

Exact finite-size scaling of the three-dimensional mean spherical film with Neumann–Dirichlet boundary conditions and surface fields at and below the film critical point

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Abstract

The three-dimensional mean spherical model confined to a film of finite thickness, free on one bounding plane and ordinary on the other, is one of the few lattice models whose finite-size scaling can be computed exactly. With surface fields on the two planes, closed forms were available only on the disordered side of the film critical point, where the spherical field is large enough for the transverse propagator to be written in hyperbolic functions. At and below that point the same finite-thickness mode sums remain perfectly well defined, but their continuation, its branch structure and its scaling functions were unknown. We show that the scaled equation of state with surface fields is a single analytic function of one variable across the whole physical range, and that its derivative is strictly positive for every real surface field. The positivity is proved on both sides of the film critical point from two determinant numerators, one bounded below by an exact rational floor and the other expanded as an entire series with manifestly positive coefficients. The equation of state therefore has exactly one real root for every real value of the scaling temperature variable: existence, uniqueness and branch selection are settled without any free-energy comparison. Surface fields do not create branches; they shift the trigonometric-to-hyperbolic boundary from zero to an exact negative value. Three further consequences are worked out in closed form. The magnetization profile continues across the boundary as an entire function and, deep in the ordered film, collapses onto the lowest transverse mode as a single field-induced condensate, while the spherical field itself crosses over from an exponential to a power-law dependence on thickness, with a sharp criterion on the field strength deciding whether an exponential window survives at all. The specific heat per spin equals one half in the bulk at and below criticality, and lies below it at order one over thickness by a strictly positive scaling function, so the sign of the confinement anomaly and the uniqueness of the equation of state are one and the same statement. The leading enhancement and suppression of local fluctuations at the free and ordinary faces are image terms of the bulk lattice Green function and are untouched by surface fields, which enter one order later; a conjectured logarithmic correction to the interior fluctuation is refuted and the cancellation responsible for its failure is identified. Every result is tested against exact finite-thickness mode sums. Not claimed here: a free-energy scaling function, hence no continued Casimir amplitude; no closed form for the interior fluctuation coefficient or for the measured

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next-to-leading specific-heat coefficient; and nothing at fixed physical surface field, where the scaling window is left behind.

1 Introduction

A film of a critical material is the simplest geometry in which confinement changes the physics. Its thickness L is the only length scale competing with the correlation length, and near the bulk critical point the two combine into scaling variables that govern every thermodynamic quantity. For a generic three-dimensional magnet those scaling functions are known only approximately, from the ε -expansion, from renormalisation-group calculations at fixed dimension, or from simulation. Universal critical amplitudes make the situation sharper still: the scaling functions are normalisation-independent in the sense of Privman and Fisher [12], so a model that can be solved exactly supplies a reference value against which approximation schemes can be tested. The mean spherical model is that solvable reference. Its spins are real-valued, its interaction is nearest-neighbour, and the constraint that fixes the length of the spin vector is imposed on the whole system at once through a single Lagrange multiplier, the spherical field. That one global field is what makes the model exactly solvable and also what makes its finite-size behaviour unusual: the field adjusts to the geometry, so the shift of the film critical point and the condensation of the field below it are properties of the film, not of the bulk.

For a film geometry — thickness L , infinite in the two remaining directions — the transverse part of the problem is a one-dimensional difference operator whose spectrum is fixed by the boundary conditions on the two bounding planes. The Neumann–Dirichlet (ND) choice, free on one plane and ordinary on the other, is the one that describes a film with two physically different walls, and it is the boundary condition under which the Gaussian and the mean-field versions of the model are known to disagree about the sign and size of the critical Casimir force [8]. Exact results for the mean spherical film have accumulated over three decades. Surface critical exponents of the spherical model were obtained in Ref. [6], and the magnetization profile in inhomogeneous fields in Refs. [1, 2]; the separation of scaling from non-scaling finite-size effects in the Gaussian and mean spherical model with free boundaries is Ref. [5]; the behaviour of systems with an asymptotically large critical shift, including the ND film at zero surface field, is Ref. [9]; the inversion of the $d = 3$ ND equation of state in closed form and the evaluation of the ND Casimir amplitude is Ref. [4]; Casimir force scaling functions for non-periodic boundary conditions in $2 < d \leq 3$ are Ref. [11]; and the review of exact Casimir results is Ref. [7].

The case that concerns this paper is the film with *surface fields*. Surface fields are what a real wall applies: preferential adsorption at the boundary of a binary liquid, or a surface coupling to a magnetic film. For the three-dimensional mean spherical film with ND boundary conditions and fields h_1, h_L on the two bounding planes, Amin [3] derived the scaled equation of state and the magnetization profile in closed form. Those formulas contain $\sqrt{y^2 - \pi^2/4}$, where $y = \phi^{1/2}L$ is the scaled spherical field, and they are therefore valid where that quantity is real: on the disordered side of the film critical point, $y > \pi/2$. The complementary regime, $0 < y \leq \pi/2$, contains the film critical point itself and the whole ordered side. At zero surface field it is settled: the trigonometric ND constraint, the absence of a logarithmic critical shift, and the below-transition condensation of the spherical field are in Ref. [9], and the ND Casimir amplitude is in Ref. [4]. At nonzero surface field it was open. The underlying finite- L mode sums do not know about the distinction: they are manifestly real and well defined on both sides, and the hyperbolic closed forms continue to trigonometric ones through an imaginary transverse momentum. What was missing was the continued equation of state with surface fields, its branch structure, the profile and specific-heat

scaling functions there, and any explicit evaluation of the scaling function at nonzero field.

This paper supplies all of that. The starting observation is that the continuation is not a case analysis. Writing $X = y^2$ and $u = X - \pi^2/4$, the combination $\cosh^2 w = \cos^2 \omega$, with $w = \sqrt{u}$ and $\omega = \sqrt{-u}$, is an entire function of u , and so are the three kernels that multiply the surface fields. The published equation of state and its continuation are therefore *one* analytic equation on the whole range $X > 0$, whose only singularity is at $X = 0$ (Section 4). Once that is recognised, the qualitative question — how many branches are there, and which one is physical — becomes a question about the sign of a single derivative. Section 5 answers it with a proof rather than a measurement: the derivative is strictly positive for every real surface field on both sides of the film critical point, because two determinant numerators are positive, one by an exact rational Taylor floor and a monotone subinterval bound, the other because it expands into an entire series whose coefficients are positive by induction. The equation of state has exactly one real root for every real ζ , and no free-energy comparison is needed to select it.

Five consequences are then worked out in closed form and checked against exact finite- L mode sums. The regime boundary is not $\zeta = 0$ but the exact field-dependent value $\zeta_0 = -(\eta_1^2/3 + \eta_1\eta_L + \eta_L^2)/4$, so a surface field moves the film across the boundary rather than creating a new branch. In the ordered film the spherical field no longer decays exponentially with thickness: it follows a power law set by the field, and whether an exponential window survives at all is decided by a sharp criterion on the field strength, with two closed-form critical values. The magnetization profile continues as an entire function of u and, deep in the ordered phase, collapses onto the shape of the lowest transverse mode: the ordered film with a surface field carries a single field-induced condensate rather than a saturated profile. The local fluctuation splits exactly into a part the constraint already fixes and an oscillatory image part, and the image part has closed-form limits at both faces — the bulk lattice Green function at the image distance, enhanced at the free face and suppressed at the ordinary one — that contain neither ζ nor the surface fields. Finally, the specific heat per spin has an exact finite- L formula whose scaling limit identifies the specific-heat scaling function as the ζ -derivative of X ; because that is the reciprocal of the derivative proved positive above, the film specific heat always lies below its bulk value at order $1/L$, and the sign of the confinement anomaly is the same statement as the uniqueness of the equation of state.

Two numbers in the literature are disputed here rather than adopted. The formulation of the problem this paper answers [13] transcribes the subtracted specific-heat constant as $\pi/2$; in the convention pinned from the exact mode sums in Section 3.5 it is $1/2$, and Section 7 gives two independent routes to that value and explains why no reachable convention produces $\pi/2$. The same source transcribes the exponent of the field-free ordered asymptotics with the growing sign; the exact solution decays, and Section 3 states the asymptotics in the form that was verified. Both disagreements are recorded in the text where they matter, because a reader who works from the published formulas needs to know which reading the numbers correspond to.

Every number quoted below was produced in high-precision arithmetic from the exact finite- L mode sums, with printed residuals, and the derivations and captured outputs behind them are archived in Ref. [10]. The figures in this paper were drawn from an independent re-implementation of the same closed forms, written for the purpose and compared digit by digit against the tabulated values; Appendix A records that comparison and the arithmetic conventions of every table.

2 Model, boundary conditions and scaling variables

2.1 The film and its constraint

Consider the mean spherical model on the lattice $\{1, \dots, L\} \times \mathbb{Z}^2$ with nearest-neighbour coupling $J > 0$ and surface fields acting only on the two bounding planes,

$$\mathcal{H} = -J \sum_{\langle ij \rangle} s_i s_j - \sum_{r=1}^L h_r^{\text{phys}} \sum_{\mathbf{x} \in \mathbb{Z}^2} s_{r,\mathbf{x}}, \quad h_r^{\text{phys}} = h_{\text{phys}}^1 \delta_{r,1} + h_{\text{phys}}^L \delta_{r,L}. \quad (1)$$

The spins s_i are real-valued, and the spherical constraint $\sum_i s_i^2 = N$ is imposed on average through a single global Lagrange multiplier. It is convenient to work with the dimensionless coupling $\tilde{K} = \beta J$ and with the surface fields in thermal units, $h_1 = \beta h_{\text{phys}}^1$ and $h_L = \beta h_{\text{phys}}^L$; the published formulas this paper continues are written in terms of

$$K = \frac{\tilde{K}}{2}, \quad (2)$$

that is, in the convention where the Boltzmann weight reads $\exp[K \sum_{ij} A_{ij} s_i s_j + \dots]$ with A the adjacency. Section 3.5 shows that this identification is not a matter of taste: it is forced by the exact finite- L constraint, and reading K literally as βJ leaves an order-one residual in the published equation of state.

The bulk critical coupling is the Watson integral of the simple-cubic lattice,

$$\tilde{K}_c = \int \frac{d^3 q}{(2\pi)^3} \frac{1}{6 - 2(\cos q_1 + \cos q_2 + \cos q_3)} = 0.252731009858662980516\dots, \quad K_c = \frac{\tilde{K}_c}{2}, \quad (3)$$

and the film critical coupling $\tilde{K}_{c,L}$ is defined in Section 3.3 as the value at which the spherical field reaches the lowest transverse eigenvalue.

2.2 The ND transverse operator

The transverse part of (1) is diagonalised by the ND mode functions and eigen-angles

$$u_L(r, k) = \frac{2}{\sqrt{2L+1}} \cos\left[\left(r - \frac{1}{2}\right)\phi_L(k)\right], \quad \phi_L(k) = \frac{\pi(2k-1)}{2L+1}, \quad k = 1, \dots, L, \quad (4)$$

which are orthonormal, $\sum_{r=1}^L u_L(r, k) u_L(r, k') = \delta_{kk'}$, the identity holding to 3.1×10^{-50} for L up to 14 and exactly at $L = 2$. The normalisation is $2/\sqrt{2L+1}$, not $\sqrt{2/(2L+1)}$: it follows from $\sum_{r=1}^L \cos^2[(r - \frac{1}{2})\phi_L(k)] = (2L+1)/4$, and every finite- L number in this paper depends on it. The transverse eigenvalues of the Laplacian are

$$\lambda_k = 2[1 - \cos \phi_L(k)], \quad \lambda_1 = 2\left[1 - \cos \frac{\pi}{2L+1}\right] \sim \left(\frac{\pi}{2L}\right)^2, \quad (5)$$

and we shall write $d_k = \lambda_k - \lambda_1 \geq 0$ throughout.

It is worth being explicit about which operator (4) diagonalises, because it fixes the boundary convention that every closed form below must respect. Reconstructing the coupling matrix from the spectrum, $T = \sum_k 2 \cos \phi_L(k) u_k u_k^\top$ gives, at $L = 5$ and $L = 6$ and to fifty digits, $T_{r,r+1} = T_{r+1,r} = 1$ for $r = 1, \dots, L-1$, a single self-loop $T_{1,1} = 1$, and $T_{r,r} = 0$ for $r \geq 2$. The ND convention is therefore exactly: the path-graph adjacency with one Neumann self-loop of weight one at the free face $r = 1$, and no self-loop at $r = L$, where the Dirichlet condition is imposed by the ghost plane $r = L+1$. That is why the lowest eigenvalue is $2 \cos \phi_L(1)$ rather than $2 \cos[\pi/(L+1)]$, and it is the origin of the half-lattice offsets that appear in the image charges of Section 8.

2.3 Scaling variables

The spherical field of the published formulas is

$$\phi = \tilde{\phi} + \lambda_1, \quad (6)$$

where $\tilde{\phi} \geq -\lambda_1$ is the bulk-referenced field that appears in the Gaussian quadratic form; with this definition $\tilde{\phi} + \lambda_k = \phi + d_k$ exactly, and $\phi = \lambda_1$ is the film critical point. Near that point the four finite-size scaling variables are

$$y = \phi^{1/2}L, \quad \eta_1 = K^{-1/2}h_1L^{3/2}, \quad \eta_L = K^{-1/2}h_LL^{1/2}, \quad \zeta = (K_{c,L} - K)L, \quad (7)$$

with $K_{c,L} = \tilde{K}_{c,L}/2$. The different powers of L in η_1 and η_L are not a choice: they follow from the distinct surface critical behaviour of the free and the ordinary face [6]. Holding η_1 and η_L fixed means the physical fields scale as $h_1 \sim L^{-3/2}$ and $h_L \sim L^{-1/2}$, and Section 9 is explicit that everything derived here lives in that window.

We shall also use, everywhere below,

$$X = y^2 = \phi L^2, \quad u = X - \frac{\pi^2}{4} = w^2 = -\omega^2, \quad z_t = (\tilde{K}_{c,L} - \tilde{K})L = 2\zeta, \quad (8)$$

so that $u > 0$ is the disordered, hyperbolic regime with $w = \sqrt{u} > 0$, and $u < 0$ the ordered, trigonometric regime with $\omega = \sqrt{-u} \in (0, \pi/2)$. The film critical point is $X = \pi^2/4$, that is $y = \pi/2$, $u = 0$.

