M^{N-1} \mathbf{w}_M$, which equals $\text{tr}(T_N^M)$ by Lemma 4.1. The three seeds come from Corollary 5.3: $c_1 = \mathbf{1}^T \mathbf{w}_M = \sum_s (-1)^{|s|} = 0$; $c_2 = \mathbf{1}^T B_M \mathbf{w}_M = \gamma_M \mathbf{1}^T e_0 = \gamma_M$ by (34); and $c_3 = \mathbf{1}^T B_M^2 \mathbf{w}_M = \gamma_M \mathbf{1}^T (\mathbf{w}_M \odot \chi_M) = \gamma_M^2$ by (35) and (28). Combining (34) with (36) gives $B_M^3 \mathbf{w}_M = B_M^2 (\gamma_M e_0) = \gamma_M B_M (\mathbf{w}_M \odot \chi_M) = \gamma_M^2 \mathbf{w}_M$, hence $c_{N+3} = \mathbf{1}^T B_M^{N+2} \mathbf{w}_M = \mathbf{1}^T B_M^{N-1} (B_M^3 \mathbf{w}_M) = \gamma_M^2 c_N$ for every $N \geq 1$. The sequence $(c_N)_{N \geq 1}$ is therefore $0, \gamma_M, \gamma_M^2, 0, \gamma_M, \gamma_M^2, \dots$, which is (13): $c_N = 0$ when $N \equiv 1 \pmod{3}$, $c_N = \gamma_M^{2k}$ when $N = 3k$ with $k \geq 1$, and $c_N = \gamma_M^{2k+1}$ when $N = 3k + 2$.

For (14), take $3 \mid M$ first, so $\gamma_M = 2$ and $\gamma_M^2 = 4$. At $N = 3k + 1$ the value is $0 = 3 \cdot 0 + 0 = 3m(N) + r(N)$. At $N = 3k$ with $k \geq 1$ it is $4^k = 3 \cdot \frac{4^k - 1}{3} + 1 = 3m(N) + r(N)$. At $N = 3k + 2$ it is $2 \cdot 4^k = 2^{2k+1} = 3 \cdot \frac{2^{2k+1} - 1}{3} + 1 = 3m(N) + r(N)$. Now take $3 \nmid M$, so $\gamma_M = -1$ and $\gamma_M^2 = 1$: the three branches give 0, 1 and -1 at $N \equiv 1, 0, 2 \pmod{3}$, which is $r(N) = 1 - (N \bmod 3)$ in the three classes. $\square$

The proof uses only the three vector identities and no spectral information about B_M . That is worth emphasising, because the eigenvalues of B_M do not determine the recursion: at $M = 3$ the polynomial $x^2 + x + 1$ does not divide $\text{charpoly}(B_3) = x^3(x-1)^2(x^3-4)$, so no argument from the eigenvalues of B_M alone can produce period three in N at every length. What the invariant space does give is the following, which is where the cube roots of 4 in $\text{charpoly}(B_3)$ come from.

Corollary 5.4 (the invariant space). *For every $M \geq 1$ the vectors*

$$b_0 = e_0, \quad b_1 = \mathbf{w}_M \odot \chi_M, \quad b_2 = \mathbf{w}_M \quad (38)$$

satisfy $B_M b_0 = b_1$, $B_M b_1 = \gamma_M b_2$ and $B_M b_2 = \gamma_M b_0$, so the space $V_M = \text{span}\{b_0, b_1, b_2\}$ is B_M -invariant with $B_M^3 = \gamma_M^2 \text{Id}$ on V_M , and in the basis (b_0, b_1, b_2) the restriction has the matrix

$$C_M = \begin{pmatrix} 0 & 0 & \gamma_M \\ 1 & 0 & 0 \\ 0 & \gamma_M & 0 \end{pmatrix}, \quad \text{charpoly}(C_M) = x^3 - \gamma_M^2. \quad (39)$$

For $M \geq 3$ the three vectors are linearly independent, so $x^3 - \gamma_M^2$ divides $\text{charpoly}(B_M)$ over $\mathbb{Z}$; at $M \leq 2$ they are not, and no divisibility is claimed.

Proof. The three relations are (35), (36) and (34). They give $B_M^3 b_0 = \gamma_M^2 b_0$, $B_M^3 b_1 = \gamma_M^2 b_1$ and $B_M^3 b_2 = \gamma_M^2 b_2$, so V_M is invariant and B_M^3 acts on it as γ_M^2 times the identity, and the matrix of the restriction in that basis has the columns given by the images, which is (39).

For the dimension, note first that $\chi_M[0] = 0$ whenever $M \geq 2$, because the all-empty state has every p_i equal to 1; hence b_1 vanishes at coordinate 0 while b_0 and b_2 both equal 1 there. Suppose $ab_0 + cb_2 = b_1$. Reading coordinate 0 gives $a + c = 0$. For $M \geq 3$ the state $s = (0, \dots, 0, 1)$, with a single occupied site, still has two cyclically consecutive zeros, so $\chi_M[s] = 0$ and $b_1[s] = 0$, while $b_0[s] = 0$ and $b_2[s] = -1$; reading that coordinate gives $c = 0$, hence $a = 0$, so $b_1 \notin \text{span}\{b_0, b_2\}$. Since b_0 and b_2 are independent, b_0 being supported on one coordinate and b_2 having every coordinate ± 1 , this gives $\dim V_M = 3$ for every $M \geq 3$. At $M = 1$ the only state with $\chi_1[s] = 1$ is $s = (1)$, so $b_1 = (0, -1) = b_2 - b_0$; at $M = 2$ a direct evaluation gives $b_1 = (0, -1, -1, 1) = b_2 - b_0$ as well. In both cases $\dim V_M = 2$ and the divisibility statement is vacuous. $\square$

Corollary 5.4 is the reason the trace recursion has period three in the width: on the three-dimensional space that the observable $\mathbf{1}^T(\cdot) \mathbf{w}_M$ sees, B_M is a permutation of the basis twisted by γ_M , and its cube is a scalar. The rest of the spectrum of B_M is invisible to that observable. Section 7 computes $\text{charpoly}(B_M)$ exactly for $M \leq 7$ and shows that the invisible part is genuinely large.

5.4 The fibre cancellation

The identities (33)–(36) have a second consequence, which scientific-author-78 [17] established by a different route and which is the input that Part II of the record had verified only for $M \leq 14$.

Proposition 5.5 (entrywise fibre cancellation). *For every $M \geq 1$, as identities of all 2^M column sums and of all 2^M entries respectively,*

$$\mathbf{1}^T B_M^3 = \gamma_M^2 \mathbf{1}^T = \alpha_M^2 \mathbf{1}^T, \quad B_M^3 \mathbf{w}_M = \gamma_M^2 \mathbf{w}_M = \alpha_M^2 \mathbf{w}_M, \quad (40)$$

where α_M , the signed count of vertex covers of $\mathcal{C}_M$ defined in Section 7, equals γ_M by Theorem 7.3.

Proof. The right identity is the computation already made in the proof of Theorem 3.2: $B_M^3 \mathbf{w}_M = B_M^2(\gamma_M e_0) = \gamma_M B_M(\mathbf{w}_M \odot \chi_M) = \gamma_M^2 \mathbf{w}_M$ by (34) and (36). For the left identity, two row-vector evaluations are needed. First, $e_0^T B_M = e_0^T D_M Q_M = e_0^T Q_M = (Q_M e_0)^T = \chi_M^T$, using $D_M e_0 = e_0$ and the computation of $Q_M e_0$ in (35). Second, $\chi_M^T B_M = \chi_M^T D_M Q_M = (\mathbf{w}_M \odot \chi_M)^T Q_M = (Q_M(\mathbf{w}_M \odot \chi_M))^T = \gamma_M \mathbf{1}^T$, using the symmetry of Q_M and the identity $Q_M(\mathbf{w}_M \odot \chi_M) = \gamma_M \mathbf{1}$ proved in (36). Chaining these with (33),

$$\mathbf{1}^T B_M^3 = (\mathbf{1}^T B_M) B_M^2 = \gamma_M e_0^T B_M^2 = \gamma_M (e_0^T B_M) B_M = \gamma_M \chi_M^T B_M = \gamma_M^2 \mathbf{1}^T. \quad (41)$$

Both sides are integer vectors of length 2^M , so this is an entrywise identity. $\square$

Section 7 proves that $\gamma_M = \alpha_M$, the signed count of vertex covers of the cycle $\mathcal{C}_M$, which is the form in which (40) is usually quoted: the third power of the width-transfer matrix scales both observable vectors by the square of a signed count of covers of one cycle. The scalar γ_M^2 is then 4 or 1, and it is the same scalar that Theorem 3.2 raises to the power $\lfloor (N-1)/3 \rfloor$.

The proof just given is not the one scientific-author-78 [17] supplied, and the difference matters for the record. That proof expands the three plaquette layers of $\mathbf{1}^T B_M^3$, applies a coverage lemma to a single plaquette, and groups the middle boundary by the union E of two of the four boundary states; every fibre then contributes $(-1)^{|E|}$ independently of the boundary state, so the sum factorises as $\alpha_M \cdot \alpha_M$ with no invariant space, no companion matrix and no eigenvector of B_M . It shares no linear algebra with the route of Section 5, which is why it counts as an independent second proof of the one matrix-level input of Theorem 3.2. Both identities in (40) are verified here entrywise on the literal B_M , all 2^M column sums and all 2^M entries, for every $M = 3, \dots, 10$, where the dimension reaches 1024 (Appendix A).

One precision is worth keeping, because it is easy to state the left identity in a form that is false. $\mathbf{1}$ is a *left* eigenvector of B_M^3 and not a right one: the row sums of B_M^3 are 4, 4, 4, 8, 4, 8, 8, -8 , which are not constant, while its column sums are all 4. The column-sum form is the one used in Theorem 3.2 and the one verified.

6 Newton's identities and the proof of the main theorem

Theorem 3.2 supplies every power sum of the eigenvalues of T_N . This section turns them into the characteristic polynomial. The step is standard but its exact form matters, because it is what removes the need for any spectral hypothesis: power sums see the eigenvalues and nothing else, so they determine the characteristic polynomial without seeing the Jordan form.

6.1 Power sums determine the characteristic polynomial

Let A be a $d \times d$ matrix over $\mathbb{C}$ with eigenvalues $\lambda_1, \dots, \lambda_d$ listed with algebraic multiplicity, and write

$$p_k(A) = \sum_{i=1}^d \lambda_i^k, \quad e_k(A) = \sum_{1 \leq i_1 < \dots < i_k \leq d} \lambda_{i_1} \cdots \lambda_{i_k}, \quad e_0(A) = 1, \quad (42)$$

so that $\det(xI - A) = \sum_{k=0}^d (-1)^k e_k(A) x^{d-k}$.

Lemma 6.1 (power sums are the traces of the powers). *For every $k \geq 1$, $p_k(A) = \operatorname{tr}(A^k)$, whether or not A is diagonalisable.*

Proof. Conjugate A to Jordan form $A = SJS^{-1}$. Then $A^k = SJ^kS^{-1}$ and $\operatorname{tr}(A^k) = \operatorname{tr}(J^k)$. Each Jordan block with eigenvalue λ is $\lambda I + N$ with N strictly upper triangular and nilpotent, so its k -th power has diagonal λ^k and strictly upper triangular remainder; summing the diagonals of the blocks gives $\sum_i \lambda_i^k = p_k(A)$. $\square$

Lemma 6.2 (Newton's identities). *For every $1 \leq k \leq d$,*

$$k e_k = \sum_{i=1}^k (-1)^{i-1} e_{k-i} p_i, \quad (43)$$

so $e_1, \dots, e_d$ are determined by $p_1, \dots, p_d$ by a recursion whose only division is by the integer $k \leq d$. Two multisets of d complex numbers with the same first d power sums have the same elementary symmetric functions and therefore the same monic characteristic polynomial.

Proof. Both sides are the standard Newton identities for the elementary symmetric functions of $\lambda_1, \dots, \lambda_d$; the recursion (43) is triangular in k with leading coefficient $k \neq 0$, so it determines e_k from $e_0, \dots, e_{k-1}$ and $p_1, \dots, p_k$ uniquely over any field of characteristic zero, in particular over $\mathbb{Q}$. The final statement follows because the monic polynomial with those roots is $\sum_k (-1)^k e_k x^{d-k}$. $\square$

When A has integer entries the p_k are integers, the e_k are a priori rationals, and the conclusion that they are integers is not needed: the polynomial $\sum_k (-1)^k e_k x^{d-k}$ is $\det(xI - A)$, which lies in $\mathbb{Z}[x]$ because A does.

6.2 Proof of the main theorem

Proof of Theorem 3.1. Fix $N \geq 1$, let $d = 2^N$, and let $\lambda_1, \dots, \lambda_d$ be the eigenvalues of the literal matrix T_N of (8), listed with algebraic multiplicity. By Lemma 6.1 and Theorem 3.2,

$$p_k(T_N) = \operatorname{tr}(T_N^k) = \begin{cases} 3m + r, & 3 \mid k, \\ r, & 3 \nmid k, \end{cases} \quad \text{for every } k \geq 1, \quad (44)$$

where $m = m(N)$, $r = r(N)$ and $n = n(N) = 2^N - 3m - r$; all four are integers and $n, m, m + r$ are non-negative.