2.4 What was known, and on which side of the boundary

For $y > \pi/2$, Ref. [3] gives the scaled equation of state

$$\frac{1}{4\pi} \ln[\cosh w] + \frac{1}{4y} \left[\eta_1^2 Y_1'(y) + 2\eta_1 \eta_L Y_2'(y) + \eta_L^2 Y_3'(y) \right] = 2\zeta, \quad w = \sqrt{y^2 - \frac{\pi^2}{4}}, \quad (9)$$

with

$$Y_1(y) = \frac{\tanh w}{w}, \quad Y_2(y) = \frac{1}{\cosh w}, \quad Y_3(y) = -w \tanh w, \quad (10)$$

and the closed-form magnetization profile

$$m_L(r) = \frac{h_1 \sinh[(L - r + 1)x] \cosh(x/2) / \sinh x + h_L \cosh[(r - \frac{1}{2})x]}{2K \cosh[(L + \frac{1}{2})x]}, \quad x = \operatorname{arccosh}[\phi/2 + \cos \phi_L(1)]. \quad (11)$$

Two things about (9)–(11) are left implicit by the published derivation and have to be settled before any continuation is attempted, because both change the answer. The first is the meaning of the prime in Y_i' ; the second is the coupling convention, that is, whether the K appearing in the prefactor of (11) and in (7) is βJ or $\beta J/2$. Sections 3.4 and 3.5 derive both from the exact finite- L mode sums: the prime is d/dy , and the convention is (2). With those two pins in place, (9) holds on exact finite- L solutions with a residual that vanishes as $1/L$, in seven cases including four with $y < \pi/2$; under the literal reading $K = \beta J$ the residual stays of order one.

3 Exact finite- L representation

3.1 The constraint

Imposing the spherical constraint through its delta-function representation turns the partition function into a Gaussian integral whose quadratic form has eigenvalues $\tilde{K}[\tilde{\phi} + 6 - 2 \cos \phi_L(k) - 2 \cos q_1 - 2 \cos q_2]$. In the limit of an infinite in-plane area, the condition $\langle \sum_i s_i^2 \rangle = N$ becomes, per site,

$$1 = \frac{1}{\tilde{K}L} \sum_{k=1}^L F(\tilde{\phi} + \lambda_k + 4) + \frac{1}{L} \sum_{r=1}^L m(r)^2, \quad (12)$$

where the in-plane integral is

$$F(z) = \int_{[-\pi, \pi]^2} \frac{d^2 q}{(2\pi)^2} \frac{1}{z - 2 \cos q_1 - 2 \cos q_2} = \frac{2}{\pi z} K_{\text{ell}}\left(\frac{16}{z^2}\right), \quad z > 4, \quad (13)$$

in the parameter convention $K_{\text{ell}}(m) = \int_0^{\pi/2} dt / \sqrt{1 - m \sin^2 t}$, and the profile is

$$m(r) = \frac{1}{\tilde{K}} [h_1 G(r, 1) + h_L G(r, L)], \quad G(r, r') = \sum_{k=1}^L \frac{u_L(r, k) u_L(r', k)}{\tilde{\phi} + \lambda_k}. \quad (14)$$

With (6) the argument of F is $\phi + d_k + 4$. The prefactor $1/\tilde{K}$ in (14) and the denominator $\tilde{\phi} + \lambda_k$ at $q = 0$ follow from the same Gaussian algebra and are not adjustable.

The closed form (13) was checked against a one-dimensional quadrature of the q_2 integral to 6.7×10^{-52} , against a two-dimensional quadrature that uses no elliptic identity to 1.2×10^{-32} , and against an independent double-precision periodic trapezoid rule to 1.7×10^{-16} . The whole constraint (12) was additionally re-derived through a completely independent double-precision path — dense diagonalisation of the $L \times L$ ND operator plus a trapezoid rule over the in-plane Brillouin zone — agreeing to 1.1×10^{-16} . Near the band bottom the only singular piece is logarithmic,

$$F(4 + c) = \frac{1}{4\pi} \ln \frac{32}{c} - \frac{c}{32\pi} \left[\ln \frac{32}{c} - 1 \right] + O(c^2 \ln c), \quad (15)$$

which agrees with the exact value to twelve significant figures at $c = 10^{-12}$. This is the term that produces the logarithm in the equation of state, and it is the reason the small- X behaviour of the scaled equation of state is logarithmic at zero field and stronger than logarithmic at nonzero field.

3.2 The exact film Green function, in both regimes

The Green function of (14) can be summed. Write $a = \phi + 2 \cos \phi_L(1)$, so that $(aI - T)G = \mathbb{K}$. In the bulk, $2 \leq r \leq L - 1$, the homogeneous recurrence $aG(r) - G(r - 1) - G(r + 1) = 0$ has the two solutions $e^{\pm x r}$ once $a = 2 \cosh x$. The two boundary rows are not the bulk recurrence: the free face carries the self-loop, $(a - 1)G(1) - G(2) = \delta$, and the ordinary face carries the ghost plane, $aG(L) - G(L - 1) = \delta$. Substituting $G(r) = A \cosh[(r - \frac{1}{2})x] + B \sinh[(r - \frac{1}{2})x]$ into the free-face condition gives $-2B \sinh(x/2) = 0$, so the cosh solution satisfies it identically and the sinh partner is excluded. Substituting $G(r) = \sinh[(L + 1 - r)x]$ into the ordinary-face condition gives $a \sinh x = \sinh 2x$, an identity. Matching the unit jump at $r = r'$ then fixes the normalisation, with min and max below always taken over (r, r') :

$$G(r, r') = \frac{\psi_L(\min) \psi_R(\max)}{\sinh x \cosh[(L + \frac{1}{2})x]}, \quad \psi_L(r) = \cosh[(r - \frac{1}{2})x], \quad \psi_R(r) = \sinh[(L + 1 - r)x], \quad (16)$$

with

$$\cosh x = \frac{a}{2} = \frac{\phi}{2} + \cos \phi_L(1). \quad (17)$$

The two boundary columns reproduce the profile kernels of (11) exactly,

$$G(r, 1) = \frac{\sinh[(L+1-r)x] \cosh(x/2)}{\sinh x \cosh[(L+\frac{1}{2})x]}, \quad G(r, L) = \frac{\cosh[(r-\frac{1}{2})x]}{\cosh[(L+\frac{1}{2})x]}, \quad (18)$$

the second because $\psi_R(L) = \sinh x$ cancels the $\sinh x$ in the normalisation. Against the mode sum, (16) holds to 1.2×10^{-49} at the worst of twenty parameter points spanning both regimes.

The point that makes this paper possible is that (16) is a rational expression in $\cosh x = a/2$. Whenever $a/2 < 1$, writing $x = i\theta$ with $\theta = \arccos(a/2) \in (0, \pi)$ turns every factor into a trigonometric one and leaves G real:

$$G(r, r') = \frac{\cos[(\min - \frac{1}{2})\theta] \sin[(L+1-\max)\theta]}{\sin \theta \cos[(L+\frac{1}{2})\theta]}, \quad \cos \theta = \frac{\phi}{2} + \cos \phi_L(1). \quad (19)$$

At finite L , in other words, the ordered side is not a continuation at all: the mode sums are manifestly real, and the trigonometric closed form is an identity rather than an extrapolation. The genuinely delicate point is where the denominator $\cos[(L+\frac{1}{2})\theta]$ vanishes, which is exactly where $a = 2 \cos \phi_L(k)$ for some k , that is, where the finite- L constraint becomes singular. Since $\cos \phi_L(1) = 1 - \pi^2/[2(2L+1)^2] + O(L^{-4})$, the condition $a/2 > 1$ is equivalent to $\phi > \pi^2/[2(2L+1)^2] + O(L^{-4})$, whose leading scaling form is $y > \pi/2$. The finite- L boundary and the scaling boundary $y = \pi/2$ differ at order L^{-3} in ϕ . Every statement below that switches between the two functional forms is made in terms of the exact finite- L quantity $a/2$; $y = \pi/2$ is used only as its leading scaling label. This is the first of the boundaries of scope recorded in Section 9.

3.3 Product identities and the exact finite- L equation of state

Two products turn the constraint sum into a closed logarithm. The first is exact for every L ,

$$\prod_{k=1}^L \lambda_k = 1, \quad (20)$$

with a residual below 1.7×10^{-48} for $L \leq 60$. The second is the determinant of the tridiagonal operator,

$$\prod_{k=1}^L (\phi + d_k) = \det(aI - T) = \frac{\cosh[(L+\frac{1}{2})x]}{\cosh(x/2)}, \quad (21)$$

verified against the recurrence $D_n = aD_{n-1} - D_{n-2}$ and against the closed form to 3.4×10^{-42} in both regimes, where the right-hand side becomes $\cos[(L+\frac{1}{2})\theta]/\cos(\theta/2)$.

The bulk and film critical couplings follow. Define

$$\tilde{K}_{c,L} = \frac{1}{L} \sum_{k=1}^L F(\lambda_k + 4). \quad (22)$$

This is exactly the coupling at which $\tilde{\phi} = 0$, that is $\phi = \lambda_1$, that is $a/2 = 1$, that is $y = \sqrt{\lambda_1} L \rightarrow \pi/2$: the film critical point, where the lowest transverse mode of the Green function goes soft. Identity

(20) makes the $\ln 32$ pieces cancel between $\tilde{K}_{c,L}$ and $\tilde{K}_c$, and the shift is

$$\tilde{K}_{c,L} - \tilde{K}_c = \frac{A}{L} + O(L^{-3}), \quad A = \lim_{L \rightarrow \infty} L(\tilde{K}_{c,L} - \tilde{K}_c) = \frac{1}{2} \left[\tilde{K}_c - \frac{\ln 2}{2\pi} - F(8) \right] = 0.0041327294443333, \quad (23)$$

with $f(c) = F(4+c) - \ln(32/c)/(4\pi)$ regular at $c = 0$ and the derivation running through the midpoint structure of the nodes $\phi_L(k)$. There is no $\ln(L)/L$ component — an independent numerical confirmation of the absence of a logarithmic critical shift for ND films [9] — and no $1/L^2$ component either: measured out to $L = 2560$, the sequence $L^2(\tilde{K}_{c,L} - \tilde{K}_c - A/L)$ falls by exactly a factor of two per doubling towards zero while $L^3(\tilde{K}_{c,L} - \tilde{K}_c - A/L)$ converges, with Richardson limit -0.0044839 . In the convention (2) the same shift reads $K_{c,L} - K_c = A/(2L)$.

Subtracting the constraint at $\phi = \lambda_1$ from the constraint at general ϕ , and using (21) for the difference of the two sums, gives an *exact* finite- L equation of state,

$$z_t = \frac{1}{4\pi} \ln \frac{\cosh[(L + \frac{1}{2})x]}{\cosh(x/2)} + \sum_{k=1}^L [f(\lambda_k) - f(\phi + d_k)] - \tilde{K} \sum_{r=1}^L m(r)^2, \quad (24)$$

$$\tilde{K} \sum_{r=1}^L m(r)^2 = \frac{h_1^2 A_1 + 2h_1 h_L A_2 + h_L^2 A_3}{\tilde{K}}, \quad A_1, A_2, A_3 \text{ defined below}, \quad (25)$$

an identity rather than an approximation: its residual is below 6×10^{-30} at every L and parameter point tested. The last term of (24) is the whole effect of the surface fields, it is $O(1/L)$ in the scaling regime, and it is a sum of squares divided by $\tilde{K} > 0$, hence nonnegative — which is why it enters z_t with a minus sign. Section 4 returns to that sign, because it is what makes the surface field help monotonicity rather than threaten it.

3.4 The boundary-column sums and their scaling limits

With $A_1 = \sum_r G(r, 1)^2$, $A_2 = \sum_r G(r, 1)G(r, L)$ and $A_3 = \sum_r G(r, L)^2$, the three sums can be carried out in closed form from $\sum_{j=1}^L \sinh^2(jx)$, $\sum_{r=1}^L \sinh[(L + \frac{3}{2} - 2r)x]$ and $\sum_{r=1}^L \cosh^2[(r - \frac{1}{2})x]$; the expressions are collected in Appendix B. They are valid in both regimes, the complex- x continuation being automatic with vanishing imaginary part at every point tested, and they agree with direct mode sums to 7.6×10^{-49} for $L \in \{5, 9, 14, 22\}$ and $\phi \in \{1.1, 0.3, 5 \times 10^{-4}, -0.5\}$.

The scaling limits are what enter the equation of state. With $w = xL$ and $C_i = \lim_L A_i/L^{4-i}$,

$$C_1(w) = \frac{\sinh 2w - 2w}{4w^3 \cosh^2 w}, \quad C_2(w) = \frac{\sinh w}{2w \cosh^2 w}, \quad C_3(w) = \frac{\sinh 2w + 2w}{4w \cosh^2 w}, \quad (26)$$

approached as $1/L$ at fixed w (relative residuals 1.8×10^{-3} to 1.2×10^{-2} at $L = 240$, degrading to 3.5×10^{-2} near $y = \pi/2$, where all three diverge like $(\pi/2 - \omega)^{-2}$), and continued by $w = i\omega$ to

$$C_1(\omega) = \frac{2\omega - \sin 2\omega}{4\omega^3 \cos^2 \omega}, \quad C_2(\omega) = \frac{\sin \omega}{2\omega \cos^2 \omega}, \quad C_3(\omega) = \frac{\sin 2\omega + 2\omega}{4\omega \cos^2 \omega}, \quad (27)$$

all real and nonnegative on $0 < \omega \leq \pi/2$. The identity that connects these to the published kernels is

$$C_i = -\frac{1}{2y} \frac{dY_i}{dy}, \quad y = \sqrt{w^2 + \frac{\pi^2}{4}}, \quad \frac{dw}{dy} = \frac{y}{w}, \quad (28)$$

checked analytically to 1.4×10^{-48} and against an independent numerical derivative to 4.1×10^{-47} in both regimes. *The prime in (9) is therefore d/dy* , and in the ordered regime (10) continues to

$$Y_1 = \frac{\tan \omega}{\omega}, \quad Y_2 = \frac{1}{\cos \omega}, \quad Y_3 = \omega \tan \omega. \quad (29)$$

3.5 The coupling convention is forced

The published profile (11) carries the prefactor $(2K)^{-1}$, the published equation of state (9) carries $1/(4y)$ in front of the field term and 2ζ on the right, and the scaling variables (7) carry $K^{-1/2}$. The derivation of Section 3.1 gives instead $1/\tilde{K}$ for the profile, $1/(2y)$ for the field term and z_t on the right. Reading K literally as βJ leaves those three factors mutually inconsistent. A single convention reconciles all three, namely (2): then $\eta_i = K^{-1/2}h_i L^p = \sqrt{2}\tilde{K}^{-1/2}h_i L^p$ supplies the missing factor of two in the field term, and $\zeta = z_t/2$ supplies the factor two on the right-hand side.

The test is decisive. Solve the exact finite- L constraint at fixed (η_1, η_L, ζ) , that is at $K = \tilde{K}_{c,L}/2 - \zeta/L$, $h_1 = \eta_1\sqrt{K}/L^{3/2}$ and $h_L = \eta_L\sqrt{K}/L^{1/2}$, and evaluate the published equation of state on the solution, once in the convention (2) and once with K read as βJ , using (29) where $y < \pi/2$. In the first reading the residual vanishes as $1/L$ with a convergent $L \times$ residual in all seven cases tested, four of them with $y < \pi/2$; in the second it stays of order one and drifts. Appendix A, Table 7 gives the seven cases. The solved values of y also converge to the scaling-limit roots: at $(\eta_1, \eta_L, \zeta) = (0, 0, 0.05)$ they are 2.407890, 2.446413, 2.466642, 2.477016, 2.482271 for $L = 20, 40, 80, 160, 320$, against the analytic root 2.487572050752277 obtained from $\ln \cosh w/(4\pi) = 2\zeta$. Every continuation performed below is therefore carried out in the variables (7)–(8) with $K = \tilde{K}/2$ and $\phi = \tilde{\phi} + \lambda_1$, and the $1/L$ residuals above are the honest size of the finite- L corrections at $L = 320$.

3.6 Field-free ordered asymptotics, with prefactor

At zero surface field and $\tilde{K} > \tilde{K}_c$, identity (21) gives $\ln \phi = -4\pi L(\tilde{K} - \tilde{K}_{c,L}) - \ln P_L + 4\pi \sum_k [f(\phi + d_k) - f(\lambda_k)]$ with $P_L = \prod_{k \geq 2} (\lambda_k - \lambda_1)$, and the last term is $o(1)$. Since $\lambda_1/\lambda_k \rightarrow (2k-1)^{-2}$ at fixed k , the product tends to the Wallis value $\prod_{k \geq 2} [1 - (2k-1)^{-2}] = \pi/4$, while $1/\lambda_1 \sim (2L+1)^2/\pi^2$, so $P_L \sim L^2/\pi$. Converting $\tilde{K}_{c,L}$ to $\tilde{K}_c$ with (23),

$$\phi \sim C L^{-2} \exp[-4\pi L(\tilde{K} - \tilde{K}_c)], \quad C = \pi e^{4\pi A} = \frac{\pi}{2} \exp[2\pi(\tilde{K}_c - F(8))] = 3.30905714 \dots \quad (30)$$

The measured combination $\phi L^2 \exp[4\pi L(\tilde{K} - \tilde{K}_c)]$ is 3.1426 at $L = 80$, 3.2763 at $L = 160$ and 3.3026 at $L = 320$, converging to the analytic C .

Two honest notes on (30). The rate $4\pi L$ belongs to the thermal convention $\tilde{K} = \beta J$; in the convention (2) of the published formulas the same asymptotics reads $\exp[-8\pi L(K - K_c)]$. And the exponent as transcribed in the problem statement [13], $\exp[-4\pi L(K_{c,3} - K)]$, grows in the ordered phase $K > K_{c,3}$, whereas the exact solution decays: the sign is $-(K - K_c)$. This paper states the asymptotics in the form it verified.

4 One equation of state in both regimes

4.1 A single analytic function of u

Because $D \equiv \cosh^2 w = \cos^2 \omega$ and $w^2 = -\omega^2 = u$, the three kernels (26), (27) and the logarithm in (24) are all single-valued functions of u , and (28) rewrites the field term of (9) as

$$\frac{1}{4y} \left[\eta_1^2 Y_1' + 2\eta_1 \eta_L Y_2' + \eta_L^2 Y_3' \right] = -\frac{1}{2} \left[\eta_1^2 C_1 + 2\eta_1 \eta_L C_2 + \eta_L^2 C_3 \right]. \quad (31)$$

The scaled equation of state is then one equation on the whole range $X > 0$,

$$\text{LHS}(X) = \frac{1}{8\pi} \ln D(u) - \frac{1}{2} Q(u) = 2\zeta, \quad D(u) = \cosh^2 w = \cos^2 \omega, \quad Q(u) = \eta^\top C(X) \eta, \quad (32)$$

with $\eta = (\eta_1, \eta_L)$ and C the symmetric 2×2 matrix with entries $C_{11} = C_1$, $C_{12} = C_{21} = C_2$, $C_{22} = C_3$. Here D is entire in u with a double zero at $X = 0$, and the C_i are single-valued in u ; the only singularity of LHS is at $X = 0$. The point $y = \pi/2$ is not a branch point, and the two functional forms are not two branches.

Equation (32) has to be evaluated, not merely written, and two cancellations would silently corrupt it deep in the ordered regime. For $|u| < 1/4$ the entire series in u are used for D and for all three C_i (Appendix B); they avoid the cancellation in $\sinh 2w - 2w$ and continue automatically to $u < 0$. For $X < \pi^2/8$ the naive $\omega = \sqrt{\pi^2/4 - X}$ and $\cos \omega$ cancel catastrophically, so everything is evaluated through

$$s = \frac{\pi}{2} - \omega = \frac{2X}{\pi + \sqrt{\pi^2 - 4X}}, \quad D = \sin^2 s, \quad \sin \omega = \cos s, \quad \sin 2\omega = \sin 2s, \quad 2\omega = \pi - 2s. \quad (33)$$

The two evaluation routes agree with each other at their switch point $X = \pi^2/8$ to 5.9×10^{-61} relative, the series branch agrees with the direct closed forms at $u = \pm 1/4$ to machine zero, and the unified $C_i(X)$ agrees with the separately implemented hyperbolic and trigonometric forms to 4.0×10^{-49} relative on a grid from $X = 0.01$ to $X = 400$.

4.2 The surface-field term is nonpositive

The sign of (31) is worth stating as a result in its own right, because the correct sign is what makes the field term help rather than threaten monotonicity, and because the opposite sign is easy to write down by mistake. There are two independent routes. At finite L , (24) carries the field as $-\tilde{K} \sum_r m(r)^2$, and $\tilde{K} \sum_r m(r)^2 = \tilde{K}^{-1} \sum_r [h_1 G(r, 1) + h_L G(r, L)]^2$ is a sum of squares; evaluated from mode sums at four (L, ϕ) points with four field configurations the values are strictly positive and match the A_1, A_2, A_3 quadratic form to 3.0×10^{-61} relative. In the scaling limit, (28) gives $-C/(2y)$ for the matrix of Y'_i , so the field term equals $-\frac{1}{2} \eta^T C \eta$, and C is positive definite on both sides of $X = \pi^2/4$ with the exact determinant identities

$$\det C = \frac{\omega^2 - \sin^2 \omega}{4\omega^4 \cos^2 \omega} \quad (u < 0), \quad \det C = \frac{\sinh^2 w - w^2}{4w^4 \cosh^2 w} \quad (u > 0), \quad (34)$$

verified to 2.4×10^{-63} . Both numerators are positive because $\sin \omega < \omega$ on $(0, \pi/2)$ and $\sinh w > w$ on $(0, \infty)$. The scaled field term is therefore strictly negative for every nonzero field, in both regimes.

4.3 The regime boundary is exactly ζ_0

At $X = \pi^2/4$, that is $u = 0$, the three kernels take the exact values

$$C_i\left(\frac{\pi^2}{4}\right) = \left(\frac{1}{3}, \frac{1}{2}, 1\right), \quad (35)$$

with zero residual at twenty digits, so

$$\text{LHS}\left(\frac{\pi^2}{4}\right) = -\frac{1}{2}\left(\frac{\eta_1^2}{3} + \eta_1 \eta_L + \eta_L^2\right) = 2\zeta_0, \quad \boxed{\zeta_0 = -\frac{1}{4}\left(\frac{\eta_1^2}{3} + \eta_1 \eta_L + \eta_L^2\right) \leq 0} \quad (36)$$

and $\zeta_0 = 0$ only when both fields vanish. Once monotonicity is established (Section 5), (36) gives the regime boundary exactly:

$$y < \frac{\pi}{2} \iff \zeta < \zeta_0, \quad y = \frac{\pi}{2} \iff \zeta = \zeta_0, \quad y > \frac{\pi}{2} \iff \zeta > \zeta_0. \quad (37)$$

Table 1: Certified roots of the scaled equation of state (32). Bracket widths are 5.7×10^{-41} to 4.6×10^{-40} in $\ln X$ and back-substitution residuals are below 5×10^{-41} . The three rows with $\zeta = \zeta_0$ return $y = \pi/2$ to 6×10^{-19} , 5×10^{-42} and machine precision respectively, which is the boundary statement (37) read as a number.

(η_1, η_L)	ζ	y	$X = y^2$	regime
(1, 0.5)	$-0.3708333333 \dots = \zeta_0 - 0.1$	1.456298476682373	2.120805253187401	$y < \pi/2$
(1, 0.5)	$-0.2708333333 \dots = \zeta_0$	1.570796326794897	2.467401100272339	$y = \pi/2$
(1, 0.5)	$-0.1708333333 \dots = \zeta_0 + 0.1$	1.743681505317413	3.040425191986001	$y > \pi/2$
(1, 0.5)	0.05	3.039536376567051	9.238781384474359	$y > \pi/2$
(0, 1.5)	$-0.6625 = \zeta_0 - 0.1$	1.498740551014240	2.246223239254468	$y < \pi/2$
(0, 1.5)	$-0.5625 = \zeta_0$	1.570796326794897	2.467401100272339	$y = \pi/2$
(0, 1.5)	$-0.4625 = \zeta_0 + 0.1$	1.662723870159960	2.764650668399716	$y > \pi/2$
(0, 0)	0.05	2.487572050752277	6.188014707683887	$y > \pi/2$
(0, 0)	-0.05	0.907398136343380	0.823371377839440	$y < \pi/2$
(0, 0)	-0.25	0.076571929781218	0.005863260430420	$y < \pi/2$

A surface field does not create branches; it shifts the trigonometric-to-hyperbolic boundary from $\zeta = 0$ to $\zeta_0 < 0$. One immediate consequence deserves emphasis: at $\zeta = 0$, that is exactly at the zero-field film critical coupling, a film carrying any surface field is already on the hyperbolic side.

Table 1 collects ten certified roots of (32), spanning both regimes, zero field, single-face field and both faces at once. The brackets are 5.7×10^{-41} to 4.6×10^{-40} wide in $\ln X$ and the residuals are below 5×10^{-41} ; an independent bisection reproduces all ten to 7.1×10^{-16} relative, and a sign-change count on a coarse grid finds exactly one change in every case. Figure 1 draws the same information as a curve, with the exact boundary ζ_0 marked on the critical level.

4.4 The zero-field limit is regular below a crossover scale

Because LHS depends on η only through the quadratic form Q , the root $X(\eta, \zeta)$ is even in η , and since $\frac{d\text{LHS}}{dX} > 0$ at the root it is analytic there. Writing $\eta = t\nu$ with $t \geq 0$ and ν fixed,

$$X(t\nu, \zeta) = X_0 + t^2 X_1 + O(t^4), \quad X_1 = \frac{\nu^\top C(X_0) \nu}{2 \frac{d\text{LHS}}{dX}|_{X_0}}, \quad X_0 = X(0, \zeta), \quad (38)$$

so the limit $\eta \rightarrow 0$ at fixed ζ is an ordinary even Taylor expansion and not a singular one. Along $\nu = (1, \frac{1}{2})$ the measured ratio $(X - X_0)/t^2$ converges to the analytic X_1 with relative error 1.1×10^{-10} at $\zeta = 0.05$, 1.8×10^{-10} at -0.05 , 1.9×10^{-9} at -0.5 and 7.5×10^{-9} at -2 , for $X_1 = 5.028575877900$, 45.16788574731 , $3.656247395605 \times 10^6$ and $8.620884372717 \times 10^{22}$ respectively.

What *is* singular is the radius of that expansion. Balancing the field term against $|2\zeta|$ gives

$$t_c(\zeta) = 2\sqrt{\frac{|\zeta|}{\nu^\top C(X_0) \nu}}, \quad (39)$$

measured as 1.00281887 at $\zeta = 0.05$, $1.45538744 \times 10^{-1}$ at -0.05 , $6.13633261 \times 10^{-6}$ at -0.5 and $5.20502864 \times 10^{-22}$ at -2 . It collapses like $e^{8\pi\zeta}$, because X_0 does. Above t_c the root is field-dominated and X becomes *linear* in $|\eta|$ rather than quadratic, so the limits $\eta \rightarrow 0$ and $\zeta \rightarrow -\infty$ do not commute. That non-commutation is the precise sense in which the zero-field ordered regime and the ordered regime at nonzero field are different problems, and it is why the results of Section 4.5 below cannot be obtained by perturbing the published zero-field answer.