Now form the multiset that Baxter's closed form predicts,

$$\Lambda' = \underbrace{\{1, \dots, 1\}}_{m+r} \cup \underbrace{\{j, \dots, j\}}_m \cup \underbrace{\{j^2, \dots, j^2\}}_m \cup \underbrace{\{0, \dots, 0\}}_n, \quad (45)$$

whose cardinality is $(m + r) + m + m + n = 2^N = d$ by (11), and whose power sums are

$$p'_k = (m + r) + m(j^k + j^{2k}) + n \cdot 0^k = \begin{cases} (m + r) + 2m = 3m + r, & 3 \mid k, \\ (m + r) + m(-1) = r, & 3 \nmid k, \end{cases} \quad (46)$$

because $j^k + j^{2k} = 2$ when $3 \mid k$ and $j^k + j^{2k} = j + j^2 = -1$ when $3 \nmid k$. These are exactly the values (44), for every $k \geq 1$ and in particular for $k = 1, \dots, d$. By Lemma 6.2 the monic characteristic polynomial of T_N equals the monic polynomial with root multiset Λ' , namely

$$(x-1)^{m+r}(x^2+x+1)^m x^n = x^n f_3(x)^m f_1(x)^r, \quad (47)$$

where the second form uses $f_3 = (x-1)(x^2+x+1)$ and holds as an identity in $\mathbb{Q}(x)$ when $r = -1$, the factor $(x-1)$ cancelling. The multiplicities of $1, j, j^2$ and 0 read off from Λ' are $m+r, m, m$ and n , which is the second statement of the theorem. $\square$

Two features of the argument are worth isolating, because they are where the earlier range restriction came from. The first is that no step uses the Jordan form of T_N : Lemma 6.1 is valid for any matrix, so the conclusion is a statement about algebraic multiplicities only. The second is that no step uses a bound on N : the recursion (43) is applied to $d = 2^N$ power sums which Theorem 3.2 supplies in closed form for every $k \leq d$, and the only objects manipulated are the integers $m(N), r(N)$ and $n(N)$. Neither T_N nor B_M is ever formed.

6.3 A worked instance

At width $N = 3$ the matrix T_3 is 8×8 , so the whole Newton step can be carried out by hand. Theorem 3.2 gives the eight power sums

$$(p_1, \dots, p_8) = (1, 1, 4, 1, 1, 4, 1, 1), \quad (48)$$

since $r(3) = 1$ and $C(3) = 4$. The recursion (43) gives, in order, $e_1 = p_1 = 1$; $2e_2 = e_1 p_1 - p_2 = 1 - 1 = 0$; $3e_3 = e_2 p_1 - e_1 p_2 + p_3 = 0 - 1 + 4 = 3$; $4e_4 = e_3 p_1 - e_2 p_2 + e_1 p_3 - p_4 = 1 - 0 + 4 - 1 = 4$; and $5e_5 = e_4 p_1 - e_3 p_2 + e_2 p_3 - e_1 p_4 + p_5 = 1 - 1 + 0 - 1 + 1 = 0$, with $e_6 = e_7 = e_8 = 0$ by the same recursion. Hence

$$(e_1, \dots, e_8) = (1, 0, 1, 1, 0, 0, 0, 0), \quad P_3(x) = x^8 - x^7 - x^5 + x^4 = x^4(x-1)^2(x^2+x+1), \quad (49)$$

which is the entry of Table 1 at $N = 3$. Computing the eight traces of the literal 8×8 matrix and running the same recursion reproduces these numbers exactly, and an independent exact characteristic-polynomial algorithm on the same matrix returns the same polynomial (Appendix A). The divisions by $k = 2, \dots, 8$ are the only non-integer operations in the argument, and each of them is exact.

6.4 No reality hypothesis is needed

Remark 6.3. It is sometimes recorded as a hypothesis that the multiplicities of j and j^2 in $\text{charpoly}(T_N)$ agree, or equivalently that the non-zero part of the spectrum is closed under complex conjugation. No such hypothesis is needed here, and none was needed in the certified range either: T_N is an integer matrix, so $P_N \in \mathbb{Z}[x]$, and complex conjugation permutes the roots of a real polynomial preserving multiplicity. Since j and $j^2 = \bar{j}$ are the only non-real numbers that can occur once the spectrum is known to lie in $\{0, 1, j, j^2\}$, their multiplicities are automatically equal. The correction was raised by scientific-author-78 [17] and is adopted throughout this paper: the multiplicity of j is $m(N)$ and that of j^2 is $m(N)$ because P_N has integer coefficients, not because a reality assumption was supplied.

6.5 The certificate is strictly stronger, and a prefix of traces is not enough

The annihilation certificate of [14] is the statement that for some $K \geq 0$,

$$T_N^K(T_N^3 - I) = 0. \quad (50)$$

It forces the spectrum into $\{0, 1, j, j^2\}$ and the non-zero part to be semisimple, so it implies Theorem 3.1 at the widths where it holds. The converse fails, and the failure is explicit.

Observation 6.4 (a 32×32 witness). Let

$$J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \quad (51)$$

so that C is the companion matrix of $x^2 + x + 1$ and J is a non-trivial Jordan block at the eigenvalue 1, and form the two 32×32 integer matrices

$$A_{\text{bad}} = J \oplus 0_{24} \oplus C^{\oplus 3}, \quad A_{\text{good}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0_{24} \oplus C^{\oplus 3}. \quad (52)$$

Then $\text{charpoly}(A_{\text{bad}}) = \text{charpoly}(A_{\text{good}}) = x^{24}(x-1)^2(x^2+x+1)^3 = P_5(x)$, the polynomial predicted by Theorem 3.1 at width five, while $A_{\text{good}}(A_{\text{good}}^3 - I) = 0$ and $A_{\text{bad}}^K(A_{\text{bad}}^3 - I) \neq 0$ for every $K \geq 0$. Moreover

$$\text{rank}(A_{\text{bad}}^3 - I) = 25 > 24 = n(5) = \text{rank}(A_{\text{good}}^3 - I), \quad (53)$$

so the characteristic polynomial cannot separate the two matrices while the exact rank of $A^3 - I$ does.

Proof. Both matrices are block diagonal with the same block sizes and the same characteristic polynomials block by block: J and I_2 both contribute $(x-1)^2$, 0_{24} contributes x^{24} , and each copy of C contributes $x^2 + x + 1$. For the certificate, $J^3 - I = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ and $J^K(J^3 - I) = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} \neq 0$ for every $K \geq 0$, so no power of A_{bad} annihilates $A_{\text{bad}}^3 - I$; on the other hand $I_2^3 - I = 0$, $C^3 = I$ and $0_{24}^3 - 0_{24} = 0$, so $A_{\text{good}}^3 - I$ is zero on every block except the nilpotent one, and $A_{\text{good}}(A_{\text{good}}^3 - I) = 0$ with $K = 1$. For the ranks, $A_{\text{good}}^3 - I = 0 \oplus (-I_{24}) \oplus 0^{\oplus 3}$ has rank 24, while $J^3 - I = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$ has rank 1 and contributes one more. $\square$

Observation 6.4 was constructed by scientific-author-62 [12] and is re-derived here from the definitions; the ranks, the characteristic polynomials and the failure of the certificate for every $K \leq 60$ are computed exactly in Appendix A, as is the same failure for every $K \leq 600$ in the record. Its content for this paper is that Theorem 3.1 is a genuine weakening of (50) – it says nothing about Jordan blocks – and that the weakening is what allows it to hold at every width. Section 9 takes up the question that the witness raises, namely which exact integer decides (50).

The second boundary of the trace route is that a *finite* prefix of the power sums cannot replace Theorem 3.2, however long. The next remark is also due to scientific-author-62 [12].

Remark 6.5 (a finite prefix of traces does not determine the polynomial). At width five, where $d = 2^5 = 32$, consider the two monic integer polynomials

$$Q_1(x) = x^{24}(x-1)^2(x^2+x+1)^3 = P_5(x), \quad Q_2(x) = x^7(x^{17}-1)(x-1)^2(x^2+x+1)^3. \quad (54)$$

Both have degree 32 and they agree on the first sixteen power sums, $p_k(Q_1) = p_k(Q_2)$ for $1 \leq k \leq 16$, while $p_{17}(Q_1) = -1$ and $p_{17}(Q_2) = 16$. The reason is that a complete set of seventeen 17th roots of unity contributes $\sum \zeta^k = 0$ to p_k for $1 \leq k \leq 16$ and 17 to p_{17} , while the seventeen zero roots it replaces contribute nothing to any p_k with $k \geq 1$. Since $16 < 17 \leq 32$, no collection of traces $\text{tr}(T_5^M)$ with $M \leq 16$ can determine $\text{charpoly}(T_5)$, and the machine verification of the trace pattern in the record, which reached $M \leq 14$, could not have sufficed at any width where $2^N \geq 17$ – however much further it was pushed. Theorem 3.2, which supplies every M at once in closed form, is exactly the missing input.

Verification of (54). Both polynomials in (54) are monic with integer coefficients and degree $7 + 17 + 2 + 6 = 32$. Write p_k for the power sums of the roots. Zero roots contribute nothing for $k \geq 1$; the factor $(x-1)^2$ contributes 2; the factor $(x^2+x+1)^3$ contributes $3(j^k + j^{2k})$, which

is 6 when $3 \mid k$ and -3 otherwise; and the factor $x^{17} - 1$ contributes $\sum_{\zeta^{17}=1} \zeta^k$, which is 0 for $1 \leq k \leq 16$ and 17 for $k = 17$. Hence for $1 \leq k \leq 16$ both polynomials give $p_k = 2 + 3(j^k + j^{2k})$, while at $k = 17$ the first gives $2 + 3(j^2 + j) = -1$ and the second gives $17 + 2 - 3 = 16$. $\square$

The same phenomenon on a smaller scale explains a warning recorded in the literature of this problem: the first six power sums of $x^7 - 1$ vanish and the seventh is 7, so a trace range that stops before the order of a root of unity cannot detect it. Both Q_1 and Q_2 above are products of cyclotomic polynomials and a power of x , and Q_2 is a perfectly plausible candidate for the characteristic polynomial of an integer matrix whose spectrum consists of roots of unity and zeros. What excludes it is not its shape but the seventeenth power sum.

6.6 Exact confirmation on the literal matrices

Theorem 3.1 is proved for every width, so no finite verification can add to it; the verification is nevertheless worth reporting, because it exercises a completely different route and would detect a misreading of the model. For every $N \leq 6$ the matrix T_N was built entry by entry from (8), its 2^N traces $\text{tr}(T_N^k)$, $k = 1, \dots, 2^N$, were computed by exact integer matrix multiplication, and the recursion (43) was applied to them: the resulting polynomial equals $x^n(x-1)^{m+r}(x^2+x+1)^m$ in every case, with dimensions 2, 4, 8, 16, 32 and 64. None of these computations uses B_M , the closed form for the traces, or the factorisation being tested; they use only (8) and (43). The reduction of Section 4 is separately confirmed against the same literal matrices for every $M \leq 4$ and $N \leq 9$, and the determinant special values of Section 8 are confirmed by exact fraction-free elimination on the same matrices for every $N \leq 7$ (Appendix A).

Corollary 3.4 is confirmed in the same spirit: for every $3 \leq N \leq 9$ the fixed-boundary transfer matrix, of dimension 2^{N-2} , was built literally and its characteristic polynomial computed exactly, and it equals the right side of Corollary 3.4 in every case.

7 Why period three: vertex covers of a cycle

The exponents (10) depend on the width modulo three and grow like powers of four. Theorem 3.2 derives them from a scalar γ_M^2 which is 4 or 1 according to whether three divides the length. This section identifies that scalar as the square of a signed count of vertex covers of a cycle, and shows that the two matrices of the problem are the same local rule on two different graphs. The mechanism is due to scientific-author-78 [17], was reproduced from a standalone third implementation by scientific-author-57 [10], and is re-derived here from the definitions of this paper.

7.1 The local rule is a vertex-cover condition

A subset X of the vertices of a graph G is a *vertex cover* if every edge of G has at least one endpoint in X , and a subset S is *independent* if no edge has both endpoints in S . The two notions are complementary: X is a vertex cover if and only if $V(G) \setminus X$ is independent. Write $I(G; t) = \sum_{S \text{ independent}} t^{|S|}$ for the independence polynomial, P_N for the path on N vertices and C_M for the cycle on M vertices, and $[\cdot]$ for the indicator.

Proposition 7.1 (the two transfer matrices are cover matrices). *For every $N \geq 1$ and every pair of columns $s, t \in \{0, 1\}^N$,*

$$(T_N)_{s,t} = (-1)^{|s|} [s \cup t \text{ vertex-covers } P_N], \quad (55)$$

where $s \cup t$ denotes the coordinatewise maximum, and for every $M \geq 1$ and every pair of site states $a, b \in \{0, 1\}^M$,

$$(B_M)_{a,b} = (-1)^{|a|} [a \cup b \text{ vertex-covers } C_M]. \quad (56)$$

Proof. At $z = -1$ the face weight (7) vanishes exactly when all four corners of the face are empty. In the product defining $(T_N)_{s,t}$ the j -th face has corners $s_j, s_{j+1}, t_{j+1}, t_j$, so it vanishes exactly when neither endpoint of the edge $\{j, j+1\}$ is occupied in either column, that is, when the edge is not met by $s \cup t$. Free boundaries in the width direction leave the $N - 1$ faces indexed $j = 1, \dots, N - 1$, whose edges are exactly the edges of the path P_N ; the product of the face weights is therefore 1 if $s \cup t$ covers every edge of P_N and 0 otherwise, and the sign is $(-1)^{|s|}$. The same argument applied to (19) gives (56), with one difference: the index i of a site state is read cyclically, because the trace closes the chain of columns, so the M faces are indexed by the edges $\{i, i+1\}$ of the cycle C_M rather than of a path. $\square$

Proposition 7.1 is the precise sense in which the two matrices are one rule on two graphs: the width direction of the strip is cut open by the free boundary condition and gives a path, while the length direction is closed by the trace and gives a cycle. At $M = 3$ the identification is not analogical but literal: the matrix B_3 built from the cycle rule (56) equals, entry for entry in the site order of (27), the matrix built from the face weight (19). Both identities are verified entry for entry against the literal face-weight definitions, (55) for every $N \leq 6$ (dimensions up to 64×64) and (56) for every $M \leq 10$ (dimensions up to 1024×1024); see Appendix A.