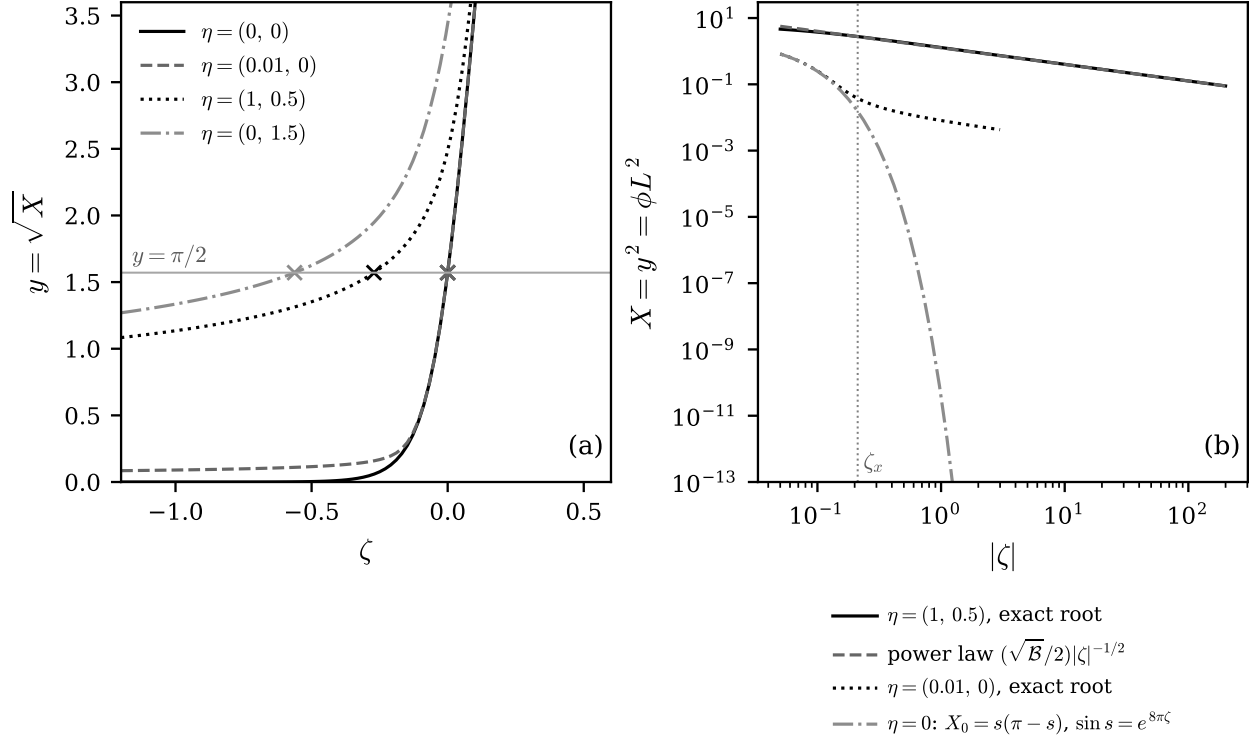


Figure 1: The unique root of the scaled equation of state (32). (a) $y = \sqrt{X}$ against ζ for four field values, with the film critical level $y = \pi/2$ drawn as a thin horizontal line and, on that line, a cross at the exact boundary ζ_0 of (36) for each field. At zero field the curve reaches $y = \pi/2$ at $\zeta = 0$ and collapses towards zero immediately below it; with any nonzero field the whole curve is displaced so that the collapse begins at $\zeta_0 < 0$, and at $\zeta = 0$ the film is already on the hyperbolic side. (b) The same root against $|\zeta|$ on logarithmic axes. The solid curve for $\eta = (1, 0.5)$ follows the field-dominated power law $(\sqrt{\mathcal{B}}/2)|\zeta|^{-1/2}$ (dashed) over the whole ordered range. The weak field $\eta = (0.01, 0)$ instead follows the zero-field exponential $X_0 = s(\pi - s)$, $\sin s = e^{8\pi\zeta}$ (dash-dotted) down to the crossover $\zeta_x = -0.2116131635$ marked by the vertical dotted line, and the power law below it. Curves in (b) are exact roots of (32) except the two asymptotes, which are analytic.

4.5 Field-modified condensation

As $X \rightarrow 0$, (33) gives $s = X/\pi + O(X^2)$ and hence $C_1 \rightarrow 2/(\pi^2 s^2)$, $C_2 \rightarrow 1/(\pi s^2)$, $C_3 \rightarrow 1/(2s^2)$, so that $Q \rightarrow \mathcal{B}/X^2$ with

$$\mathcal{B} = 2\eta_1^2 + 2\pi\eta_1\eta_L + \frac{\pi^2}{2}\eta_L^2, \quad (40)$$

and the ordered equation of state becomes

$$\frac{1}{4\pi} \ln \frac{X}{\pi} - \frac{\mathcal{B}}{2X^2} + o(1) = 2\zeta. \quad (41)$$

Dominant balance replaces the exponential of (30) by a power law,

$$X \sim X_{\text{asy}} = \frac{\sqrt{\mathcal{B}}}{2} |\zeta|^{-1/2}, \quad X = X_{\text{asy}} \left[1 - \frac{\ln(X_{\text{asy}}/\pi)}{16\pi|\zeta|} + O(|\zeta|^{-2} \ln^2 |\zeta|) \right]. \quad (42)$$

Against certified roots at $\eta = (1, 0.5)$, for which $\mathcal{B} = 6.375293203725963$, the leading law is accurate to +0.45% at $\zeta = -10$, +0.069% at -100 , $+1.1 \times 10^{-3}\%$ at -10^4 and $+1.6 \times 10^{-5}\%$

at -10^6 ; including the logarithmic correction improves these to $+4.3 \times 10^{-2}\%$, $+4.7 \times 10^{-3}\%$, $+4.9 \times 10^{-5}\%$ and $+4.9 \times 10^{-7}\%$. The same holds at $\eta = (1, 0)$ with $\mathcal{B} = 2$, at $\eta = (0, 1.5)$ with $\mathcal{B} = 11.103304951225528$ and at $\eta = (0.1, 0.1)$ with $\mathcal{B} = 0.132179875077243$. At zero field, (41) gives instead $\sin s = e^{8\pi\zeta}$ and $X_0 = s(\pi - s)$, the scaling image of (30).

Which of the two governs the true root is decided by a sharp criterion. Both asymptotics decrease with $|\zeta|$ and the exponential does so far faster, so a crossover ζ_x exists only if $X_0(\zeta_0) > X_{\text{asy}}(\zeta_0)$, and $X_{\text{asy}}(\zeta_0)$ is independent of $|\eta|$ along each axis: $\sqrt{6} = 2.449489742783178$ along $\eta = (\varepsilon, 0)$ and $\pi/\sqrt{2} = 2.221441469079183$ along $\eta = (0, \varepsilon)$. Solving the criterion gives the two critical field strengths in closed form,

$$\varepsilon_c = \sqrt{\frac{-3 \ln \sin s_c}{2\pi}} = 0.065489295445993 \quad \text{along } \eta = (\varepsilon, 0), \quad s_c = \frac{\pi - \sqrt{\pi^2 - 4\sqrt{6}}}{2}, \quad (43)$$

$$\varepsilon_c = \sqrt{\frac{-\ln \sin s_c}{2\pi}} = 0.142940888254194 \quad \text{along } \eta = (0, \varepsilon), \quad s_c = \frac{\pi - \sqrt{\pi^2 - 2\sqrt{2}\pi}}{2}, \quad (44)$$

the factor of three in (43) and its absence in (44) reflecting $\zeta_0 = -\varepsilon^2/12$ and $-\varepsilon^2/4$ on the two axes. Bisection on the criterion reproduces both closed forms to 1.4×10^{-61} .

Both behaviours are realised. At $\eta = (0.01, 0)$, below both critical values, the crossover is at $\zeta_x = -0.2116131635$: above it the true root follows the field-free exponential to $+1.8 \times 10^{-4}$ at $\zeta = -10^{-3}$, $+2.6 \times 10^{-4}$ at -10^{-2} , $+1.7 \times 10^{-3}$ at -0.05 and $+1.9 \times 10^{-2}$ at -0.1 , while below it the exponential is wrong by $+0.71$ at $\zeta = -0.3$ and the power law by $+0.14$. At $\eta = (1, 0.5)$, far above both, $X_0(\zeta_0) = 3.474147 \times 10^{-3}$ against $X_{\text{asy}}(\zeta_0) = 2.425878$, so there is no crossover at all and the power law governs the entire ordered regime, with deviations $+2.0 \times 10^{-2}$ at $\zeta = -0.3$, $+2.4 \times 10^{-2}$ at -0.5 , $+2.1 \times 10^{-2}$ at -1 , $+1.4 \times 10^{-2}$ at -2 , $+7.6 \times 10^{-3}$ at -5 , $+2.6 \times 10^{-3}$ at -20 and $+6.9 \times 10^{-4}$ at -100 , while the field-free form is wrong by three orders of magnitude or overflows. Figure 1(b) shows the two structures on one set of axes.

In finite- L variables, inside the scaling window where η is fixed and so $h_1 \sim L^{-3/2}$, $h_L \sim L^{-1/2}$, the ordered film crosses over from (30) to

$$\phi \sim \frac{\sqrt{\mathcal{B}}}{2} L^{-3/2} (K - K_{c,L})^{-1/2}, \quad \eta \neq 0, \quad (45)$$

an exponential-to-power-law change of the thickness dependence. This is a statement about the scaling window: at fixed *physical* surface field, η itself grows with L , the model leaves the window, and (45) cannot be the whole answer. Section 9 records that regime as open rather than extrapolating into it.

5 Conventions, monotonicity and branch selection

5.1 The variables in which the statement is made

A uniqueness statement is only as good as the variables it is stated in, and Section 3.5 showed that the published ones admit exactly one consistent reading. Everything in this section is therefore carried out in the variables (7)–(8), with $K = \tilde{K}/2$, $\phi = \tilde{\phi} + \lambda_1$, $\zeta = z_t/2$, and with the prime in the published kernels understood as d/dy through (28). In those variables the scaled equation of state is (32), its left-hand side is a single analytic function of X on $(0, \infty)$, and the question of how many branches it has is the question of how many times $\text{LHS}(X)$ can take the value 2ζ . That is a question about the sign of one derivative, and it has a different answer at finite L and in the scaling limit.

5.2 Finite L : no branch problem exists

Before the scaling limit, the statement is stronger and simpler. The left-hand side of (12),

$$\Phi(\phi) = \frac{1}{\tilde{K}L} \sum_{k=1}^L F(\phi + d_k) + \frac{1}{L} \sum_{r=1}^L m(r)^2, \quad (46)$$

is strictly decreasing in ϕ : $F'(z) < 0$ by (13), and $\sum_r m(r)^2 = \tilde{K}^{-2} \sum_k [h_1 u_L(1, k) + h_L u_L(L, k)]^2 / (\phi + d_k)^2$ is manifestly strictly decreasing. Numerically Φ falls monotonically from $+2.27 \times 10^8$ to -0.711 at $L = 8$ and from $+3.13 \times 10^6$ to -0.643 at $L = 20$ over $\phi \in [10^{-6}, 4]$, so it brackets exactly one root. The mechanism is exact: with $A_{ij} = \sum_r G(r, i)G(r, j) = (G^2)_{ij}$ and the resolvent identity $dG/d\phi = -G^2$,

$$\frac{d}{d\phi} \left[\frac{2h^\top A h}{\tilde{K}} \right] = -\frac{4h^\top G^3 h}{\tilde{K}} < 0, \quad (47)$$

G being positive definite because all its weights $\phi + d_k$ are positive. A central finite difference of the left side agrees with the right side to 8.0×10^{-26} relative at three values of L . *The finite- L equation of state therefore has a unique solution for every $(\tilde{K}, h_1, h_L)$, and at finite L there is no branch selection problem at all.*

5.3 The scaling limit: a certified positive derivative

Differentiating (32),

$$\frac{d\text{LHS}}{dX} = \frac{D'}{8\pi D} + \frac{1}{2} \eta^\top [-C'(X)] \eta, \quad \frac{D'}{8\pi D} = \begin{cases} \frac{\tan \omega}{8\pi \omega}, & X < \frac{\pi^2}{4}, \\ \frac{\tanh w}{8\pi w}, & X > \frac{\pi^2}{4}, \end{cases} \quad (48)$$

where $D' = dD/du$ and the first term is strictly positive for every $X > 0$ in both regimes. Since $u = w^2 = -\omega^2$,

$$C'(X) = -\frac{1}{2\omega} \frac{dC}{d\omega} \quad (X < \pi^2/4), \quad C'(X) = +\frac{1}{2w} \frac{dC}{dw} \quad (X > \pi^2/4), \quad (49)$$

so the second term of (48) is nonnegative for every η if and only if $dC/d\omega$ is positive semidefinite on the ordered side and dC/dw is negative semidefinite on the disordered one. Both are true, and both are proved rather than sampled.

5.4 Strict monotonicity and uniqueness

Proposition 1 (Strict monotonicity and uniqueness). *Let $\eta_1, \eta_L \in \mathbb{R}$ and let $\text{LHS}(X)$ be defined by (32) with D and C_i given by (26), (27) and their series continuations. Then $\frac{d\text{LHS}}{dX} > 0$ for every $X > 0$. Moreover $\text{LHS}(X) \rightarrow -\infty$ as $X \rightarrow 0^+$ and $\text{LHS}(X) \rightarrow +\infty$ as $X \rightarrow \infty$, so for every real ζ the scaled equation of state (32) has exactly one solution $X > 0$, and (37) holds.*

Proof. The first term of (48) is strictly positive on the whole range, so it suffices to show that $-C'(X)$ is positive semidefinite; strict positivity of $\frac{d\text{LHS}}{dX}$ then follows from the first term alone, uniformly in η .