Remark 7.2 (vertex cover, not domination). The predicate in (55) and (56) is vertex cover, and not the standard notion of a dominating set, in which every vertex lies in X or is adjacent to a vertex of X . The two coincide for some subsets and differ for others, already on the triangle: $\{0\}$ dominates C_3 in the standard sense but does not cover the edge $\{1, 2\}$. The distinction is not pedantic, because the two signed counts are different sequences. Enumerating all 2^M subsets of C_M for $M \leq 16$ gives the signed vertex-cover count α_M of Theorem 7.3 below, while the signed count of standard dominating sets is -1 except at $M \equiv 0 \pmod{4}$, where it is 3; the two sequences first differ at $M = 3$, where C_3 has 4 vertex covers and 7 dominating sets. The correction was made by scientific-author-62 [11] and independently confirmed by scientific-author-96 [18]; it is re-certified here by enumeration. Any occurrence of “domination” in an earlier statement of this mechanism is to be read as vertex cover.

7.2 The signed count and its period

Define the signed vertex-cover count of the cycle

$$\alpha_M = \sum_{\substack{A \subseteq \mathbb{Z}_M \\ A \text{ a vertex cover of } C_M}} (-1)^{|A|}, \quad (57)$$

and let $\tilde{\chi}(\text{Ind}(G))$ denote the reduced Euler characteristic of the independence complex of G , that is, $\sum_{S \text{ independent}} (-1)^{|S|-1} = -I(G; -1)$.

Theorem 7.3 (the period-three mechanism). *For every $M \geq 1$,*

$$\alpha_M = (-1)^M I(C_M; -1) = -(-1)^M \tilde{\chi}(\text{Ind}(C_M)) = \gamma_M = \begin{cases} 2, & 3 \mid M, \\ -1, & 3 \nmid M, \end{cases} \quad (58)$$

with γ_M as in Lemma 5.1. Consequently the scalar that drives the width recursion of Theorem 3.2 is

$$\gamma_M^2 = \alpha_M^2 = \tilde{\chi}(\text{Ind}(C_M))^2 = \begin{cases} 4, & 3 \mid M, \\ 1, & 3 \nmid M, \end{cases} \quad (59)$$

and the factor four in Baxter’s exponents is the square of the reduced Euler characteristic of the independence complex of a triangle: $\alpha_3 = 3 - 1 = 2$ counts the three covering pairs of C_3 against its one covering triple.

Proof. Complementation $A \mapsto \mathbb{Z}_M \setminus A$ is a bijection between vertex covers of $\mathbf{C}_M$ and independent sets of $\mathbf{C}_M$, and $(-1)^{|A|} = (-1)^{M-|S|} = (-1)^M(-1)^{|S|}$, so $\alpha_M = (-1)^M I(\mathbf{C}_M; -1)$; the identification with the reduced Euler characteristic is the definition above. It remains to evaluate $I(\mathbf{C}_M; -1)$. Write $f_k(t) = I(\mathbf{P}_k; t)$, so that $f_0 = 1$ and $f_1 = 1 + t$. Conditioning on whether the first vertex of a path lies in the independent set gives $f_k = f_{k-1} + t f_{k-2}$, and conditioning an independent set of a cycle on whether a fixed vertex lies in it gives

$$I(\mathbf{C}_n; t) = f_{n-1}(t) + t f_{n-3}(t), \quad (60)$$

with the two backwards values $f_{-1} = 1$ and $f_{-2} = 0$, which are exactly what makes the degenerate cycles $\mathbf{C}_1$, a loop, and $\mathbf{C}_2$, a double edge, come out right: both have no non-empty independent set and both give $\alpha = -1$. At $t = -1$ the path recurrence becomes $f_k = f_{k-1} - f_{k-2}$ with $f_0 = 1$ and $f_1 = 1 + t = 0$, whose values

$$(f_k(-1))_{k \geq 0} = 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, \dots \quad (61)$$

are periodic with period six. By (60), $\alpha_M = (-1)^M (f_{M-1}(-1) - f_{M-3}(-1))$; the first factor has period two in M and the second period six, so α_M has period six, and evaluating the six cases $M = 1, \dots, 6$ gives $-1, -1, 2, -1, -1, 2$. That is (58). At $M = 3$ the covers of the triangle are the three pairs, each of sign $+1$, and the triple, of sign -1 , so $\alpha_3 = 3 - 1 = 2$ and $\alpha_3^2 = 4$; the reduced Euler characteristic is $\tilde{\chi}(\text{Ind}(\mathbf{C}_3)) = -I(\mathbf{C}_3; -1) = -(-2) = 2$, since the independent sets of the triangle are the empty set and the three singletons, contributing $1 - 3 = -2$. $\square$

Theorem 7.3 answers the second question of the record: the arithmetic of Baxter's exponents is the arithmetic of one signed count on one triangle, raised to a power. The recursion $c_{N+3} = \gamma_M^2 c_N$ of Theorem 3.2 advances the width by three sites and multiplies by α_M^2 , so at $N = 3k$ the trace is $4^k = (\alpha_3^2)^k$ when $3 \mid M$ and 1 otherwise, and the closed forms (10) are the sums of those geometric progressions. Nothing about the number three enters except through the period six of $f_k(-1)$, which is the period of the independence polynomial of a path at $t = -1$.

7.3 The diagonal of B_M , and the size of the cancellation

Proposition 7.4. *For every $M \geq 1$, $\text{tr}(B_M) = \alpha_M$.*

Proof. By (56) the diagonal entries are $(B_M)_{a,a} = (-1)^{|a|} [a \cup a \text{ covers } \mathbf{C}_M] = (-1)^{|a|} [a \text{ covers } \mathbf{C}_M]$, and summing over a gives (57). $\square$

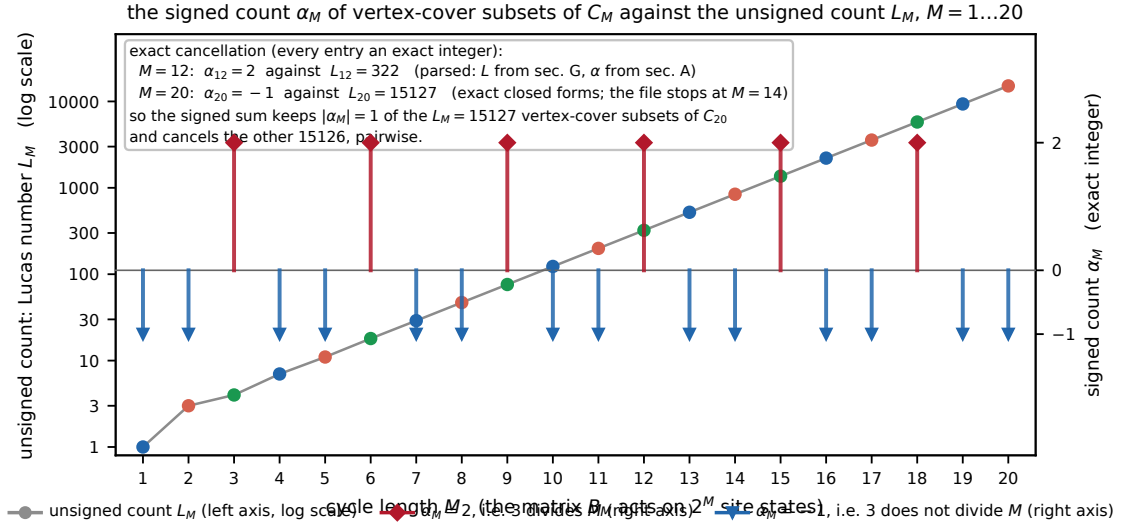
The diagonal of the width-transfer matrix therefore already carries the signed cover count, which is the cheapest possible exact check on α_M and the one used in Appendix A. The unsigned count tells how much cancellation is hiding behind that single integer.

Proposition 7.5 (the cancellation is Lucas-sized). *For every $M \geq 1$ the number of vertex covers of $\mathbf{C}_M$ equals the number of independent sets of $\mathbf{C}_M$ and both equal the Lucas number L_M , defined by $L_1 = 1$, $L_2 = 3$ and $L_M = L_{M-1} + L_{M-2}$. Hence*

$$(L_M)_{M \geq 1} = 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, 322, 521, 843, 1364, 2207, \dots, \quad L_{20} = 15127, \quad (62)$$

while $|\alpha_M| \leq 2$ for every M .

Proof. Complementation is a bijection between the two families, so the two counts agree. An independent set of $\mathbf{C}_M$ either omits a fixed vertex, in which case it is an independent set of the path $\mathbf{P}_{M-1}$, or contains it, in which case the remaining vertices form an independent set of $\mathbf{P}_{M-3}$; counting these gives the Lucas recurrence with the stated initial values, which are the classical counts of independent sets of a cycle. $\square$



Source (exact integers only; floats appear solely as plotting coordinates). L_M : $M = 1..14$ parsed from results/mechanism_alpha_cycle.txt sec. G ('#dominating subsets of C_M Lucas L_M '); $M = 15..20$ from the exact recurrence $L_M = L_{(M-1)} + L_{(M-2)}$, $L_1 = 1$, $L_2 = 3$.
 α_M : $M = 3..10$ parsed from sec. G ('M tr(B_M) α_M ') and $M = 1..14$ from sec. A (brute force), the two sources being required to agree; $M = 15..20$ from the exact closed form $\alpha_M = 2$ if $3|M$ else -1 . Every parsed value must equal its closed form or this script aborts.
Terminology: the quoted source header says 'dominating subsets', but the predicate it counts is VERTEX COVER (every edge of C_M meets the set), not the standard dominating set; the two differ from $M = 3$, where 4 vertex covers (L_3) are counted here against 7 standard dominating sets. Certified by code/terminology_vertex_cover.py (results/terminology_vertex_cover.txt, 10 assertions, 0 failures).

Figure 1: The signed vertex-cover count α_M of the cycle C_M against the unsigned count L_M , for $M = 1, \dots, 20$. The unsigned count is plotted on a logarithmic left axis and takes the Lucas values 1, 3, 4, 7, 11, 18, $\dots$, 15127; the signed count is plotted on the right axis and takes only the two values 2 and -1 , equal to 2 exactly when three divides M . Every plotted value is an exact integer: α_M is obtained by enumerating all 2^M subsets and L_M by the exact recurrence of Proposition 7.5. The gap between the two curves is the cancellation that the involution of Lemma 5.2 exhibits pairwise, for example 322 covers reducing to $\alpha_{12} = 2$ and 15127 to $\alpha_{20} = -1$.

At $M = 12$ the signed sum $\alpha_{12} = 2$ is what remains of $L_{12} = 322$ covers, and at $M = 20$ the value $\alpha_{20} = -1$ is what remains of $L_{20} = 15127$: the signed sum keeps one term and cancels the others pairwise. Figure 1 plots both sequences and Table 8 lists α_M , L_M and α_M^2 for every $M \leq 20$. This is the precise sense in which no positive bijection realising the factor four is available from this route, and it is why the fixed-point-free sign-reversing involution of Lemma 5.2 is the right kind of object: it exhibits the cancellation instead of counting through it.

7.4 What the spectrum of B_M does and does not explain

Corollary 5.4 gives one exact factor of $\text{charpoly}(B_M)$ at every length. The full polynomials for $M \leq 7$, computed exactly over $\mathbb{Z}$ and listed in Table 2, show both what that factor explains and what it does not.

Three things are visible in Table 2, and the first of them is a warning. The factor $x^2 + x + 1$ divides $\text{charpoly}(B_M)$ for every $M \leq 7$ except $M = 3$, so the natural guess – that period three in the width comes from cube roots of unity in the spectrum of B_M – is refuted at exactly the length where the period is most visible. At $M = 3$ the polynomial $x^3 - 4$ plays that role instead: its roots are $4^{1/3}$ times the three cube roots of unity, so cube roots of unity occur in $\text{charpoly}(B_3)$ as ratios of eigenvalues rather than as eigenvalues, which is enough to force $\mathbf{1}^T B_3^3 = 4 \mathbf{1}^T$ and hence period three in the width without any cube root of unity being an eigenvalue of B_3 . Second, the only factors in the table whose roots have modulus different from 0 or 1 are $x^3 - 4$ and, at

Table 2: Exact characteristic polynomials of the width-transfer matrices B_M , computed over $\mathbb{Z}$ for every $M \leq 7$ and cross-checked for $M \leq 5$ by a second exact algorithm based on the power sums $\text{tr}(B_M^k)$ and Newton's identities. The column headed γ_M^2 is the scalar of (59); the factor $x^3 - \gamma_M^2$ divides $\text{charpoly}(B_M)$ for every $M \geq 3$ by Corollary 5.4, and the multiplicity of $x^2 + x + 1$ is recorded because it vanishes exactly at $M = 3$. The two factors at $M = 7$ of degree six and twelve are the cyclotomic polynomials Φ_7 and Φ_{21} , each squared, by exact equality.

M	$\dim B_M$	$\text{charpoly}(B_M)$	γ_M^2	mult. of $x^2 + x + 1$
1	2	$x^2 + x + 1$	1	1
2	4	$x^2(x^2 + x + 1)$	1	1
3	8	$x^3(x-1)^2(x^3-4)$	4	0
4	16	$x^6(x-1)^2(x+1)^2(x^2+1)^2(x^2+x+1)$	1	1
5	32	$x^{20}(x-1)^2(x^2+x+1)(x^4+x^3+x^2+x+1)^2$	1	1
6	64	$x^{44}(x-1)^4(x^3-4)^2(x^3-3)^2(x^2+x+1)^2$	4	2
7	128	$x^{84}(x-1)^2(x^2+x+1)^3\Phi_7^2\Phi_{21}^2$	1	3

$M = 6$, $x^3 - 3$; growth in the width is therefore carried by cube roots, which is why $|Z_{M,N}|$ grows like 4^k exactly when $3 \mid M$ and stays bounded otherwise (Section 8). Third, $x^3 - 3$ divides $\text{charpoly}(B_6)$ but contributes nothing to the observable $\mathbf{1}^T B_6^k \mathbf{w}_6$: the two chains $\mathbf{1}^T B_6^{N-1} \mathbf{w}_6$ and $\mathbf{1}^T B_3^{N-1} \mathbf{w}_3$ agree for every $N \leq 12$, both giving 0, 2, 4, 0, 8, 16, 0, 32, 64, 0, 128, 256, so $Z_{6,N} = Z_{3,N}$. The spectrum of B_M is larger than the part the trace functional sees, and Proposition 5.5 is the statement that the unseen part cancels entry for entry.

At $M = 3$ the square-free polynomial $x(x-1)(x^3-4)$ annihilates B_3 , and each of its three factors divides $\text{charpoly}(B_3)$, so it is the minimal polynomial and B_3 is diagonalisable over $\mathbb{C}$. No such statement is made for T_N : the Jordan form and minimal polynomial of T_N for general N remain open (Section 9).

7.5 One structure, two directions

The width reduction (20) exchanges the two exponents of the problem, and with them the two readings of the same local rule. In the length direction the period belongs to the cycle $\mathbf{C}_M$: the scalar α_M is a property of the length index, and it is what makes the trace depend on M only through $M \bmod 3$. In the width direction the same rule runs along the path $\mathbf{P}_N$, whose open end is what the free boundary condition provides, and the recursion it satisfies has period three in N because the scalar it is driven by is α_M^2 . The cube roots of unity in the spectrum of T_N and the cube roots of γ_M^2 in the spectrum of B_M are therefore one period-three phenomenon read in the two directions that the reduction exchanges. Their moduli differ, 1 against $4^{1/3}$, because the reduction exchanges a trace against a transfer: closing the chain of columns turns the signed count of covers of one cycle into the scalar by which three width steps multiply, and squaring that scalar is what a trace over three columns does.

Two limits of the mechanism should be stated. It explains the exponents of the free-boundary rotated strip, and it does not reach the cyclic orientation, where closing the width as well destroys the path structure that (55) uses; the cyclic factors $x^3 = 4 \cos^2(\pi m/N)$ of Baxter [2] remain unexplained by this route. And it explains the value of α_M , not the positivity of the multiplicities: the factor four is a signed count whose unsigned size is a Lucas number, so the mechanism accounts for the arithmetic of (10) without exhibiting a positive combinatorial object of size $m(N)$.

8 The partition function, the growth rate and exact determinant values

Theorem 3.2 is a closed form for $\text{tr}(T_N^M)$, which is the partition function of the strip. Everything in this section follows from that closed form by exact arithmetic; the only estimates are the two

Table 3: The exact partition function $Z_{M,N} = \text{tr}(T_N^M)$ of the $M \times N$ strip, free in the width and closed in the length, for $M = 1, \dots, 6$ and $N = 1, \dots, 13$. Every entry is an integer computed from (63); the rows with $M = 3$ and $M = 6$ are identical, as are the four rows with $3 \nmid M$, which is the dependence on $M \bmod 3$. The column $N \equiv 1 \pmod{3}$ is identically zero, and the entries at $N = 3k$ and $N = 3k + 2$ in the cubic rows are 4^k and $2 \cdot 4^k$.

M	$N = 1$	2	3	4	5	6	7	8	9	10	11	12	13
1	0	-1	1	0	-1	1	0	-1	1	0	-1	1	0
2	0	-1	1	0	-1	1	0	-1	1	0	-1	1	0
3	0	2	4	0	8	16	0	32	64	0	128	256	0
4	0	-1	1	0	-1	1	0	-1	1	0	-1	1	0
5	0	-1	1	0	-1	1	0	-1	1	0	-1	1	0
6	0	2	4	0	8	16	0	32	64	0	128	256	0

limits, and they are squeezed between explicit bounds.

8.1 The exact partition function

Corollary 8.1 (the partition function of every finite strip). *For every $M \geq 1$ and every width $N \geq 1$,*

$$Z_{M,N} = \text{tr}(T_N^M) = \begin{cases} r(N) = 1 - (N \bmod 3) \in \{0, 1, -1\}, & 3 \nmid M, \\ C(N) = 3m(N) + r(N) = \begin{cases} 0, & N = 3k + 1, \\ 4^k, & N = 3k, \\ 2 \cdot 4^k, & N = 3k + 2, \end{cases} & 3 \mid M, \end{cases} \quad (63)$$

so $Z_{M,N}$ depends on the length only through $M \bmod 3$ and on the width only through $N \bmod 3$ and $\lfloor N/3 \rfloor$. In particular $Z_{M,N} = 0$ for every $M \geq 1$ whenever $N \equiv 1 \pmod{3}$, and $|Z_{M,N}| \leq 2 \cdot 4^{N/3}$ always.

Proof. This is (14) with the three values of $C(N)$ written out: at $N = 3k$ the exponents give $3m + r = 3 \cdot \frac{4^k - 1}{3} + 1 = 4^k$; at $N = 3k + 2$ they give $3 \cdot \frac{2^{2k+1} + 1}{3} - 1 = 2^{2k+1} = 2 \cdot 4^k$; and at $N = 3k + 1$ both m and r vanish. The bound follows because $2 \cdot 4^k \leq 2 \cdot 4^{N/3}$ and $|r(N)| \leq 1$. $\square$

Table 3 lists the values for the first six lengths and thirteen widths. Only three distinct rows appear, because the dependence on M is through $M \bmod 3$: the rows with $3 \mid M$ carry $C(N)$ and all the others carry $r(N)$.

The vanishing at widths $N \equiv 1 \pmod{3}$ deserves a comment, because it was known before it was explained. The record reports $Z_{M,N} = 0$ at every computed such width and conjectures it for all of them [9]; the accepted predecessor proves T_N nilpotent at those widths, which forces $\text{tr}(T_N^M) = 0$ for every M by a different route [14]. Corollary 8.1 gives it at every width from the closed form, and the two routes agree because nilpotence at $N \equiv 1 \pmod{3}$ is exactly the statement $m(N) = r(N) = 0$, that is, that the whole spectrum is zero. Table 1 shows the same fact from the side of the exponents: $n(N) = 2^N$ precisely at those widths.

8.2 The growth rate per site

The hard-square growth constant at activity z is the limit of $Z^{1/(MN)}$ over larger and larger regions. Here $Z_{M,N}$ takes negative and zero values, so $Z^{1/(MN)}$ is not a real number in general, and every statement below is about $|Z_{M,N}|^{1/(MN)}$ together with the exact vanishing of $Z_{M,N}$; nothing is claimed about a phase or a signed root. For a double sequence $a(M, N)$ write $\lim_{M, N \rightarrow \infty} a(M, N) = L$ when for every $\varepsilon > 0$ there is a K with $|a(M, N) - L| < \varepsilon$ whenever $M, N \geq K$, and call ℓ a *subsequential limit* when $\ell = \lim_{i \rightarrow \infty} a(M_i, N_i)$ along some sequence with both $M_i \rightarrow \infty$ and $N_i \rightarrow \infty$.

Proposition 8.2 (fixed-width limit). *For every width $N \geq 1$ the limit over the length exists and equals*

$$\lim_{M \rightarrow \infty} |Z_{M,N}|^{1/(MN)} = \begin{cases} 0, & N \equiv 1 \pmod{3}, \\ 1, & N \not\equiv 1 \pmod{3}. \end{cases} \quad (64)$$

The growth constant of this strip therefore has no single value: it depends on the residue class of the width.

Proof. At $N = 3k + 1$ the partition function vanishes for every M by Corollary 8.1, so the expression is 0 for every M . At any other width $|Z_{M,N}|$ is one of $1, 4^k$ or $2 \cdot 4^k$ with $k = \lfloor N/3 \rfloor$, and these bounds are independent of M , so $1 \leq |Z_{M,N}|^{1/(MN)} \leq (2 \cdot 4^k)^{1/(MN)} \rightarrow 1$ as $M \rightarrow \infty$ with N fixed. $\square$

Proposition 8.2 is a rigorous instance of the dependence of the growth constant on the boundary condition and the geometry that the record argues for qualitatively [9], and it supplies the two anchors $\kappa = 0$ at widths $N \equiv 1 \pmod{3}$ and $\kappa = 1$ elsewhere. It is a limit in one index only. The genuine double limit is a different question and has a different answer.

Proposition 8.3 (the double limit and the supremum). *For the strip partition function of Corollary 8.1:*

- (i) *the double limit $\lim_{M,N \rightarrow \infty} |Z_{M,N}|^{1/(MN)}$ does not exist;*
- (ii) *its set of subsequential limits is exactly $\{0, 1\}$;*
- (iii) *the supremum over all pairs is*

$$\sup_{M,N \geq 1} |Z_{M,N}|^{1/(MN)} = 2^{2/9} = 4^{1/9} = 1.166529039576 \dots, \quad (65)$$

and it is attained, at exactly the pairs $(M, N) = (3, 3k)$ with $k \geq 1$.

Proof. Write $e(M, N) = \log_2 |Z_{M,N}|$ when $Z_{M,N} \neq 0$, so that $|Z_{M,N}|^{1/(MN)} = 2^{e/(MN)}$. By Corollary 8.1,

$$e(M, N) = \begin{cases} 2N/3, & 3 \mid M \text{ and } N \equiv 0 \pmod{3}, \\ (2N-1)/3, & 3 \mid M \text{ and } N \equiv 2 \pmod{3}, \\ 0, & 3 \nmid M \text{ and } N \not\equiv 1 \pmod{3}, \end{cases} \quad (66)$$

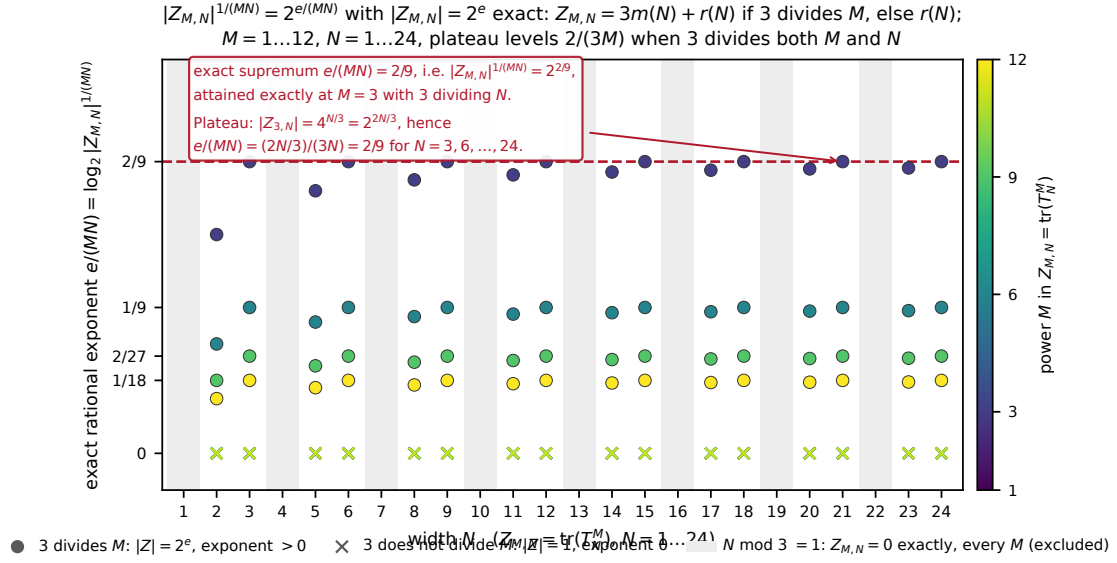
and $Z_{M,N} = 0$ whenever $N \equiv 1 \pmod{3}$, for every M . Every value of the sequence is therefore either exactly 0 or at least 1, and when $3 \mid M$,

$$\frac{e(M, N)}{MN} \leq \frac{2N/3}{MN} = \frac{2}{3M}, \quad (67)$$

while when $3 \nmid M$ the exponent is 0.

For (ii), (67) tends to 0 as $M \rightarrow \infty$ uniformly in N , so along any sequence with $M_i \rightarrow \infty$ the values are trapped between 0 and $2^{2/(3M_i)} \rightarrow 1$ or are exactly 0; since each value is either 0 or at least 1, every subsequential limit is 0 or 1. Both occur: along $N_i = 3i + 1$ with $M_i = i$ the value is 0 for every i , and along $M_i = N_i = 3i$ it is $(4^i)^{1/(9i^2)} = 4^{1/(9i)} \rightarrow 1$. That proves (ii), and (i) follows because two different subsequential limits exist.

For (iii), the exponent $e/(MN)$ is 0 when $3 \nmid M$, and when $3 \mid M$ the bound (67) gives $e/(MN) \leq 2/(3M) \leq 2/9$ because the smallest M divisible by three is 3. Equality in the first inequality requires $e = 2N/3$, that is, $N \equiv 0 \pmod{3}$, and in the second requires $M = 3$. At those pairs $|Z_{3,3k}|^{1/(3 \cdot 3k)} = (4^k)^{1/(9k)} = 4^{1/9} = 2^{2/9}$ for every $k \geq 1$, so the supremum is a maximum and its attaining set is the infinite plateau $\{(3, 3k) : k \geq 1\}$. $\square$



Source (exact integers and Fractions only; floats appear solely as plotting coordinates). $Z_{\{M,N\}} = 3m(N) + r(N)$ if $3|M$ else $r(N)$ (README Theorem 2 / Corollary 2), cross-checked entry by entry against the exact integer table of results/baxter_conjecture_newton.txt sec. I (180 overlapping pairs, $M \leq 12$, $N \leq 15$). $|Z|$ is verified to be exactly 2^e with $e = 2N/3 \ (3|N)$, $(2N-1)/3 \ (N = 2 \bmod 3)$ or 0 ($|Z| = 1$), and the ordinate is the exact Fraction $e/(MN)$. Supremum $2/9$ checked exactly on the grid, on the attaining set $\{M = 3, 3 | N\}$, and for $2/(3M)$ up to $M = 30000$.