Ordered side. A direct differentiation of (27), which can be checked symbolically with zero residual, gives

$$\det\left(\frac{dC}{d\omega}\right) = \frac{P(\omega)}{4\omega^6 \cos^3 \omega}, \quad P(x) = 2x \sin x (3 - x^2) - \cos x (3 \sin^2 x + 3x^2 + x^4), \quad (50)$$

with $P(x) = x^8/15 - 11x^{10}/1050 + 13x^{12}/16200 + O(x^{14})$, so the determinant has the expected eighth-order zero at $\omega = 0$ and is positive near it. On $(0, 1]$, standard alternating Taylor bounds — $\sin x$ bounded below by its eleventh-order and above by its fifth-order alternating polynomials, $\cos x$ bounded above by its eighth-order one — give $P(x) \geq x^8 Q(x^2)$ with

$$Q(t) = \frac{11}{168} - \frac{2923t}{302400} + \frac{787t^2}{1247400} - \frac{1877t^3}{79833600} + \frac{t^4}{2016000} - \frac{t^5}{193536000}, \quad (51)$$

and on $0 \leq t \leq 1$, retaining the negative terms at $t = 1$ and dropping the positive ones,

$$Q(t) \geq \frac{356292047}{6386688000} = 0.0557866686144681 > 0. \quad (52)$$

On $[1, \pi/2]$ each factor of (50) is monotone in the needed direction: $x \sin x$ is increasing because $\sin x + x \cos x > 0$ there, $3 - x^2$ is decreasing and positive, $\cos x$ is decreasing and positive, and $3 \sin^2 x + 3x^2 + x^4$ is increasing. Hence on any subinterval $[a, b]$,

$$P \geq 2a \sin a (3 - b^2) - \cos a (3 \sin^2 b + 3b^2 + b^4), \quad (53)$$

each factor bounded in the direction that makes the estimate rigorous rather than illustrative. Over 4096 equal subintervals the minimum of (53) is 0.0555261197867, attained on $[1, 1.0001394]$. Together with $P > 0$ on $(0, 1]$ this gives $P(\omega) > 0$ on the whole of $(0, \pi/2)$, hence $\det(dC/d\omega) > 0$.

Disordered side. Continuing (50) by $\omega \rightarrow iw$ gives $P(iw) = -\Phi_h(w)$ with

$$\Phi_h(w) = 2w \sinh w (3 + w^2) - \cosh w (3 \sinh^2 w + 3w^2 - w^4), \quad \det\left(\frac{dC}{dw}\right) = \frac{-\Phi_h(w)}{4w^6 \cosh^3 w}, \quad (54)$$

the continuation holding symbolically with zero residual. Using $3 \sinh^2 w = \frac{3}{2}(\cosh 2w - 1)$ and $\cosh w \cosh 2w = (\cosh 3w + \cosh w)/2$, the numerator rearranges to

$$-\Phi_h(w) = \frac{3}{4} \cosh 3w - \frac{3}{4} \cosh w + (3w^2 - w^4) \cosh w - (6w + 2w^3) \sinh w, \quad (55)$$

and expanding each term and collecting powers of w produces an entire series in w^2 whose first four coefficients vanish and whose remaining coefficients are all strictly positive:

$$-\Phi_h(w) = \sum_{n \geq 4} b_n \frac{w^{2n}}{(2n)!}, \quad b_n = \frac{3}{4} (9^n - 1) + 12n^2 - 18n - 4n(n-1)(2n-1)^2. \quad (56)$$

The vanishing of $b_0 = b_1 = b_2 = b_3$ is the mirror of the eighth-order contact found on the ordered side, $-\Phi_h(w) = w^8/15 + 11w^{10}/1050 + 13w^{12}/16200 + O(w^{14})$. Positivity of b_n for $n \geq 4$ is proved, not scanned. Writing $4n(n-1)(2n-1)^2 = 16n^2(n-1)^2 + 4n(n-1)$ splits

$$b_n = \left[\frac{3}{4} (9^n - 1) - 16n^2(n-1)^2 \right] + [8n^2 - 14n], \quad (57)$$

and the second bracket is nonnegative for $n \geq 2$. For the first, at $n = 4$ one has $\frac{3}{4}(9^4 - 1) = 4920 \geq 2304 = 16 \cdot 4^2 \cdot 3^2$; and if it holds at n then $\frac{3}{4}(9^{n+1} - 1) = 9 \cdot \frac{3}{4}(9^n - 1) + 6 \geq 144n^2(n-1)^2 + 6$,

which is at least $16(n+1)^2n^2$ provided $9(n-1)^2 \geq (n+1)^2$, that is $3(n-1) \geq n+1$, that is $n \geq 2$. Both the base case and the induction step are exact integer arithmetic. Hence $-\Phi_h(w) > 0$ for every $w > 0$ and $\det(dC/dw) > 0$ for every $w > 0$, with no sampling and no interval arithmetic.

Diagonal signs. A positive determinant alone leaves the common sign of the two eigenvalues open. It is fixed by the finite- L identity (47): $dA/d\phi = -2G^3$ is negative definite at every finite L in both regimes, and $C(X)$ is the scaling limit of the corresponding normalised A_i , so the limiting diagonal entries of dC/dX are nonpositive. A real symmetric 2×2 matrix with positive determinant and nonpositive diagonal entries has two negative eigenvalues — positive determinant forces the two eigenvalues to share a sign, and a shared positive sign would make the trace, hence at least one diagonal entry, positive. Therefore dC/dw is positive definite on $(0, \pi/2)$ and dC/dw is negative definite on $(0, \infty)$, and by (49) $-C'(X)$ is positive definite for every $X > 0$ on both sides of $\pi^2/4$. This gives $\frac{d\text{LHS}}{dX} > 0$.

Limits and conclusion. As $X \rightarrow 0^+$, (33) gives $D = \sin^2 s$ with $s \sim X/\pi$, so $\ln D/(8\pi) \rightarrow -\infty$; if $\eta \neq 0$ then $Q \rightarrow \mathcal{B}/X^2$ by (40) and $-Q/2 \rightarrow -\infty$ as well, so $\text{LHS} \rightarrow -\infty$ in every case. As $X \rightarrow \infty$, $\ln D/(8\pi) \sim \sqrt{X}/(4\pi) \rightarrow +\infty$ while $Q \rightarrow 0$ because all three kernels decay, so $\text{LHS} \rightarrow +\infty$. Strict monotonicity and continuity on $(0, \infty)$ then give exactly one root for every real ζ , and evaluating at $X = \pi^2/4$ with (35) gives (36) and hence (37). $\square$

5.5 Why a sampled margin is not a substitute

The positivity of $\frac{d\text{LHS}}{dX}$ can also be supported numerically, by taking its minimum over a grid. That measurement is retained here only as a consistency check, and the reason it cannot carry the claim is instructive. Over a grid reaching $X = 10^{14}$ the minimum of $\frac{d\text{LHS}}{dX}$ is 3.9788736×10^{-9} and is *identical for six different field values* (η_1, η_L) , including mixed-sign ones. That is because the minimum sits at the top of the grid, where the field term has died and $\frac{d\text{LHS}}{dX} \rightarrow \tanh w/(8\pi w) = 1/(8\pi\sqrt{X})$; and indeed $1/(8\pi\sqrt{10^{14}}) = 3.9788736 \times 10^{-9}$. Over the narrower range $X \in [10^{-14}, 10^4]$ the same quantity is $+3.98 \times 10^{-4}$. A sampled margin measures where the grid stops, not where the model becomes singular. The certificate of Proposition 1 replaces it, and the two independent corroborations that were used to build confidence in the certificate — an entrywise derivative-sign argument on the ordered side, maximum -2.69×10^{-1} for $C'_i(u)$ and minimum $+4.21 \times 10^{-3}$ for $\det C'(u)$, and a sixty-digit check on 99 interior points including the mixed-sign fields $(2, -1)$ and $(-2, 1)$ with minimum $d\text{LHS}/dy = 0.234403628611$ — are recorded here as measurements, not as evidence.

5.6 Consequences: branch selection and the sign of the confinement anomaly

Three consequences follow immediately and are used throughout the rest of the paper. No multiple real branches exist, so no free-energy comparison and no continuity criterion is needed to select one: the physical root is the only root. The regime boundary is therefore the exact number ζ_0 of (36) rather than a transition between two solutions, and a surface field moves the film across that boundary instead of creating a new branch. And the quantity $\frac{d\text{LHS}}{dX}$ whose positivity was just certified is, up to the factor two of (74), the reciprocal of the specific-heat scaling function of Section 7; the uniqueness of the equation of state and the sign of the confinement anomaly are consequently not two results but one.

5.7 A disputed transcription: the subtracted specific-heat constant

One convention question remains, and it is worth separating from the physics because it changes a number that appears in the published scaling form. The specific heat per spin of the film approaches

a bulk value as $L \rightarrow \infty$, and the scaling form writes the approach as that value minus $K_c^2 X'/L$. The problem statement [13] transcribes the subtracted constant as $\pi/2$. It is $1/2$. The bulk value follows from the $(L/2) \ln \tilde{K}$ Gaussian normalisation of (64) alone, and the bulk constraint function diverges logarithmically at criticality, so $c_\infty = \frac{1}{2}$ exactly at and below $\tilde{K}_c$; no rescaling of $\tilde{K}$ against K , of ζ against z_t , or of the fields can turn $1/2$ into $\pi/2$, because c_L is invariant under $\tilde{K} = 2K$ as Section 7 shows. Two independent routes to $1/2$ are given there, together with the exact finite- L formula whose validation pins the two further convention points — fixed dimensionless field, and bond energy only — on which the value depends. The discrepancy is recorded rather than adopted, and the scaling function X' , which is the substantive content, is unaffected by it.

6 Magnetization profile

6.1 The scaling function in both regimes

From (14) and (18), with $xL \rightarrow w$ in the scaling limit,

$$m(r) = \frac{M(\rho)}{2\sqrt{KL}} + O(L^{-1}), \quad \rho = \frac{r}{L}, \quad M(\rho) = \eta_1 f_1(\rho) + \eta_L f_L(\rho), \quad (58)$$

with

$$f_1 = \frac{\sinh[(1-\rho)w]}{w \cosh w} \longrightarrow \frac{\sin[(1-\rho)\omega]}{\omega \cos \omega}, \quad f_L = \frac{\cosh(\rho w)}{\cosh w} \longrightarrow \frac{\cos(\rho\omega)}{\cos \omega}, \quad (59)$$

or, as single entire functions of u written with $P(z) = \cosh \sqrt{z} = \cos \sqrt{-z}$ and $R(z) = \sinh(\sqrt{z})/\sqrt{z} = \sin \sqrt{-z}/\sqrt{-z}$,

$$f_1(\rho) = (1-\rho) \frac{R((1-\rho)^2 u)}{P(u)}, \quad f_L(\rho) = \frac{P(\rho^2 u)}{P(u)}. \quad (60)$$

This is the scaling limit of the published profile (11), continued across $y = \pi/2$ with no extra input. At $X = \pi^2/4$ exactly, (60) gives $f_1(\rho) = 1 - \rho$ and $f_L(\rho) = 1$ — at $\rho = 0.3$, $f_1 = 0.7000000000000000$ and $f_L = 1.0000000000000000$ to fifteen digits — so $M(\rho) = \eta_1(1 - \rho) + \eta_L$; and the two functional forms join continuously, $f_1(0.3) = 0.697083483839284$ at $u = +0.01$ against 0.702940246667100 at $u = -0.01$. At the ordinary face the profile is pinned for every ζ : $f_1(1) = 0$ and $f_L(1) = 1$ give $M(1) = \eta_L$ exactly, a useful check on any numerical implementation.

6.2 Comparison with the exact finite film

Table 2 compares (58) with the profile built from exact finite- L mode sums at $\eta = (1, 0.5)$, in both regimes. The closed-form Green function itself reproduces the mode sums to 3.0×10^{-61} and 4.3×10^{-60} at $L = 12$, so the deviation measured in the table is the $O(1/L)$ correction of (58), not an error in the finite- L input. It is largest at the ordinary face, $\rho = 1$, it falls as $1/L$, and the product $L \times \text{dev}$ drifts slowly upward — the same slow drift that appears in the finite- L approach of X_L to X (Appendix A, Table 9) and that Section 9 records as unexplained. No cleaner definition of ρ was found: the shifts that would remove the leading correction are $r \rightarrow r - \frac{1}{2}$ in f_L and $r \rightarrow r + 1$ in f_1 , which are different, so $\rho = r/L$ is the natural common variable and the residual $O(1/L)$ term is genuine.

Table 2: Relative deviation of the scaling profile (58) from the exact finite- L profile built from mode sums, at $\eta = (1, 0.5)$. $X_L = \phi_L L^2$ is the finite- L value of the scaling variable and X its certified scaling-limit root. “Interior” excludes the two layers adjacent to each face. The deviation is $O(1/L)$ and is largest at the ordinary face.

ζ	L	X_L	X	max rel. dev.	$L \times \text{dev}$	interior only
0.05	12	8.2958739482	9.238781384474	1.583×10^{-1}	—	—
0.05	40	8.9212423010	9.238781384474	5.357×10^{-2}	+2.1429	5.121×10^{-2}
0.05	160	9.1560886477	9.238781384474	1.402×10^{-2}	+2.2435	1.387×10^{-2}
-0.5	12	1.7497355215	1.829032591936	2.826×10^{-1}	—	—
-0.5	40	1.8052718592	1.829032591936	8.987×10^{-2}	+3.5950	7.997×10^{-2}
-0.5	160	1.8231002046	1.829032591936	2.291×10^{-2}	+3.6648	2.223×10^{-2}

6.3 Deep in the ordered film: a single field-induced condensate

As $X \rightarrow 0$, (33) gives $s \rightarrow 0$ with $s \sim X/\pi$, and both kernels collapse onto the same shape,

$$s f_1(\rho) \rightarrow \frac{2}{\pi} \cos \frac{\rho\pi}{2}, \quad s f_L(\rho) \rightarrow \cos \frac{\rho\pi}{2}, \quad (61)$$

verified at $X = 10^{-10}$ and $\rho = 0.25$ to eleven digits, 0.58815997768851 against 0.58815997768240 and 0.92387953251433 against 0.92387953251129, with similar agreement at $\rho = 0.5$ and 0.75. Since $s \sim X/\pi \sim (\sqrt{B}/2\pi)|\zeta|^{-1/2}$ by (42),

$$M(\rho) \sim \frac{2}{\sqrt{B}} (2\eta_1 + \pi\eta_L) \cos \frac{\rho\pi}{2} |\zeta|^{1/2}, \quad m(r) \sim \frac{(2\eta_1 + \pi\eta_L) \cos(\rho\pi/2) |\zeta|^{1/2}}{\sqrt{B} \sqrt{KL}}. \quad (62)$$

At $\eta = (1, 0.5)$, $\rho = 0.4$, the measured M is 22.795011770011 at $\zeta = -100$ and 228.750162395014 at $\zeta = -10^4$, against (62) with ratios 0.996178542099 and 0.999674865619; Figure 2(b) draws the same collapse.