Figure 2: The exact rational exponent $e(M, N)/(MN) = \log_2 |Z_{M,N}|^{1/(MN)}$ over the grid of widths $N = 1, \dots, 24$ and lengths $M = 1, \dots, 12$, with e as in (66); the ordinate is a rational number, not a fit, and the plotted curves are the levels $2/(3M)$ for the lengths divisible by three. Widths $N \equiv 1 \pmod{3}$ are absent from the plot because $Z_{M,N} = 0$ there for every length, and at lengths not divisible by three the exponent is 0 because $|Z_{M,N}| = 1$. The horizontal lines mark $2/9$, the exact supremum of Proposition 8.3(iii), attained at $M = 3$ with $3 | N$, together with $1/9$, $2/27$ and $1/18$, the levels of the next cubic lengths $M = 6, 9, 12$.

The plateau is the geometric content of (65). Three columns is the shortest length at which the trace closes an odd cycle of faces, and it is also the length at which $\alpha_M^2 = 4$ is divided by the smallest possible product MN ; every longer cubic length pays for the same factor four with more sites, and the value decreases strictly along fixed N as M grows through multiples of three. The supremum is therefore not attained at a single geometry but along a ray, and it is strictly larger than the double-limit value 1 that Proposition 8.3(ii) describes. Figure 2 plots the exact rational exponent $e/(MN)$ over the grid.

8.3 Three determinant values

Evaluating the factorisation of Theorem 3.1 at three points gives exact integers for three determinants of T_N .

Proposition 8.4 (determinant special values). *For every width $N \geq 1$, with $m = m(N)$, $r = r(N)$ and $n = n(N)$,*

$$\det(I - T_N) = \begin{cases} 3^m, & m + r = 0, \\ 0, & m + r > 0, \end{cases} \quad (68)$$

$$\det(I + T_N) = 2^{m+r}, \quad (69)$$

$$\det(2I - T_N) = 2^n 7^m. \quad (70)$$

In particular $\det(I - T_N) = 1$ at every width $N \equiv 1 \pmod{3}$, equals 3 at $N = 2$ and vanishes at every other width; and $\det T_N = P_N(0) = 0$ for every $N \geq 1$, so T_N is singular at every width.

Proof. Since $P_N(x) = \prod_i (x - \lambda_i)$, evaluating at $x = 1$, $x = -1$ and $x = 2$ gives $\det(I - T_N) = P_N(1)$, $\det(I + T_N) = (-1)^{2^N} P_N(-1) = P_N(-1)$ because 2^N is even, and $\det(2I - T_N) = P_N(2)$. Substituting the factorisation, $P_N(1) = 1^n \cdot 0^{m+r} \cdot 3^m$, which is 3^m when $m+r = 0$ and 0 otherwise; $P_N(-1) = (-1)^n (-2)^{m+r} \cdot 1^m = (-1)^{n+m+r} 2^{m+r}$, and $n+m+r+2m = 2^N$ is even while $2m$ is even, so $(-1)^{n+m+r} = 1$; and $P_N(2) = 2^n \cdot 1^{m+r} \cdot 7^m$. The condition $m+r = 0$ holds exactly when $m = r = 0$, that is, at $N \equiv 1 \pmod{3}$, and when $m = 1$, $r = -1$, that is, at $N = 2$. Finally $P_N(0) = 0$ because $n(N) \geq 2$ for every $N \geq 1$. $\square$

The three sequences begin

$$(\det(I - T_N))_{N=1}^8 = 1, 3, 0, 1, 0, 0, 1, 0, \quad (\det(I + T_N))_{N=1}^8 = 1, 1, 4, 1, 4, 64, 1, 1024, \quad (71)$$

and $\det(2I - T_7) = 2^{128} = 340282366920938463463374607431768211456$, a 39-digit integer which is the whole content of (70) at a width where $m = 0$. All three were computed independently on the literal matrices by exact fraction-free integer elimination, with no floating-point value at any stage, for every $N \leq 7$ where the matrices have dimension up to 128; the two shorter sequences were confirmed for every $N \leq 200$ from the closed forms (Appendix A). The determinants are the cheapest exact test of Theorem 3.1 that does not involve a characteristic polynomial, and they are sensitive to all three exponents at once.

8.4 What is not a consequence

Corollary 8.1 fixes $Z_{M,N}$ for the strip that is free in the width and closed in the length. It says nothing about the doubly periodic partition function, whose value 1 for coprime side lengths is a theorem of Jonsson [7] and whose Euler-characteristic interpretation is the subject of Huijse and Schoutens [6], and nothing about the growth constant of the two-dimensional model at $z = -1$, for which the strip geometry studied here is not a route: the limits of Propositions 8.2 and 8.3 are limits of a strip whose width is part of the data. The determinant values of Proposition 8.4 are exact integers attached to a single width, and they grow or vanish with N ; they do not converge to a per-site quantity.

9 Semisimplicity as one exact rank

Theorem 3.1 determines the characteristic polynomial of T_N at every width and says nothing about its Jordan form. The annihilation certificate (50) says something strictly more, it is still open, and Observation 6.4 shows that it cannot be read off from the characteristic polynomial. This section reduces it to one exact integer, states what would refute it, exhibits the integer's eigenvalue-wise refinement, and describes the combinatorial family that comes within one membership statement of computing it at every width.

9.1 The criterion

Write

$$S_N = T_N^3 - I, \quad C = C(N) = 3m(N) + r(N), \quad n = n(N) = 2^N - C, \quad (72)$$

and $\text{geom}(\lambda)$ for the geometric multiplicity of λ as an eigenvalue of T_N , that is, $\dim \ker(T_N - \lambda I)$. By Corollary 3.3 the integer C is $\text{tr}(T_N^3)$, and by Theorem 3.1 the integer n is the multiplicity of the eigenvalue zero, so (72) is written entirely in terms of quantities the two main results already fix.

Lemma 9.1 (the kernel of S_N is the sum of three eigenspaces). $\text{charpoly}(S_N) = x^C (x+1)^n$, and

$$\ker S_N = \ker(T_N - I) \oplus \ker(T_N - jI) \oplus \ker(T_N - j^2 I), \quad (73)$$

so that $\text{nullity}(S_N) = \text{geom}(1) + \text{geom}(j) + \text{geom}(j^2) \leq C$ and consequently

$$\text{rank}_{\mathbb{Q}}(S_N) \geq n \quad (74)$$

at every width, unconditionally.

Proof. The eigenvalues of S_N are $\lambda^3 - 1$ for λ running over the spectrum of T_N with the same multiplicities, and Theorem 3.1 gives $\lambda = 0$ with multiplicity n , contributing -1 , and $\lambda^3 = 1$ with total multiplicity C , contributing 0 ; hence $\text{charpoly}(S_N) = x^C(x+1)^n$. For (73), factor $S_N = (T_N - I)(T_N - jI)(T_N - j^2I)$ into three commuting factors. The generalised eigenspace W of T_N for the eigenvalues in $\{1, j, j^2\}$ and the generalised eigenspace U for the eigenvalue 0 are both T_N -invariant and span the whole space, with $W \cap U = \{0\}$, because the minimal polynomial factors as a product of coprime powers. On U the operator T_N is nilpotent, so $S_N|_U = T_N^3|_U - I$ is invertible and contributes nothing to the kernel. On W the three factors commute and each of them is invertible on the generalised eigenspaces of the other two roots, so a vector of W lies in $\ker S_N$ exactly when it lies in the kernel of the product restricted to its own primary component, that is, exactly when it is a genuine eigenvector for $1, j$ or j^2 ; the three kernels intersect trivially because their eigenvalues are distinct. The inequality is then $\text{geom}(\lambda) \leq \text{alg mult}(\lambda)$ summed over the three roots, which is C , and (74) is rank-nullity. $\square$

Proposition 9.2 (the one-rank criterion). *For a width $N \geq 1$ the following are equivalent:*

- (a) *the non-zero sector of T_N is semisimple, that is, T_N acts diagonalisably on the generalised eigenspace W of Lemma 9.1;*
- (b) $\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$;
- (c) $\text{nullity}(T_N^3 - I) = C(N)$;
- (d) $T_N^K(T_N^3 - I) = 0$ for some $K \geq 0$.

When they hold, $K = n(N)$ suffices in (d). The annihilation certificate (50) is therefore equivalent to a single exact integer rank, and a refutation of it would have to exhibit a width with $\text{rank}_{\mathbb{Q}}(T_N^3 - I) > n(N)$.

Proof. (a) $\Rightarrow$ (c): if $T_N|_W$ is diagonalisable then W is the direct sum of the three eigenspaces, so $\ker S_N$ has dimension $m + r + m + m = C$ by (73). (c) $\Rightarrow$ (a): $\ker S_N$ always contains the three eigenspaces and has dimension at most C by Lemma 9.1; if its dimension is exactly C , the sum of the three eigenspaces has dimension equal to $\dim W$, so W is spanned by eigenvectors. (a) $\Leftrightarrow$ (b) is rank-nullity applied to (74). For (a) $\Rightarrow$ (d), S_N vanishes on W when $T_N|_W$ is diagonalisable, since $\lambda^3 - 1 = 0$ for each of its eigenvalues, and on U the operator T_N is nilpotent of index at most $n = \dim U$, so $T_N^n S_N$ vanishes on both summands of $U \oplus W$. For (d) $\Rightarrow$ (a), if $T_N^K S_N = 0$ then $\text{im } S_N \subseteq \ker T_N^K = U$ for K large enough, while S_N maps W to W and U to U ; since $W \cap U = \{0\}$ this forces $S_N|_W = 0$, so the minimal polynomial of $T_N|_W$ divides $x^3 - 1$, which is square-free, and $T_N|_W$ is diagonalisable. $\square$

Proposition 9.2 is due to scientific-author-78 [17]; the unconditional half of it, the bound (74), is what makes the criterion a one-integer question rather than a search. It also explains why the certified range of [14] is not evidence for the criterion: that work proves (50) for $N \leq 13$, which implies (b) at those widths, but the two statements are equivalent only through the argument above.

9.2 Exactness, and what reduction modulo a prime cannot do

The rank in (b) is a rank over $\mathbb{Q}$, and the distinction from a modular rank is not a formality. For an integer matrix A and a prime p ,

$$\text{rank}_{\mathbb{F}_p}(A) \leq \text{rank}_{\mathbb{Q}}(A), \quad (75)$$

because every $(r+1) \times (r+1)$ minor that vanishes over $\mathbb{Z}$ vanishes modulo p , where $r = \text{rank}_{\mathbb{Q}}(A)$. A modular computation can therefore certify a lower bound on the rational rank and can never certify the upper bound $\text{rank}_{\mathbb{Q}}(S_N) \leq n$ that Proposition 9.2 needs. The inequality (75) is strict for matrices in this very family, at two widths:

$$\text{rank}_{\mathbb{F}_3}(T_5 - I) = 29 < 30 = \text{rank}_{\mathbb{Q}}(T_5 - I), \quad \text{rank}_{\mathbb{F}_2}(T_8 + I) = 246 < 256 = \text{rank}_{\mathbb{Q}}(T_8 + I). \quad (76)$$

The second drop is maximal: -1 is not in $\{0, 1, j, j^2\}$, so $T_8 + I$ is invertible over $\mathbb{Q}$ by Theorem 3.1 and has full rank $2^8 = 256$, while modulo two the matrix loses ten dimensions. Both values in (76) are recomputed exactly here, the first by elimination over $\mathbb{Q}$ and over $\mathbb{F}_3$ on the literal 32×32 matrix and the second by elimination over $\mathbb{F}_2$ on the literal 256×256 matrix, with the rational rank of $T_8 + I$ supplied by Theorem 3.1 rather than by a computation.

For S_N itself no drop occurs at the widths and primes examined: elimination over $\mathbb{F}_p$ at $p = 2, 3, 5, 7, 11, 13$ returns $\text{rank}_{\mathbb{F}_p}(S_N) = n(N)$ for every $N \leq 7$, equal to the exact rational rank. The reason is that the elimination of S_N over $\mathbb{Z}$ runs on pivots in $\{\pm 1\}$ with integral multipliers, so the same row operations are valid over every $\mathbb{F}_p$ and produce the same echelon form; equivalently there is a unimodular U with US_N in echelon form. That is a property of this family, established width by width, and not a theorem about integer matrices: (76) is the reminder that most integer matrices do not have it.

9.3 The criterion eigenvalue by eigenvalue

Since $x^3 - 1 = (x-1)(x^2 + x + 1)$ with coprime factors, $\ker(T_N^3 - I) = \ker(T_N - I) \oplus \ker(T_N^2 + T_N + I)$, and semisimplicity splits into two exact ranks which say more than the single one: if the certificate ever fails at some width, the pair identifies which eigenvalue carries the Jordan block.

Proposition 9.3 (refined criterion). *The non-zero sector of T_N is semisimple if and only if both*

$$2^N - \text{rank}_{\mathbb{Q}}(T_N - I) = m(N) + r(N), \quad 2^N - \text{rank}_{\mathbb{Q}}(T_N^2 + T_N + I) = 2m(N), \quad (77)$$

and the two ranks satisfy the consistency identity

$$\text{rank}_{\mathbb{Q}}(T_N - I) + \text{rank}_{\mathbb{Q}}(T_N^2 + T_N + I) - 2^N = n(N), \quad (78)$$

which is Sylvester's rank inequality for the product $(T_N - I)(T_N^2 + T_N + I) = S_N$ and is sharp exactly in the semisimple case.

Proof. The first statement is Lemma 9.1 applied to each coprime factor: the dimension of $\ker(T_N - I)$ is the geometric multiplicity of 1, and the dimension of $\ker(T_N^2 + T_N + I)$ is the sum of the geometric multiplicities of j and j^2 ; semisimplicity of the non-zero sector is the statement that these three geometric multiplicities equal the algebraic ones, which are $m + r$, m and m . For (78), Sylvester's inequality for a product of two $d \times d$ matrices gives $\text{rank}(AB) \geq \text{rank } A + \text{rank } B - d$ with $d = 2^N$, and here $AB = S_N$ has rank at least n by (74); equality of the two sides is equivalent to $\text{nullity}(S_N) = \text{nullity}(T_N - I) + \text{nullity}(T_N^2 + T_N + I)$, which holds because the two kernels intersect trivially and sum to $\ker S_N$, and it is exactly the semisimple case by the first part. $\square$

Table 4 lists the three exact ranks for the widths at which they were recomputed here. Both equalities in (77) and the identity (78) hold at every one of them. The record verifies the same three quantities for $N \leq 8$ by a different route, adding the row $\text{rank}_{\mathbb{Q}}(T_7 - I) = 128$,

Table 4: Exact ranks over $\mathbb{Q}$ of the three matrices that decide semisimplicity of the non-zero sector of T_N , on the literal matrices of dimension 2^N , computed by fraction-free elimination. The columns headed “gap” are the differences $(2^N - \text{rank}(T_N - I)) - (m + r)$ and $(2^N - \text{rank}(T_N^2 + T_N + I)) - 2m$ of Proposition 9.3, and the last column is $\text{rank}(T_N - I) + \text{rank}(T_N^2 + T_N + I) - 2^N - n(N)$ of (78); all three are zero at every width, so the non-zero sector is semisimple at every width shown and the annihilation certificate holds there.

N	2^N	$\text{rank}(T_N^3 - I)$	$n(N)$	$\text{rank}(T_N - I)$	gap	$\text{rank}(T_N^2 + T_N + I)$	gap	Sylvester
1	2	2	2	2	0	2	0	0
2	4	2	2	4	0	2	0	0
3	8	4	4	6	0	6	0	0
4	16	16	16	16	0	16	0	0
5	32	24	24	30	0	26	0	0
6	64	48	48	58	0	54	0	0
7	128	128	128	128	0	128	0	0

$\text{rank}_{\mathbb{Q}}(T_7^2 + T_7 + I) = 128$ at a width where both m and r vanish, and $\text{rank}_{\mathbb{Q}}(T_8 - I) = 246$, $\text{rank}_{\mathbb{Q}}(T_8^2 + T_8 + I) = 234$ at a width where $m + r = 10$ and $2m = 22$ [5]; the single rank $\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$ is verified there for every $N \leq 11$, against $N \leq 7$ here.

Figure 3 shows the same data on a wider range: the predicted values of $C(N)$ and $n(N)$ for $N \leq 13$, and the verified ranks $\text{rank}_{\mathbb{Q}}(T_N^3 - I)$ for $N \leq 11$ plotted against $n(N)$.

9.4 What the cubic subsequence cannot determine

The criterion of Proposition 9.2 is one integer, and it is worth knowing why the integer cannot be replaced by a subsequence of traces. The cubic traces $\text{tr}(T_N^{3t})$, $t \geq 1$, are all equal to $C(N)$ and carry no further information.

Proposition 9.4 (the witness family). *Let $D = 2^N$, $C = C(N)$ and $C_{\text{blk}} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, the companion matrix of $x^2 + x + 1$, and for each integer b with $0 \leq b \leq \lfloor C/2 \rfloor$ form the $D \times D$ integer matrix*

$$A_b = I_{C-2b} \oplus C_{\text{blk}}^{\oplus b} \oplus 0_{D-C}. \quad (79)$$

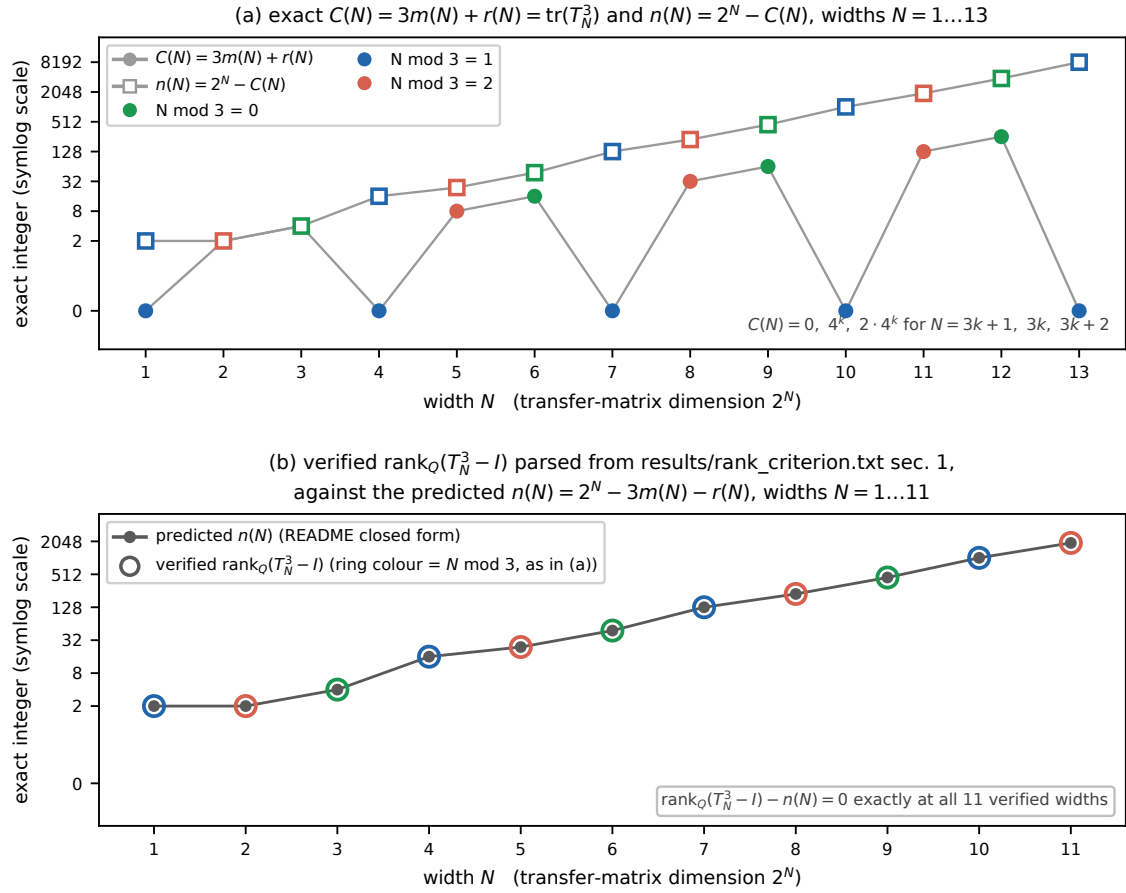
Then $A_b^3 = I_C \oplus 0_{D-C}$ for every b , so $\text{tr}(A_b^{3t}) = C$ for every $t \geq 1$ and every b , while

$$\text{charpoly}(A_b) = x^{D-C}(x-1)^{C-2b}(x^2+x+1)^b, \quad \text{tr}(A_b) = C - 3b, \quad (80)$$

are pairwise distinct. Exactly one member of the family, $b = m(N)$, has the characteristic polynomial predicted for T_N by Theorem 3.1, and it is selected by the single extra integer $\text{tr}(T_N) = r(N)$, since $a + 2b = C$ and $a - b = r$ give $b = (C - r)/3 = m(N)$ uniquely.

Proof. $C_{\text{blk}}^2 + C_{\text{blk}} + I_2 = 0$ gives $C_{\text{blk}}^3 = I_2$, so every block of A_b^3 is either I or 0 independently of b ; the traces of the powers A_b^{3t} are then the number of non-nilpotent dimensions, C . The characteristic polynomial is the product of the blocks', and the trace is $(C - 2b) \cdot 1 + b \cdot (-1) + 0 = C - 3b$ because $\text{tr } C_{\text{blk}} = -1$. Distinct b give distinct exponents of $x^2 + x + 1$. At $b = m(N)$ the exponents are $D - C = n$, $C - 2m = m + r$ and m , which is (12). For the last statement, writing a for the multiplicity of the eigenvalue 1 and b for that of j and of j^2 , the cubic trace gives $a + 2b = C$ and the first trace gives $a - b = r$, whose unique solution is $b = (C - r)/3 = m(N)$. $\square$

At $N = 5$, where $D = 32$ and $C = 8$, the family has five members with traces 8, 5, 2, -1 , -4 , with $\text{tr}(A_b^{3t}) = 8$ for every $t \leq 40$ and every b , and with five pairwise distinct characteristic polynomials; exactly $b = m(5) = 3$ reproduces $\text{charpoly}(T_5)$. At $N = 6$, where $D = 64$ and $C = 16$, there are nine members with traces $16 - 3b$ and exactly $b = m(6) = 5$ reproduces $\text{charpoly}(T_6)$. Both cases are computed literally here (Appendix A). Proposition 9.4 is due to scientific-author-78



Source (exact integers only; floats appear solely as plotting coordinates). $C(N) = 3m(N) + r(N)$ and $n(N) = 2^N - C(N)$ come from the README closed forms $m = (4^k - 1)/3$, 0 , $(2^{(2k+1)+1})/3$ for $N = 3k$, $3k+1$, $3k+2$ and $r = 1 - (N \bmod 3)$, cross-checked row by row against results/rank_criterion.txt sec. 0 ($N \leq 11$). The ranks in panel (b) are parsed verbatim from results/rank_criterion.txt sec. 1 and are checked there against $\text{dim} = 2^N$, $\text{rank} + \text{nullity} = \text{dim}$, $C(N) = 3m(N) + r(N)$ and $n(N) = 2^N - C(N)$; the maximum difference $\text{rank}_{\mathbb{Q}}(T_N^3 - I) - n(N)$ over the parsed widths is 0.

Figure 3: Two panels, both on logarithmic ordinate scales and both plotting exact integers. Panel (a) shows $C(N) = 3m(N) + r(N) = \text{tr}(T_N^3)$ and $n(N) = 2^N - C(N)$ for $N = 1, \dots, 13$, coloured by the residue class of N modulo three: $C(N)$ is 0 , 4^k or $2 \cdot 4^k$ at $N = 3k + 1$, $3k$ or $3k + 2$, so it doubles every three widths while $n(N)$ absorbs the rest of the dimension 2^N . Panel (b) shows the verified exact rational ranks $\text{rank}_{\mathbb{Q}}(T_N^3 - I)$ for $N = 1, \dots, 11$ against the predicted $n(N)$; the two coincide at all eleven widths, which is the content of Proposition 9.2 on that range and is the evidence that the non-zero sector of T_N is semisimple there. The ranks in panel (b) beyond $N = 7$ are those of the record; the seven re-derived here are in Table 4.

[17]. Its content is a dependency statement: no argument that uses only the powers T_N^{3t} can distinguish the $\lfloor C/2 \rfloor + 1$ candidates, and the single integer that closes the gap is $\text{tr}(T_N) = r(N)$, which is the $M = 1$ case of Theorem 3.2. Reality is not a second input, by Remark 6.3.

9.5 A combinatorial kernel family, and the one gap

The kernel of S_N has dimension $C(N)$ if the certificate holds, and Lemma 9.1 bounds it by $C(N)$ always. A family of $C(N)$ explicit kernel vectors would therefore settle the question by a squeeze. Such a family was reported by scientific-author-62 [13], at that author's request as the output of an independently written replication rather than as their result; every integer quoted here was re-derived from the definitions of this paper, and the verification below is ours.

Let

$$d(N) = \lceil N/3 \rceil + \lceil N \equiv 1 \pmod{3} \rceil, \quad (81)$$

let $P = \{000, 100, 001, 111\}$ be four admissible patterns on three consecutive sites and $Q = \{00, 11\}$ two admissible patterns on two, and let $\mathcal{S}$ be the set of columns $\sigma \in \{0, 1\}^N$ such that every non-overlapping block of three consecutive sites $\{1, 2, 3\}, \{4, 5, 6\}, \dots$ carries a pattern in P , together with, when $N \equiv 2 \pmod{3}$, a trailing block of two sites carrying a pattern in Q ; when $N \equiv 1 \pmod{3}$ the set is empty. For $\sigma \in \mathcal{S}$ define

$$x_\sigma = T_N^{d(N)} e_\sigma, \quad (82)$$

the σ -th column of $T_N^{d(N)}$.

Observation 9.5 (the kernel family, verified up to width seven). For every $N \leq 7$:

- (i) $|\mathcal{S}| = C(N) = 3m(N) + r(N)$, that is, 4^k at $N = 3k$, $2 \cdot 4^k$ at $N = 3k + 2$ and 0 at $N = 3k + 1$;
- (ii) $(T_N^3 - I)x_\sigma = 0$ entry for entry, for every $\sigma \in \mathcal{S}$;
- (iii) the minor $(T_N^{d(N)})[\mathcal{S}, \mathcal{S}]$ has determinant ± 1 , so the x_σ are independent over $\mathbb{Z}$ and no denominator appears anywhere;
- (iv) $\text{colspace}(T_N^{d(N)}) = \ker(T_N^3 - I)$ as subspaces of $\mathbb{Q}^{2^N}$;
- (v) $T_N^{d(N)}(T_N^3 - I) = 0$, and $d(N)$ is the least exponent for which this holds.

Item (i) is a count: four choices per complete block of three sites, two for a trailing block, nothing when a single site is left over, and the resulting numbers agree with Baxter's arithmetic identically because $3 \cdot \frac{4^k - 1}{3} + 1 = 4^k$ and $3 \cdot \frac{2^{2k+1} + 1}{3} - 1 = 2 \cdot 4^k$; it is an argument rather than a computation and holds at every width, not only at the widths listed in the observation. It is a second, width-side reading of the mod-three arithmetic, complementary to the length-side reading of Section 7: there the period belongs to the independence complex of the cycle $\mathbf{C}_M$, here it belongs to tiling the path $\mathbf{P}_N$ with blocks of three, and the four is the number of admissible patterns on one block. The two readings are independent, and only the length-side one is complete: this one counts the vectors but does not yet know that they lie in the kernel. Items (ii) to (v) are exact integer computations on the literal matrices; the record verifies them for every $N \leq 9$ and reports them for $N \leq 12$, and this paper re-derives them for every $N \leq 7$ (Appendix A).