The physical content is that an ordered film with a surface field does not saturate. Its profile is a single condensate in the lowest transverse mode $u_L(r, 1) \sim \cos(\rho\pi/2)$, whose amplitude grows as $|\zeta|^{1/2} L^{-1/2}$ without bound as the film is taken further below its critical point. The $|\zeta|^{1/2}$ growth is the same dominant balance that produces the power law (42) for the spherical field, and the $\cos(\rho\pi/2)$ shape is the same lowest-mode structure that the exact finite- L constraint suggests for the fixed-physical-field regime recorded as open in Section 9.3. That the profile is dominated by the $k = 1$ mode deep in the ordered phase is also the reason the single-mode argument succeeds here and fails for the local fluctuation in Section 8.

7 Specific heat

7.1 Definition, and two convention points that decide the answer

The specific heat per spin in units of k_B , with $N = AL$ spins, is

$$c_L = \frac{C}{k_B N} = -\frac{\tilde{K}^2}{L} \frac{d^2(\beta F/A)}{d\tilde{K}^2} \Big|_{h_1, h_L, L}. \quad (63)$$

Because $\tilde{K} = 2K$, this is also $-K^2 d^2(\beta f)/dK^2$: $d/dK = 2 d/d\tilde{K}$ and $K^2 = \tilde{K}^2/4$ cancel. The specific heat is therefore convention-independent, which matters here because the coupling convention was exactly what had to be pinned in Section 3.5.

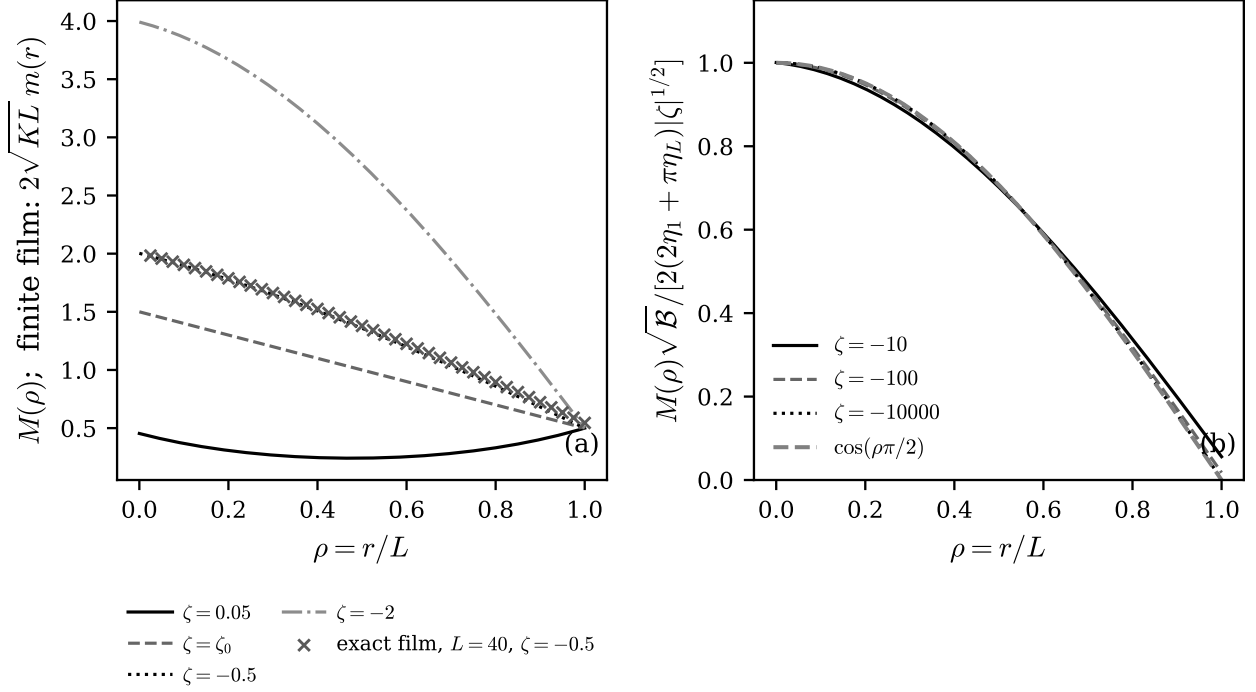


Figure 2: Magnetization profile scaling function at $\eta = (1, 0.5)$. (a) $M(\rho)$ of (58) at $\zeta = 0.05$ (hyperbolic side), at the exact regime boundary $\zeta = \zeta_0 = -0.2708333333$ where $M = \eta_1(1 - \rho) + \eta_L$, and at $\zeta = -0.5$ and -2 on the trigonometric side; the crosses are the exact finite film at $L = 40$, $\zeta = -0.5$, plotted as $2\sqrt{KL}m(r)$ against $\rho = r/L$, and they lie on the scaling curve to within the $O(1/L)$ deviation of Table 2. Every curve ends at $M(1) = \eta_L = 0.5$, which is exact. (b) Deep in the ordered film the same profiles, normalised by the asymptotic amplitude $(2/\sqrt{B})(2\eta_1 + \pi\eta_L)|\zeta|^{1/2}$ of (62), collapse onto $\cos(\rho\pi/2)$, the shape of the lowest transverse mode; the three curves are $\zeta = -10$, -100 and -10^4 .

Two points have to be fixed before anything is derived, and both are visible in the checks below. First, the derivative in (63) is taken at fixed *dimensionless* h_1, h_L , the fields that appear in $\eta_1 = K^{-1/2}h_1L^{3/2}$ and $\eta_L = K^{-1/2}h_LL^{1/2}$. Holding the physical field fixed instead would make h proportional to β and add a term that does not scale. Second, with that choice the internal energy is the bond energy alone: the field term of (1) does not contribute to $-d\ln Z/d\beta$ because h is held fixed, and forgetting this produces an extra $-S/2$ in the internal energy and a wrong specific heat.

7.2 Free energy and an exact finite- L formula

The representation already pinned in Section 3.1 gives the free energy per area in closed form,

$$\frac{\beta F}{A} = -\frac{\tilde{K}L(\tilde{\phi} + 6)}{2} + \frac{L}{2} \ln \tilde{K} + \frac{1}{2} \sum_{k=1}^L I(c_k) - \frac{S}{2}, \quad c_k = \phi + d_k, \quad S = h_1 m_1 + h_L m_L, \quad (64)$$

$$I(c) = \int \frac{d^2 q}{(2\pi)^2} \ln(c + \hat{q}^2) = I(0) + \int_0^c F(4+t) dt, \quad I(0) = \frac{4G}{\pi} = 1.1662436161232751206, \quad (65)$$

with G Catalan's constant. $I(0)$ is the classical Mahler-measure value and is re-derived rather than assumed, by direct two-dimensional quadrature of $I(c)$ at $c = 1$ and $c = 4$ against $I(0) + \int_0^c F(4+t) dt$,

agreement 0.0 at twenty digits in both cases. It is a pure constant and drops out of every $\tilde{K}$ -derivative, so it is never load-bearing for c_L ; it matters only for the absolute value of $\beta F/A$.

The saddle-point condition $d(\beta F/A)/d\tilde{\phi} = 0$ is *identically* the constraint (12): substituting $\sum_k F(4 + c_k) = \tilde{K}[L - \sum_r m_r^2]$ into $(\tilde{K}L)/2 - \frac{1}{2} \sum_k F(4 + c_k) - (\tilde{K}/2) \sum_r m_r^2$ gives zero term by term. Consequently the total $\tilde{K}$ -derivative along the physical solution equals the partial derivative at fixed $\tilde{\phi}$, which is what makes the reduction below exact rather than asymptotic. Two independent routes to the internal energy per area then agree identically,

$$\frac{\beta U}{A} = \tilde{K} \frac{d(\beta F/A)}{d\tilde{K}} \Big|_{\tilde{\phi}} = -\frac{\tilde{K}L(\tilde{\phi} + 6)}{2} + \frac{L}{2} + \frac{S}{2} = -\frac{\tilde{K}}{2} \frac{\text{Tr}(AQ^{-1}) + m^\top A m}{A}, \quad (66)$$

the second using $\text{Tr}(AQ^{-1})/A = [(6 + \tilde{\phi})\Sigma - L]/\tilde{K}$ with $\Sigma = \sum_k F(4 + c_k)$, which follows because the adjacency eigenvalue is $2 \cos \phi_L(k) + 2 \cos q_1 + 2 \cos q_2 = 6 + \tilde{\phi} - (c_k + \hat{q}^2)$, and $m^\top A m = (\tilde{\phi} + 6) \sum_r m_r^2 - S/\tilde{K}$ from $A = (\tilde{\phi} + 6)\mathbb{K} - Q/\tilde{K}$. The constraint turns $\sum_r m_r^2$ back into Σ and the two routes coincide. This is a check on the algebra rather than on the arithmetic, so it is recorded rather than measured.

Differentiating (66) along the saddle gives the exact finite- L specific heat per spin, with $\phi' = d\phi/d\tilde{K}$ at fixed h_1, h_L, L ,

$$c_L = \frac{1}{2} + \frac{\tilde{K}^2}{2} \phi' + \frac{S}{L} + \frac{\mathcal{A}\phi'}{2L}, \quad (67)$$

$$\mathcal{A} = h^\top G^2 h = \sum_{k=1}^L \frac{\hat{h}(k)^2}{(\phi + d_k)^2}, \quad \hat{h}(k) = u_L(1, k)h_1 + u_L(L, k)h_L, \quad (68)$$

$$\phi' = \frac{\tilde{K}L + \mathcal{A}/\tilde{K}}{\tilde{K}\Sigma' - 2 \sum_k \hat{h}(k)^2/(\phi + d_k)^3}, \quad \Sigma' = \sum_{k=1}^L F'(\phi + d_k). \quad (69)$$

Here ϕ' is implicit differentiation of the exact finite- L constraint, using $d\Sigma/d\tilde{K} = -\Sigma/\tilde{K}^2 + \Sigma'\phi'$ and $d\mathcal{A}/d\phi = -2h^\top G^3 h$ from (47). No asymptotics enters: (67) is an identity at every finite L . The derivative $F'(4 + c)$ is evaluated in closed form from (13) with $dK_{\text{ell}}/dm = [E(m) - (1 - m)K_{\text{ell}}(m)]/[2m(1 - m)]$, and from the series $F' = -1/(4\pi c) - [\ln(32/c) - 2]/(32\pi)$ below $c = 10^{-(\text{dps}-4)}$ where the elliptic argument would round; against a numerical derivative of (13) the relative difference is 5.1×10^{-50} for $c = 10^{-6}, \dots, 4$, and series against elliptic at $c = 10^{-20}$ it is 3.5×10^{-31} .

Every ingredient of (67) is validated at $L = 20$ in four scaling-window cases: constraint residual 2.7×10^{-51} ; $\mathcal{A}$ from the mode sum against the closed forms of Section 3.4, relative 7.0×10^{-49} ; m_1, m_L from the mode sum against the closed-form columns (18), relative 4.3×10^{-49} ; ϕ' from implicit differentiation against a central difference of the solved ϕ in $\tilde{K}$, relative 2.0×10^{-22} . End to end, c_L from (67) against $-(\tilde{K}^2/L)d^2(\beta F/A)/d\tilde{K}^2$ with $\beta F/A$ rebuilt from mode sums and the quadrature of $\int_0^c F(4 + t)dt$ and differentiated by a five-point stencil at $h = 10^{-9}\tilde{K}$: relative 4.3×10^{-33} . That end-to-end check is what pins the two convention points above. An earlier version of the compact formula divided the last term of (67) by $\tilde{K}$ as well; it reproduced the zero-field case exactly, where $\mathcal{A} = 0$, and missed the field case by 1.9×10^{-3} , precisely $\mathcal{A}\phi'(1 - 1/\tilde{K})/(2L)$. The discrepancy is the reason the check is run with fields present.

7.3 The bulk reference is exactly one half at and below criticality

Let $W_3(\tilde{\phi}) = \int_0^\pi (dq/\pi) F(6 - 2\cos q + \tilde{\phi})$ be the bulk constraint function, so that $W_3(\tilde{\phi}_\infty) = \tilde{K}$ and

$$c_\infty = \frac{1}{2} + \frac{\tilde{K}^2}{2W_3'(\tilde{\phi}_\infty)}. \quad (70)$$

At $\tilde{K} = \tilde{K}_c$ and below, $\tilde{\phi}_\infty = 0$ and $W_3'(0)$ diverges logarithmically, so

$$c_\infty = \frac{1}{2} \quad \text{exactly, at and below } \tilde{K}_c. \quad (71)$$

The one half is the $(L/2) \ln \tilde{K}$ Gaussian normalisation per spin and it is the whole bulk answer at criticality. Just above $\tilde{K}_c$, $\tilde{\phi}_\infty = 16\pi^2(\tilde{K}_c - \tilde{K})^2$ and

$$c_\infty = \frac{1}{2} - 16\pi^2 \tilde{K}_c^2 (\tilde{K}_c - \tilde{K}) + O(t^2 \ln t), \quad t \equiv \tilde{K}_c - \tilde{K}, \quad (72)$$

the $\alpha = -1$ kink: finite, with a slope discontinuity and no divergence. The measured slope $(\frac{1}{2} - c_\infty)/(\tilde{K}_c - \tilde{K})$ is 7.4607751, 9.7488368, 10.051665 and 10.082929 at $\tilde{K} = \tilde{K}_c(1 - \varepsilon)$ for $\varepsilon = 10^{-1}, \dots, 10^{-4}$, against the predicted $16\pi^2 \tilde{K}_c^2 = 10.086414$. Independently, $W_3(10^{-12})$ against $\tilde{K}_c - \sqrt{\tilde{\phi}/(4\pi)}$ has residual -1.2×10^{-14} , the quadrature tolerance of a logarithmically singular integrand, which fixes $\tilde{K}_c = W_3(0)$ as the Watson integral independently of the finite- L sums.

7.4 The scaling function is the ζ -derivative of X

In the scaling window $\phi = X/L^2$ with $X = X(\eta_1, \eta_L, \zeta)$, and $d\zeta/d\tilde{K} = -L/2$ while the η_i contribute only at $O(L^{-2})$ because $d\eta_i/d\tilde{K} = -\eta_i/(2\tilde{K})$ is $O(1)$. Hence $\phi' = -X_\zeta/(2L) + O(L^{-2})$, where $X_\zeta \equiv dX/d\zeta$. The other two terms of (67) are subleading by power counting: $m_r \sim L^{-1/2}$ with $h_1 \sim L^{-3/2}$ and $h_L \sim L^{-1/2}$ give $S \sim L^{-1}$, hence $S/L \sim L^{-2}$; and $\mathcal{A} = \tilde{K}^2 \sum_r m_r^2 \sim L^{-1}$ with $\phi' \sim L^{-1}$ give $\mathcal{A}\phi'/(2L) \sim L^{-3}$. Therefore

$$c_L = \frac{1}{2} - \frac{\tilde{K}_c^2}{4} \frac{1}{L} \frac{dX}{d\zeta} + O(L^{-2}) = \frac{1}{2} - \frac{K_c^2}{L} X'(\eta_1, \eta_L, \zeta) + O(L^{-2}), \quad X' \equiv \frac{dX}{d\zeta}, \quad (73)$$

using $K_c = \tilde{K}_c/2$. This is the scaling form quoted in the literature, with the prime identified as $d/d\zeta$ and the subtracted constant identified as $1/2$ in the convention pinned here. The problem statement [13] transcribes that constant as $\pi/2$. No convention reachable from the definitions above reproduces $\pi/2$, because the bulk value follows from the $(L/2) \ln \tilde{K}$ Gaussian normalisation alone and is exactly $1/2$ at and below $\tilde{K}_c$ (71). The constant is recorded here as a discrepancy rather than adopted. The scaling function X' , which is the substantive content, is convention-independent and unaffected.

Because (32) is one analytic equation with certified positive derivative, X' is closed-form on both sides of $y = \pi/2$:

$$X' = \frac{dX}{d\zeta} = \frac{2}{\frac{d\text{LHS}}{dX}}, \quad (74)$$

with $\frac{d\text{LHS}}{dX}$ given by (48). Proposition 1 then gives the main physical statement of this section: *X' is finite and strictly positive for every real ζ and every real (η_1, η_L) , in both regimes.* Consequently $c_L < 1/2$ at order $1/L$ for every surface field and every temperature in the scaling window. Confinement always lowers the film specific heat below its bulk value and surface fields never reverse that; the uniqueness proof of the equation of state and the sign of the specific-heat anomaly are one and the same statement.

7.5 Closed forms and the four limits

At zero field, (74) reduces to

$$X' = \frac{16\pi\omega}{\tan\omega} \quad (u < 0), \quad X' = \frac{16\pi w}{\tanh w} \quad (u > 0), \quad (75)$$

agreeing with the general two-regime formula to 8.0×10^{-31} relative at $\zeta = 0, \pm 0.05, -0.5, -2, 0.5$, four of the six cases agreeing exactly at fourteen printed digits.

At the film critical point, $\eta = 0$ and $\zeta = 0$, both branches give

$$X' = 16\pi = 50.26548245743669, \quad c_L = \frac{1}{2} - \frac{16\pi K_c^2}{L} + O(L^{-2}) = \frac{1}{2} - \frac{4\pi \tilde{K}_c^2}{L} + O(L^{-2}), \quad (76)$$

so the specific-heat scaling function is continuous at $y = \pi/2$.

At nonzero field it is continuous across the regime boundary as well. At $\zeta = \zeta_0$, where $y = \pi/2$, evaluating (74) at $\zeta_0 \pm 10^{-8}$ gives 4.303030060876927, 4.303030271751036 and 4.303030482625162 for $\eta = (1, 0.5)$; 2.532322697, 2.532322771 and 2.532322845 for $(0, 1.5)$; and 27.3515191, 27.3515236, 27.3515281 for the mixed-sign field $(2, -1)$. The spread is the linear response to the 10^{-8} step, not a jump.

Deep in the ordered film at nonzero field, differentiating (42),

$$X' \longrightarrow \frac{\sqrt{B}}{4} |\zeta|^{-3/2}, \quad (77)$$

with relative deviation 1.15×10^{-2} , 1.86×10^{-3} and 2.55×10^{-4} at $\zeta = -10, -100, -1000$ for $\eta = (1, 0.5)$, and the same $|\zeta|^{-1}$ approach for $(0, 1.5)$ and $(2, -1)$. The specific-heat anomaly therefore *decays* as $|\zeta|^{-3/2}/L$ deep in the ordered film: the field-induced condensate of Section 6 rounds it away, in contrast to the field-free exponential condensation regime where (75) gives $X' \rightarrow 16\pi\omega/\tan\omega \rightarrow 16\pi^2/(2s)$ and grows.

In the bulk disordered regime at zero field the equation of state reduces to $\ln \cosh w/(4\pi) = 2\zeta$, so

$$w = \operatorname{arccosh}(e^{8\pi\zeta}) \quad \text{exactly}, \quad X' = \frac{16\pi w}{\tanh w} = 128\pi^2\zeta + 16\pi \ln 2 + O(e^{-16\pi\zeta}), \quad (78)$$

the exact inversion agreeing with the general two-regime formula to 5.6×10^{-39} at $\zeta = 1$ and to 1.8×10^{-36} beyond, and the asymptote to 9.4×10^{-20} at $\zeta = 1$ and 1.8×10^{-36} beyond. The additive constant is $+16\pi \ln 2 = +34.841377444858$, arising from $w = 8\pi\zeta + \ln 2$ at large ζ . It is worth flagging, because it is exactly the kind of constant an asymptotic matching argument gets wrong: any constant offset inside $(\zeta - \text{const})$ is invisible in a ratio that tends to 1, since both sides are $128\pi^2\zeta$ to leading order, so ratios approaching unity are not evidence for the offset. A plausible-looking but incorrect value, $-128\pi^2 A = -5.2364$ with A from (23), was produced that way and was exposed only because the bulk-referenced extraction of the next subsection failed to converge. The offset that is measurable is recorded below.

Substituting (78) into (73) reproduces the bulk kink (72) at fixed $\tilde{K} < \tilde{K}_c$, because the film-referenced variable then runs as $\zeta_L = (\tilde{K}_c - \tilde{K})L/2 + A/2$, that is $\tilde{K}_c - \tilde{K} = (2\zeta_L - A)/L$. Measured directly at $\eta = 0$, $\tilde{K} = \tilde{K}_c(1 - 10^{-3})$: $\zeta_L - (\tilde{K}_c - \tilde{K})L/2$ is 0.002066142735, 0.002066308951 and 0.002066350745 at $L = 100, 200, 400$ against $A/2 = 0.002066364722$, and the scaling prediction $\frac{1}{2} - K_c^2 X'(\zeta_L)/L$ approaches the exact finite- L c_L with relative error 3.4×10^{-4} , 1.2×10^{-4} and 5.6×10^{-5} while $c_L \rightarrow c_\infty = 0.49745963252759$. With the wrong offset A instead of $A/2$ the same relative errors are 9.9×10^{-4} , 4.7×10^{-4} and 2.5×10^{-4} , a clean $1/L$ tail: the offset is measurable and it is $A/2$. The film formula therefore has the correct bulk limit including the kink, which is an independent check on the normalisation of c_L .

Table 3: Specific heat: the exact finite- L identity (67) against the scaling prediction of (73), with $K_c^2 = 0.01596824083605402$. Columns are the five cases; entries are printed to eight significant figures. The three “dev ratio” rows are $\text{dev}(L)/\text{dev}(L/2)$ with $\text{dev} = L(c_L - \frac{1}{2}) + K_c^2 X'$ of (80), equal to $1/2$ for a clean $O(1/L)$ approach: every case converges at the predicted rate, on both sides of $y = \pi/2$, at zero field and at mixed-sign field. The $L = 160$ entries for the last four cases were produced by the independent re-implementation of Appendix A; all other entries are common to both.

(η_1, η_L)	(1, 0.5)	(1, 0.5)	(0, 0)	(0, 1.5)	(2, -1)
ζ	0.05	-0.5	0	-0.2	0.3
X'	84.901240	1.8286102	50.265482	13.103026	413.06214
$-K_c^2 X'$	-1.3557234	-0.029199689	-0.80265133	-0.20923228	-6.5958757
$L(c_L - \frac{1}{2})$ at $L = 20$	-1.1840726	-0.044958027	-0.75471705	-0.19887706	-4.4167295
$L(c_L - \frac{1}{2})$ at $L = 40$	-1.2629444	-0.037056494	-0.77792095	-0.20330946	-5.2928424
$L(c_L - \frac{1}{2})$ at $L = 80$	-1.3073837	-0.033121619	-0.79008592	-0.20606867	-5.8735116
$L(c_L - \frac{1}{2})$ at $L = 160$	-1.3310358	-0.031158932	-0.79631732	-0.20759776	-6.2139072
dev ratio 40/20	0.541	0.499	0.516	0.572	0.598
dev ratio 80/40	0.521	0.499	0.508	0.534	0.554
dev ratio 160/80	0.511	0.500	0.504	0.517	0.529

7.6 Validation against exact finite films

Table 3 compares the exact finite- L identity (67), with no asymptotics anywhere, against the prediction $-K_c^2 X'$ of (73), in five cases spanning both regimes, zero field, and a mixed-sign field. The finite- L entries approach the prediction at exactly the rate the scaling form requires: the successive-difference ratios of the deviation, the last three rows of the table, are $1/2$ to within a few percent in every case, which is the signature of convergence at rate $1/L$. The approach is slowest where X is largest, $\eta = (2, -1)$ with $y = 8.76$, as expected since the neglected $O(L^{-2})$ terms grow with X ; the ratio $L(c_L - \frac{1}{2})/(-K_c^2 X')$ there is still 0.670 at $L = 20$ but 0.890 at $L = 80$ and 0.942 at $L = 160$. Figure 3 draws both the scaling function and this convergence. The $L = 160$ entries for the last four cases were produced by the independent re-implementation described in Appendix A; the corresponding deviation ratios, in the last row of the table, continue to hold at $1/2$.

7.7 The next-to-leading coefficient, and what it is not

Writing the scaling form with one more term,

$$c_L = \frac{1}{2} - \frac{K_c^2}{L} X'(\eta_1, \eta_L, \zeta) + \frac{C_2(\eta_1, \eta_L, \zeta)}{L^2} + O(L^{-3}), \quad (79)$$

$$q(L) = L \left[L \left(c_L - \frac{1}{2} \right) + K_c^2 X' \right] = C_2 + \frac{A_2}{L} + \frac{A_3}{L^2} + \dots, \quad (80)$$

with c_L from the exact identity (67) and X' from the closed form, C_2 can be extracted two independent ways that must agree: Richardson, $C_2 \approx 2q(2L) - q(L)$, and an exact three-point fit of $q = C_2 + A_2/L + A_3/L^2$ on the top three rungs of a ladder $L = 40, 80, 160, 320, 640$. Table 4 gives the result. The successive-difference ratios of q tend to $1/2$ in every case, which is the statement that the leading correction to c_L really is $O(1/L)$ and the next one $O(1/L^2)$: there is no $\ln(L)/L^2$ term in the specific heat either. The approach ratios of $L(c_L - \frac{1}{2})$ to $-K_c^2 X'$ reach 0.995373, 1.01676, 0.998015, 0.997999 and 0.984869 at $L = 640$, and q is stable under the working precision, 3.992749577021 at $L = 320$ identically at forty, fifty and sixty digits.

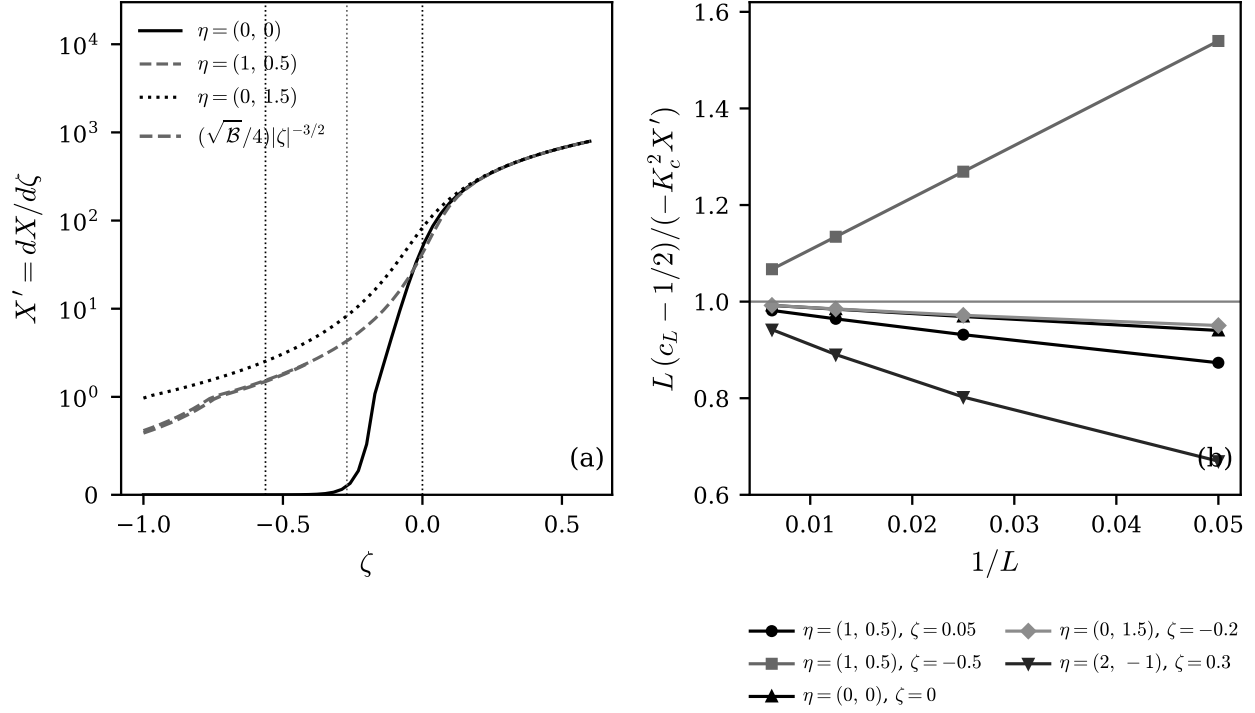


Figure 3: Specific heat. (a) The scaling function $X' = dX/d\zeta$ of (74) against ζ for $\eta = (0, 0)$, $(1, 0.5)$ and $(0, 1.5)$, on a symlog scale, with a dotted vertical line at each field's exact regime boundary ζ_0 of (36) and, dashed in grey over $-1 \leq \zeta \leq -0.4$, the deep-ordered asymptote $(\sqrt{B}/4)|\zeta|^{-3/2}$ of (77) at $\eta = (1, 0.5)$. Every curve is continuous across its own ζ_0 , which is the continuity of the specific-heat scaling function at the film critical point; the zero-field curve instead rises as $\zeta \rightarrow 0^-$, the field-free condensation regime. (b) The ratio $L(c_L - 1/2)/(-K_c^2 X')$ against $1/L$ for the five cases of Table 3, with c_L from the exact finite- L identity (67); all five tend to 1 at rate $1/L$, the mixed-sign case $\eta = (2, -1)$ most slowly because its $X = 76.7$ is the largest.