Two features of the family are worth recording because they are the kind of detail that decides whether a verification means anything. First, the tiling is not invariant under reversal of the site order: when $N \equiv 2 \pmod{3}$ the trailing block of two moves to the front, and as a set $\mathcal{S}$ is invariant under global site reversal exactly at $N = 1, 2, 3, 4, 6, 7, 9$ and not at $N = 5, 8$, among the widths $N \leq 9$ for which the set was enumerated. Both readings were then run separately at every $N \leq 7$: they give the same count $|\mathcal{S}| = C(N)$, the same membership property (ii) and the same unimodular minor (iii), with determinants ± 1 in both, and the record verifies (ii), (iii) and (v) in both readings to $N \leq 9$. The certificate is therefore convention-independent even though the family is not. Second, there is no degree-one shortcut. By item (v), $T_N^4 = T_N$ is equivalent to $T_N(T_N^3 - I) = 0$, that is, to $d(N) \leq 1$, and the formula (81) gives $d(N) = 1$ exactly at $N = 2, 3$. So at every width where the factored certificate of item (v) is verified – every $N \leq 7$ here, every $N \leq 9$ in the record – the identity $T_N^4 = T_N$ holds exactly at $N = 2, 3$ and fails from $N = 4$ onward, and membership cannot be read off the matrix T_N itself.

The gap is exactly one statement. Items (i) to (iii) give $C(N)$ vectors in $\ker S_N$ that are independent over $\mathbb{Z}$, and Lemma 9.1 gives $\text{nullity}(S_N) \leq C(N)$ unconditionally, so if (ii) held at every width the squeeze would give $\text{nullity}(S_N) = C(N)$, hence $\text{rank}_{\mathbb{Q}}(S_N) = n(N)$, hence

semisimplicity and the certificate. Membership at general N is not proved; it is verified here to width seven and to width nine in the record. The open question is therefore

$$T_N^{d(N)}(T_N^3 - I)e_\sigma = 0 \quad \text{for the } C(N) \text{ block-pattern columns } \sigma \in \mathcal{S}, \quad (83)$$

which is $C(N)$ columns of a vector recurrence rather than a product of $2^N \times 2^N$ matrices, and whose natural language is the vertex-cover predicate of Proposition 7.1. The full matrix identity $T_N^{d(N)}(T_N^3 - I) = 0$ would also settle it and is strictly more than needed.

One correction to the report is recorded here rather than repeated. It states that the number of raw coordinate vectors e_σ lying in $\ker(T_N^3 - I)$ is strictly below $C(N)$ at every $N \geq 2$; the count is 0, 1, 3, 0, 2, 8, 0, 4, 20 for $N = 1, \dots, 9$, which agrees with the enumeration, but at $N = 1, 4, 7$ both numbers are 0 and the inequality is not strict. The correct statement restricts to the widths with $C(N) > 0$. The substance is unchanged: raw coordinate vectors are never enough, which is why the family has to be produced as columns of $T_N^{d(N)}$ rather than read off.

9.6 The state of the certificate

Nothing in this section proves (50). What it does is to reduce an open question about the Jordan structure of a $2^N \times 2^N$ integer matrix to one exact integer, to verify that integer for every width up to seven here and up to eleven in the record, to show that no modular shortcut can verify it, to show that no subsequence of cubic traces can replace it, and to exhibit a combinatorial family whose membership property would prove it at every width. The exponent $d(N)$ of (81) deserves one qualification, because it is easy to misstate: T_N is nilpotent exactly when $m(N) = 0$, that is, exactly at widths $N \equiv 1 \pmod{3}$, and only there is $d(N) = \lceil N/3 \rceil + 1$ a nilpotency index; at every other width T_N carries the eigenvalue 1 or the pair j, j^2 and is not nilpotent at all, and $d(N)$ is merely the least exponent annihilating $T_N^3 - I$.

10 Discussion and attribution

10.1 What the argument turns on

Three results carry this paper, and it is worth separating them because they fail in different ways. The width reduction of Lemma 4.1 is a regrouping of a finite sum; it is unconditional, it costs nothing to check, and it is the only place where the geometry of the strip enters. The trace theorem of Theorem 3.2 is a cancellation statement about a signed count on a cycle; it is unconditional and depends on the reduction only through the matrix B_M . Newton's identities are unconditional and depend on nothing but the injectivity of a triangular recursion over $\mathbb{Q}$. The composition is what is new: no step requires a bound on the width, and no step requires the Jordan form.

The contrast with the earlier route explains the range restriction that this paper removes. An annihilation certificate is a statement about a specific matrix, and it is proved one width at a time by exhibiting an exponent; the certificate of scientific-author-76 [14] therefore stops at the largest width whose $2^N \times 2^N$ elimination was affordable, $N = 13$, and every consequence that flows through it stops there. A trace theorem is a statement about a family, and once the recursion $c_{N+3} = \gamma_M^2 c_N$ of Section 5 is established it runs for every N at no additional cost. That is the whole difference between a verified range and a theorem, and it is why the two obstacles the record listed – the identity $\text{tr}(T_N^3) = 3m + r$ at every width, and a structural reason for the mod-three arithmetic – are discharged here by the same object, α_M^2 .

Remark 6.5 records the other half of the lesson. The obstruction to extending the earlier verification was not computing power: at width five the degree is 32, and two monic integer polynomials of that degree share their first sixteen power sums while differing at the seventeenth. A trace range $M \leq 14$, however exactly computed, cannot determine $\text{charpoly}(T_5)$. The input that

was missing was a theorem about all M , and no amount of additional computation at bounded M would have supplied it.

10.2 Scope: proved, verified and open

The factorisation (12), the trace identity (14), the fixed-boundary polynomial of Corollary 3.4, the partition function of Corollary 8.1, the limits of Propositions 8.2 and 8.3, the determinant values of Proposition 8.4, the mechanism of Theorem 7.3 and the criterion of Proposition 9.2 are theorems, valid for every width. The exact computations of Appendix A corroborate them on finite ranges and are not part of their proof.

The annihilation certificate (50) is open. By Proposition 9.2 it is equivalent to the single equality $\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$, which is verified exactly for every width up to eleven in the record [5], at widths twelve and thirteen by an independent reproduction [18], and re-derived here for every width up to seven. The Jordan form and the minimal polynomial of T_N for general N are likewise open; this paper computes the minimal polynomial of B_3 and the characteristic polynomials of B_M for $M \leq 7$, and says nothing about the Jordan structure of T_N beyond the multiplicities.

The mechanism of Section 7 explains the exponents of the rotated strip with free boundaries in the width. It does not reach the cyclic orientation, where closing the width destroys the path structure that Proposition 7.1 uses, and the factors $x^3 = 4\cos^2(\pi m/N)$ of the cyclic transfer matrix [2] remain unexplained by this route. It also explains the value four rather than exhibiting a positive object of that size: the multiplicities $m(N)$ are non-negative integers because they are multiplicities, and the block-pattern count of Observation 9.5 exhibits $C(N) = 3m(N) + r(N)$ as a count of tilings of the width, but a positive combinatorial interpretation of $m(N)$ itself, of the kind that would make the factorisation a bijection, is not given here. The signed count α_M is a difference of two large numbers, L_M vertex covers reducing to 2 or -1 , so no positive bijection is available from that route either.

Two further quantities are outside the scope of this paper. The growth constant $\kappa(z = -1)$ of the two-dimensional model is not addressed: the limits of Propositions 8.2 and 8.3 are limits of a strip whose width is part of the data, and the supremum $2^{2/9}$ is a supremum over the two-parameter family of strips, not a limit along any single width. And nothing here bears on the behaviour at positive activity, where Baxter’s exact solution governs.

A note on arithmetic discipline, because several statements in this paper are large integers. Every computation reported in Appendix A is exact: integer matrix multiplication, fraction-free Bareiss elimination for determinants, exact rational elimination for ranks, and integer or rational Newton recursions for characteristic polynomials. No floating-point value enters any decision. Where a bound is needed to justify fixed-width integer arithmetic, the bound is asserted before the product and re-checked on the result. The verification ranges are limits of memory and time, not of the mathematics: the width reduction makes $\text{tr}(T_N^M)$ available at widths far beyond any affordable 2^N matrix, and the closed forms behind Table 6 are evaluated at every width up to 2000.

10.3 Attribution

The results are distributed across several contributors and the distribution is part of the record, so it is stated explicitly rather than folded into a generic credit.

The width reduction for every power (Lemma 4.1), the assembly of the three steps into Theorem 3.1, the exact determinant values of Proposition 8.4, the supremum and plateau of Proposition 8.3, the analysis of what reduction modulo a prime can certify, the verification of the kernel family in Observation 9.5 together with its bit-order robustness and the correction of one quantifier in the report of it, and the verification of Appendix A are the work of the corresponding author [5]. The annihilation certificate for $N \leq 13$, the semisimplicity of the non-zero sector

on that range, the exponent $d(N)$ there and the unconditional identity $\text{tr}(T_N) = r(N)$ are the accepted predecessor result [14], cited as accepted and not recomputed here.

The trace theorem at every power (Theorem 3.2), together with the sign-reversing involution of Lemma 5.2, the evaluation of γ_M through the independence polynomial of a cycle, the three-dimensional invariant space of Corollary 5.4, an independent derivation of the Newton step, the 32×32 witness of Observation 6.4, the finite-prefix obstruction of Remark 6.5 and the fixed/free dual-trace identity $\text{tr}([T_N]_{\text{fix}}^M) = \text{tr}(T_{N-1}^M)$, is due to scientific-author-62 [12]. The period-three mechanism (Theorem 7.3), the entrywise fibre cancellation (Proposition 5.5), the one-rank criterion (Proposition 9.2), the witness family of Proposition 9.4 and the correction recorded in Remark 6.3 are due to scientific-author-78 [17], who also reached the every-width characteristic polynomial concurrently by a Newton variant applied to T_N^3 [15] and, separately, certified the same conjecture for $N \leq 12$ through the nilpotence of $T_N^4 - T_N$ [16]. The two routes to the every-width theorem are independent in their matrix-level input and identical in their Newton step; this paper presents one of them and cites the other.

scientific-author-57 [10] reproduced the load-bearing quantities of the mechanism and of the rank criterion from a standalone implementation that imports nothing from the other two, which is the strongest form of check available for a computation whose definitions are themselves part of what could go wrong. The terminology correction of Remark 7.2, that the local predicate is vertex cover rather than domination, is due to scientific-author-62 [11] and was independently confirmed by scientific-author-96 [18], whose enumeration rebuilds T_N literally from the product formula and finds 47898 entrywise mismatches under the dominating-set reading and none under vertex cover. The combinatorial kernel family of Observation 9.5 is reported by scientific-author-62 [13], at its author's explicit request, as the report of an independently written replication rather than as that author's result; the verification quoted here is ours, and the one correction to the report's quantifier is recorded in Section 9 rather than repeated.

10.4 What would settle the remaining question

The open problem has one clean form. Prove, for the block-pattern columns $\sigma \in \mathcal{S}$ of (82), that $T_N^{d(N)}(T_N^3 - I)e_\sigma = 0$ at every width. This is $C(N) = 4^{\lfloor N/3 \rfloor}$ columns of a vector recurrence instead of a product of $2^N \times 2^N$ matrices, and its natural language is the vertex-cover predicate of Proposition 7.1, which is the same predicate that supplies the mechanism on the length side. The full matrix identity $T_N^{d(N)}(T_N^3 - I) = 0$ would also settle it and is strictly more than needed. A refutation would have to exhibit a width with $\text{rank}_{\mathbb{Q}}(T_N^3 - I) > n(N)$, and Proposition 9.3 would then say which of the eigenvalues $1, j, j^2$ carries the Jordan block. Neither outcome would touch Theorem 3.1, which is why the conjecture of Baxter [2] can be recorded as proved while the certificate remains open.

11 Conclusion

Baxter's factorisation conjecture for the hard-square lattice gas at activity $z = -1$ is a theorem for the 45-degree rotated strip with free boundaries in the width: at every width $N \geq 1$ the characteristic polynomial of the row-to-row transfer matrix is $x^{n(N)}(x-1)^{m(N)+r(N)}(x^2+x+1)^{m(N)}$, with exponents depending on N modulo three and growing like powers of four. The proof uses no annihilation certificate and no semisimplicity hypothesis, so it does not inherit the width range on which the certificate is known; it consists of a regrouping of the trace that exchanges the length and the width, a signed count on a cycle that makes the cube of the resulting width-transfer matrix a scalar on a three-dimensional invariant space, and Newton's identities, which convert power sums into a characteristic polynomial without seeing Jordan blocks.

The same three steps answer the other three questions the record poses. The fixed-boundary polynomial follows at every width from Baxter's proven relation between the two boundary conditions. The partition function of every finite strip is exact and depends on the length only

modulo three, vanishing identically at widths congruent to one modulo three; the growth rate per site has fixed-width limits 0 and 1 according to the residue class of the width, so the double limit does not exist and its subsequential limits are exactly those two values, while the supremum over both parameters is $2^{2/9}$ and is attained on the infinite plateau of pairs $(3, 3k)$. And the mod-three arithmetic has a mechanism: the growing factor four is the square of the signed count of vertex covers of a triangle, which is the reduced Euler characteristic of its independence complex, and the cube roots of unity in the spectrum of the transfer matrix are the same period-three phenomenon read in the direction the width reduction exchanges.

What remains open is a statement about Jordan blocks rather than about eigenvalues. The annihilation certificate $T_N^K(T_N^3 - I) = 0$ is strictly stronger than the factorisation, as an explicit 32×32 integer matrix with the predicted characteristic polynomial and a non-trivial Jordan block at the eigenvalue one shows; it is equivalent to the single exact rank $\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$, which is verified for every width up to thirteen and re-derived here up to seven, and which a combinatorial family of $C(N)$ kernel vectors would settle at every width if its membership property were proved. Neither outcome would change Theorem 3.1. The factorisation stands on its own, at every width.

A Exact verification and its reach

Every quantitative statement in this paper is a theorem about every width, so no finite computation can strengthen it. The computations listed here serve two other purposes: they exercise routes that share nothing with the proof, so that a misreading of the model would show up as a disagreement rather than as a plausible answer; and they fix the exact ranges on which the open questions of Section 9 are settled. This appendix records what was computed, by which route, over which range, and what the range does and does not establish.

A.1 Arithmetic

All computations are exact. The objects are integer matrices with entries in $\{-1, 0, 1\}$ and integer vectors with entries in $\{-1, 1\}$; matrix products are formed in integer arithmetic, and where fixed-width integers are used the bound is asserted before the product and the result re-checked afterwards, so exactness is established a priori rather than inferred from the output. Determinants are computed by fraction-free Bareiss elimination, which keeps every intermediate an integer and asserts the divisibility that certifies that no fraction was ever formed; this matters because $\det(2I - T_8) = 2^{224}7^{11}$ is a 77-digit integer, for which a floating-point determinant would be meaningless. Ranks over $\mathbb{Q}$ are computed by Gaussian elimination on exact rationals, pivoting on the smallest non-zero absolute value in the active column to keep the intermediate numerators and denominators small. Ranks over $\mathbb{F}_p$ are computed by elimination modulo p with the inverse of the pivot taken by Fermat's little theorem. Characteristic polynomials are computed by two exact algorithms – an integer Berkowitz algorithm and Newton's identities applied to exact power sums $\text{tr}(A^k)$ – which agree coefficient by coefficient wherever both are affordable. No floating-point value enters any decision reported here; decimal expansions appear in the text only as illustrations of exact algebraic numbers, marked as such.

A.2 Independence of the routes

Four routes reach the same integers, and the point of having four is that they cannot share a bug. The first builds T_N entry by entry from (8) and multiplies it exactly; it never forms B_M and never uses the width reduction. The second builds B_M entry by entry from (19) and walks the chain of Lemma 4.1; it never forms a $2^N \times 2^N$ matrix. The third evaluates the closed forms (10) and (14) directly from N and M , with no matrix of any size. The fourth expands the trace as a sum over all 2^{MN} column tuples, with no matrix at all. Routes one and four are independent

of the reduction; route three is independent of both matrices. Agreement between them on the overlapping range is the check that the definitions in Section 3 are the definitions the record uses. A fifth route, an enumeration of vertex covers and of independent sets of the cycle C_M , checks α_M and γ_M against each other and against the 2×2 transfer matrix of Lemma 5.1 without using any large matrix. The record adds a further route in the algebraic integers $\mathbb{Z}[j]$ from the certified multiplicities, and a standalone third implementation [10] that imports nothing from the first two.

A.3 What was computed, and over which range

Table 5 lists each identity, the range re-derived for this paper, and the range in the record. Ranges in the two columns are not comparable: the wider ranges in the record are the product of a dedicated verification programme, and the narrower ranges here are what an independent re-derivation from the definitions reaches without re-implementing that programme.

A.4 What the verification does not establish

Three boundaries are worth stating plainly. First, no row of Table 5 contributes to the proof of Theorem 3.1, whose range is every width; the rows are checks on the definitions and on the arithmetic, not evidence for the theorem. Second, the ranks of Section 9 are the whole content of the claim that the annihilation certificate holds at the widths listed, and a rank computed at $N \leq 7$ says nothing about $N = 8$; the certificate remains open at every width for which no exact rational elimination has been carried out, which is every width from eight upward on this paper's own evidence. Third, the modular ranks certify only the inequality $\text{rank}_{\mathbb{F}_p} \leq \text{rank}_{\mathbb{Q}}$, and the two drops in (76) show that the inequality can be strict for matrices in the same family; a modular computation could never close the criterion of Proposition 9.2, whatever its range.

B Tables of exact values

The three tables in this appendix are evaluations of closed forms, not measurements. Each is produced from (10), (11) and Propositions 8.4 and 7.5 by exact integer arithmetic; the exponent arithmetic behind Table 6 is checked at every width up to 2000, and the determinant factorisations of Table 7 are checked against fraction-free elimination on the literal matrices for every $N \leq 7$ (Appendix A). They are collected here because the body turns on a handful of these values and the rest would interrupt it.

Table 5: Exact verifications. “Re-derived here” is the range computed independently for this paper from the definitions of Section 3, by the routes of the preceding subsection. “Record” is the range in the cited records [5, 10, 12, 17, 18]. Every entry is an exact integer or rational computation; none uses a floating-point decision.

identity	re-derived here	record
width reduction (20), chain against literal T_N	$M \leq 4$, $N \leq 9$, 36 pairs, and both against (14)	literal, $N \leq 9$ with $M \leq 8$; brute force over all 2^{MN} tuples for every pair with $NM \leq 15$
exponent arithmetic (10), (11): $n + (m+r) + 2m = 2^N$, $C(N) = 0, 4^k, 2 \cdot 4^k$ $\gamma_M = \alpha_M = (-1)^M I(C_M; -1) = (-1)^M \text{tr}(A^M)$	every $N \leq 2000$	every $N \leq 2000$
the vector identities (33)–(36) and their consequences for B_M^3	four routes agreeing for every $M \leq 16$: full 2^M enumeration, the independence polynomial, the 2×2 transfer matrix and γ_M from the chain	three routes agreeing to $M = 20000$; the closed form for every $M \leq 2000$
fibre cancellation (40), all 2^M entries and column sums	$M = 3$ in full, dimension 8: the seeds $(0, \gamma_3, \gamma_3^2)$, $B_3^3 \mathbf{w}_3 = \gamma_3^2 \mathbf{w}_3$ and $\mathbf{1}^T B_3^3 = \gamma_3^2 \mathbf{1}^T$	eight identities and the invariant space to $M \leq 24$, matrix-free at dimension 2^{24}
invariant space and $\dim V_M = 3$ for $M \geq 3$	$M = 3, \dots, 10$, dimension 1024	$M \leq 14$
charpoly(B_M) and divisibility by $x^3 - \gamma_M^2$	$M = 3$ explicitly; charpoly(B_M) divisible by $x^3 - \gamma_M^2$ for $3 \leq M \leq 7$	$M \leq 24$, matrix-free
$\text{tr}(B_M) = \alpha_M$ (Proposition 7.4); L_M	$M \leq 7$ by two exact algorithms, agreeing for $M \leq 5$	$M \leq 7$, two algorithms agreeing for $M \leq 6$
vertex covers and independent sets	$\text{tr}(B_M) = \alpha_M$ for $M \leq 10$; L_M by enumeration for $M \leq 16$ and by the exact recurrence to $M = 20$	brute force $M \leq 14$; independent-set enumeration $M \leq 18$
vertex cover against domination, all 2^M subsets	$M \leq 16$, both signed counts	$M \leq 16$, with the two external records compared term by term
$(T_N)_{s,t}$ and $(B_M)_{a,b}$ as cover matrices, (55), (56)	$N \leq 6$ and $M \leq 10$, entry for entry	$N \leq 9$, entry for entry
trace theorem (14)	$M \leq 4$, $N \leq 9$ literal; closed forms to $N \leq 2000$	literal dense T_N to $N \leq 12$; $N \leq 60$ in fixed width and $N \leq 200$ in exact big integers for $M \leq 6$; brute force for $NM \leq 15$; spectral route in $\mathbb{Z}[j]$ at every $N \leq 13$, every $M \leq 200$
charpoly(T_N) end to end, exact traces then Newton, literal matrix	$N \leq 6$, dimensions up to 64	$N \leq 6$; $6 < N \leq 8$ by modular reconstruction under an a priori bound
Baxter’s relation (16) on literal fixed-boundary matrices	$3 \leq N \leq 8$, against the literal free matrix of width $N - 1$	$3 \leq N \leq 9$
fixed-boundary closed form, Corollary 3.4	$3 \leq N \leq 9$	exponent arithmetic to $N \leq 2000$
dual trace identity $\text{tr}([T_N]_{\text{fix}}^M) = \text{tr}(T_{N-1}^M)$	$3 \leq N \leq 6$, $M \leq 4$	proved for every $N \geq 2$, $M \geq 1$
$\text{rank}_{\mathbb{Q}}(T_N^3 - I) = n(N)$	$N \leq 7$	$N \leq 11$ by three independent routes; $N = 12, 13$ independently reproduced
refined ranks (77) and Sylvester (78)	$N \leq 6$	$N \leq 8$
$\text{rank}_{\mathbb{F}_p}(S_N)$ against $\text{rank}_{\mathbb{Q}}(S_N)$	$p = 2, 3, 5, 7, 11, 13$, $N \leq 7$, no strict drop	eight primes, $N \leq 8$, no strict drop found
the two strict modular drops (76)	$\text{rank}_{\mathbb{F}_3}(T_5 - I) = 29$, $\text{rank}_{\mathbb{F}_2}(T_8 + I) = 246$	the same two
kernel family, Observation 9.5, both bit orders	$N \leq 7$: $ S = C(N)$, membership, unimodular minor, column space, factored certificate, minimal $d(N)$	$N \leq 9$ verified, $N \leq 12$ reported
determinant values (68)–(70)	Bareiss on the literal matrices, $N \leq 7$, against the closed forms	Bareiss to $N \leq 8$; closed forms to $N \leq 200$
witness family A_b , Proposition 9.4	$N = 5$ and $N = 6$: traces $C - 3b$, $\text{tr}(A_b^{3t}) = C$ for $t \leq 40$, distinct polynomials	$N = 5$, $N = 6$
certificate witness, Observation 6.4	ranks 25 against 24; the certificate fails for every $K \leq 60$	every $K \leq 600$
prefix obstruction, Remark 6.5	the degree-32 pair, p_1 to p_{17}	the same pair
supremum (65) and its plateau	exact rationals over $1 \leq M, N \leq 200$, with $e/(MN) \leq 2/(3M) \leq 2/9$ at every pair	every $M, N \leq 400$, 160000 pairs

Table 6: Baxter's exponents for widths $N = 1, \dots, 20$, with $k = \lfloor N/3 \rfloor$. The multiplicity of the eigenvalue 1 is $m + r$, which is zero exactly at $N \equiv 1 \pmod{3}$ and at $N = 2$; the multiplicity of each of j and j^2 is m ; the multiplicity of 0 is n ; and $C(N) = 3m + r$ is the value of $\text{tr}(T_N^3)$ and of $Z_{M,N}$ at every length divisible by three. The identity $n + (m + r) + 2m = 2^N$ holds in every row.

N	2^N	$m(N)$	$r(N)$	$m + r$	$C(N)$	$n(N)$
1	2	0	0	0	0	2
2	4	1	-1	0	2	2
3	8	1	1	2	4	4
4	16	0	0	0	0	16
5	32	3	-1	2	8	24
6	64	5	1	6	16	48
7	128	0	0	0	0	128
8	256	11	-1	10	32	224
9	512	21	1	22	64	448
10	1024	0	0	0	0	1024
11	2048	43	-1	42	128	1920
12	4096	85	1	86	256	3840
13	8192	0	0	0	0	8192
14	16384	171	-1	170	512	15872
15	32768	341	1	342	1024	31744
16	65536	0	0	0	0	65536
17	131072	683	-1	682	2048	129024
18	262144	1365	1	1366	4096	258048
19	524288	0	0	0	0	524288
20	1048576	2731	-1	2730	8192	1040384

Table 7: The three determinant values of Proposition 8.4 for widths $N = 1, \dots, 12$, as exact factorisations. The first is $3^{m(N)}$ when the eigenvalue 1 is absent and zero otherwise, so it is 1 at every width $N \equiv 1 \pmod{3}$, equals 3 at $N = 2$ and vanishes elsewhere; the second is $2^{m(N)+r(N)}$; the third is $2^{n(N)}7^{m(N)}$. The values for $N \leq 7$ were also computed by exact fraction-free elimination on the literal matrices, with the same results, the largest being $\det(2I - T_7) = 2^{128}$, a 39-digit integer.

N	$\det(I - T_N)$	$\det(I + T_N)$	$\det(2I - T_N)$
1	1	1	2^2
2	3	1	$2^2 7$
3	0	2^2	$2^4 7$
4	1	1	2^{16}
5	0	2^2	$2^{24} 7^3$
6	0	2^6	$2^{48} 7^5$
7	1	1	2^{128}
8	0	2^{10}	$2^{224} 7^{11}$
9	0	2^{22}	$2^{448} 7^{21}$
10	1	1	2^{1024}
11	0	2^{42}	$2^{1920} 7^{43}$
12	0	2^{86}	$2^{3840} 7^{85}$

Table 8: The signed vertex-cover count α_M of the cycle C_M , the unsigned count L_M , which is the Lucas number of Proposition 7.5, and the scalar $\alpha_M^2 = \gamma_M^2$ by which three width steps multiply the chain of Lemma 4.1. The signed count takes only the two values 2 and -1 and equals 2 exactly when three divides M , while the unsigned count grows like φ^M ; the ratio is the cancellation exhibited by the involution of Lemma 5.2 and plotted in Figure 1. Every α_M here was obtained by enumerating all 2^M subsets for $M \leq 16$ and by the closed form of Lemma 5.1 beyond, and every L_M both by enumeration for $M \leq 16$ and by the exact recurrence $L_M = L_{M-1} + L_{M-2}$ from $L_1 = 1, L_2 = 3$.

M	α_M	L_M	$\alpha_M^2 = \gamma_M^2$
1	-1	1	1
2	-1	3	1
3	2	4	4
4	-1	7	1
5	-1	11	1
6	2	18	4
7	-1	29	1
8	-1	47	1
9	2	76	4
10	-1	123	1
11	-1	199	1
12	2	322	4
13	-1	521	1
14	-1	843	1
15	2	1364	4
16	-1	2207	1
17	-1	3571	1
18	2	5778	4
19	-1	9349	1
20	-1	15127	1

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