Three qualifications belong with Table 4, and none of them is cosmetic.

C_2 is convention-bound; X' is not. The variable ζ is defined through the *exact* film critical coupling, $\zeta = (\tilde{K}_{c,L} - \tilde{K})L/2$. Referencing it to the bulk $\tilde{K}_c$ instead, $\zeta_b = (\tilde{K}_c - \tilde{K})L/2$, gives $\zeta = \zeta_b + A/2 + O(L^{-2})$ at every L , so the two conventions differ by a pure shift of the argument, $C_{2,b}(\zeta_b) = C_2(\zeta_b + A/2)$. Verified: $C_{2,b}(0.05) = 4.23755156819$ against $C_2(0.05 + A/2) = 4.23755153184$, relative difference 8.6×10^{-9} , and $C_{2,b}(-0.5) = -0.313931800942$ against $C_2(-0.5 + A/2) = -0.313931800893$, relative difference 1.5×10^{-10} ; while the same ladders give $C_{2,b}(0.05) = 4.2376$ against $C_2(0.05) = 4.0364$ and $C_{2,b}(-0.5) = -0.31393$ against $C_2(-0.5) = -0.31319$. A number quoted for C_2 without saying which critical coupling defines ζ is not usable.

The shift is the *only* difference between the conventions, for two structural reasons. The film critical shift has no $1/L^2$ term: $L(\tilde{K}_{c,L} - \tilde{K}_c)$ is 0.0041299944793, 0.00413203742754, 0.0041325537613, 0.00413268579238, 0.00413271851436, 0.00413272670971 and 0.00413272876041 at $L = 40, \dots, 2560$, converging to the analytic A of (23), so $\tilde{K}_{c,L} = \tilde{K}_c + A/L + O(L^{-3})$. And the naive bulk-referenced extraction, in which the leading term is evaluated at $X'(\zeta_b)$ rather than $X'(\zeta_b + A/2)$, does not converge to a different C_2 at all: q grows linearly, with $q(L)/L \rightarrow K_c^2[X'(\zeta_b) - X'(\zeta_b + A/2)]$, measured as -0.0311 at $L = 640$ against the predicted -0.037716245 and still approaching it. A wrong offset in ζ therefore announces itself as a

Table 4: The $O(L^{-2})$ coefficient C_2 of (79), extracted from the exact finite- L identity on the ladder $L = 40, \dots, 640$. “Ratio” is the successive-difference ratio of q over the top three rungs, tending to $1/2$ in every case. The mixed-sign point is the least settled and should be quoted as $C_2 \approx 64.8$, not 64.85.

(η_1, η_L)	ζ	X	Richardson $L=320$	3-point fit	ratio
(1, 0.5)	0.05	9.238781384474359	4.036141534	4.03636542172	0.508
(1, 0.5)	-0.5	1.829032591935519	-0.3131933909	-0.31319273338	0.507
(0, 0)	0	2.46740110027234	1.021777701	1.0217863712	0.503
(0, 1.5)	-0.2	4.531512998736936	0.2701852781	0.27020069102	0.505
(2, -1)	0.3	76.71454726880523	64.82084327	64.8483003509	0.523

non-converging q , never as a plausible-looking C_2 . That diagnostic is what exposed the incorrect asymptotic constant discussed above.

Finally, C_2 is a numerical extraction from the exact finite- L identity, not an analytic derivation, and it inherits the film-referenced ζ . Its growth with X is real and is the quantitative form of the narrowing scaling window already visible in Table 3: C_2/X' is 0.0475, -0.171, 0.0203, 0.0206 and 0.157 across the five cases, while C_2/X is 0.437, -0.171, 0.414, 0.0596 and 0.845, so no simple universal amplitude accounts for the spread. No closed form for C_2 is claimed.

8 Local fluctuation

8.1 An exact split

The connected local fluctuation is the diagonal element of the inverse quadratic form,

$$\langle s_r^2 \rangle_c = \langle s_r^2 \rangle - \langle s_r \rangle^2 = \frac{1}{\widetilde{K}} \sum_{k=1}^L u_L(r, k)^2 F(4 + \phi + d_k), \quad \sum_{k=1}^L u_L(r, k)^2 = 1, \quad (81)$$

and the spherical constraint (12) is exactly the statement that $\langle s_r^2 \rangle_c + m(r)^2$ averages to 1 over r . Both statements are checked from mode sums, the completeness relation to 8.0×10^{-50} at $L = 8, 20, 60$ and the constraint to 3.9×10^{-33} at $L = 20, 40, 60$ in three scaling-window cases including $\zeta = -0.5$. The F -weighting is what prevents reuse of the closed form (16): G carries weights $1/(\phi + d_k)$ while (81) carries $F(4 + \phi + d_k)$, so the product identities of Section 3.3 do not apply and the mode sum has to be analysed directly.

Write $N = 2L + 1$ and $n_r = 2r - 1$. Using $\cos^2 t = [1 + \cos 2t]/2$ with $2t = n_r \phi_L(k)$, and the constraint in the form $\widetilde{K}^{-1} L^{-1} \sum_k F(4 + \phi + d_k) = 1 - L^{-1} \sum_r m(r)^2$, the sum splits exactly,

$$\langle s_r^2 \rangle_c = \frac{2L}{N} \left[1 - \frac{1}{L} \sum_{r'} m(r')^2 \right] + S(r, L), \quad S(r, L) = \frac{2}{N\widetilde{K}} \sum_{k=1}^L \cos[n_r \phi_L(k)] F(4 + \phi + d_k). \quad (82)$$

This is algebra, not an approximation: against the direct mode sum the worst residual is 3.1×10^{-33} at $r = 1, 2, L/2, L - 1, L$ for $L = 20, 40, 60, 80$ in three field configurations and four values of ζ , and that residual is the tolerance of the closed form used for the field term rather than of the identity. The content of (82) is that everything the constraint already fixes is r -independent, so the entire r -dependence sits in the oscillatory sum S . That is what makes the scaling limit tractable.

8.2 Image boundary layers at both faces

Define, for real $a \geq 0$, the bulk simple-cubic lattice Green function at layer separation a , continued away from integer a ,

$$\mathcal{G}(a) = \frac{1}{\pi} \int_0^\pi \cos(aq) F(6 - 2 \cos q) dq = \int \frac{d^3 q}{(2\pi)^3} \frac{e^{iaq_3}}{6 - 2 \sum_i \cos q_i}. \quad (83)$$

The integrand carries the logarithmic singularity of (15) at $q = 0$, which is subtracted analytically using $c = 2 - 2 \cos q = 4 \sin^2(q/2)$, so that $\ln(32/c) = \ln 8 - 2 \ln \sin(q/2)$. The normalisation is then fixed by an independently derived constant rather than by this quadrature: $\mathcal{G}(0) = \tilde{K}_c = 0.252731009858663003026002026614$, the Watson integral, with residual 2.3×10^{-33} .

At the free face, for fixed r and $L \rightarrow \infty$ inside the scaling window, $n_r \phi_L(k) \rightarrow aq$ with $a = 2r - 1$ and $q = \pi(2k - 1)/N$, the sum $(2/N) \sum_k$ becomes $\pi^{-1} \int_0^\pi dq$, $\phi \rightarrow 0$, $\tilde{K} \rightarrow \tilde{K}_c$, and the bracket in (82) tends to 1 because the field term is $O(1/L)$. At the ordinary face, for $d = L + 1 - r$ fixed, $n_r = N - 2d$, so $\cos[n_r \phi_L(k)] = \cos[\pi(2k - 1) - 2d\phi_L(k)] = -\cos[2d\phi_L(k)]$ *exactly at every finite L* , and the image term changes sign. Hence

$$\langle s_r^2 \rangle_c \longrightarrow \begin{cases} 1 + \frac{\mathcal{G}(2r - 1)}{\tilde{K}_c}, & \text{free (Neumann) face, } r \text{ fixed,} \\ 1 - \frac{\mathcal{G}(2d)}{\tilde{K}_c}, & \text{ordinary (Dirichlet) face, } d \text{ fixed.} \end{cases} \quad (84)$$

The sign flip is checked from the identity rather than assumed, to 8.7×10^{-51} at $d = 1$ and 2.8×10^{-51} at $d = 2$ for $L = 120$. The physical reading is that the free face enhances local fluctuations and the ordinary face suppresses them, and that enhancement and suppression are the two signs of one method-of-images structure.

Why the image distances are $2r - 1$ and $2d$ is worth spelling out, because a continuum argument gets one of them wrong. The ND eigenfunctions are $\cos[(r - \frac{1}{2})q]$, so the free surface sits at the plane $r = \frac{1}{2}$ and the image of layer r is at $1 - r$, at distance $2r - 1$; a Neumann plane reflects with a plus sign. The Dirichlet condition is imposed by zeroing the spin on the ghost plane $r = L + 1$, so the image of layer r is at $2(L + 1) - r$, at distance $2d$; a Dirichlet plane reflects with a minus sign. The distances therefore come out as $2r - 1$ and $2d$ rather than $2r$ and $2d - 1$: that is the half-lattice offset of Section 2.2, and it is derived here from the exact finite- L mode sum rather than assumed. Table 5 compares (84) with Richardson extrapolation of the exact finite- L sums from $L = 60, 120, 240$.

Neither limit in (84) contains ζ or η . At fixed r , $\phi = X/L^2 \rightarrow 0$ for every finite scaling variable, so the image term sees only the critical propagator. Measured at $r = 1$, $L = 240$, across five configurations $(\eta_1, \eta_L, \zeta) = (0, 0, 0.05), (0, 0, -0.5), (1, 0.5, 0.05), (1, 0.5, -0.5), (0, 1.5, -0.2)$, the values are 1.33756, 1.34928, 1.33584, 1.31784 and 1.32780 against the L -independent target 1.34054: a spread of ± 0.011 that is entirely the $O(1/L)$ correction. The leading boundary layer is therefore universal across the whole scaling window, including the ordered side, and *surface fields do not change the leading enhancement or suppression of local fluctuations at either face; they enter one order later in $1/L$* . The first correction, by contrast, is not universal at all: it differs by a factor of two hundred between $\eta = 0$ and $\eta = (1, 0.5)$ at $\zeta = -0.5$ in the interior.

For r and $L + 1 - r$ both of order L the sum S contains both images together with the repeated reflections between the two planes. That ladder is not resummed here, and it is the reason the interior coefficient of the next subsection is measured rather than derived.

Table 5: Boundary-layer limits of the local fluctuation at the film critical point ($h = 0$, $\tilde{K} = \tilde{K}_{c,L}$, $\phi = \lambda_1$), against Richardson extrapolation $2v(240) - v(120)$ of the exact finite- L mode sums from $L = 60, 120, 240$. The free face converges monotonically from below with differences halving; the ordinary face converges an order of magnitude better because no soft $k = 1$ weight sits at that end. The residual $\sim 7 \times 10^{-6}$ at the free face is the next order in $1/L$, uniform in r .

face	index	closed form (84)	Richardson	difference
free (r fixed)	$r = 1$	1.34053732955	1.340529575	7.8×10^{-6}
free (r fixed)	$r = 2$	1.10898991059	1.108982403	7.5×10^{-6}
free (r fixed)	$r = 3$	1.06370834882	1.063701328	7.0×10^{-6}
free (r fixed)	$r = 5$	1.03509781439	1.035092498	5.3×10^{-6}
ordinary (d fixed)	$d = 1$	0.830296588589	0.8302964531	1.4×10^{-7}
ordinary (d fixed)	$d = 2$	0.919721496437	0.9197209951	5.0×10^{-7}
ordinary (d fixed)	$d = 3$	0.947118883246	0.9471177736	1.1×10^{-6}

8.3 The interior, and a refuted logarithmic correction

For $\rho = r/L$ fixed in $(0, 1)$ both image distances grow with L and $\mathcal{G}(a) \sim 1/(4\pi a)$, so $S \rightarrow 0$ and

$$\langle s_r^2 \rangle_c = 1 + \frac{V(\rho; \zeta, \eta)}{L} + o(L^{-1}). \quad (85)$$

The natural conjecture is that the soft $k = 1$ mode produces a logarithmic correction, $\langle s_r^2 \rangle_c = 1 + c_1(\rho) \ln(L)/L + V/L + o(L^{-1})$ with

$$c_1(\rho) = \frac{\cos^2(\rho\pi/2)}{2\pi\tilde{K}_c}, \quad (86)$$

which is what the $k = 1$ term of the mode sum alone gives, using $F(4+\phi) = [2 \ln L + \ln(32/X)]/(4\pi) + O(\phi \ln \phi)$ and $(4/N) \cos^2[(r - \frac{1}{2})\phi_L(1)] \rightarrow 2 \cos^2(\rho\pi/2)/L$.

It is false. The sequence $L[\langle s_r^2 \rangle_c - 1]$ does not grow like $\ln L$; it converges, with successive differences halving exactly over $L = 40, 80, 160, 320$ at twelve points, which is the signature of a pure $1/L$ correction and the exclusion of any $\ln(L)/L$ term. The apparent c_1 obtained by differencing against $\ln L$ is small, of both signs, and shrinking with L : -0.0164 , -0.0064 and -0.0176 at $\rho = 0.25, 0.5, 0.75$ for $\eta = (0, 0)$, $\zeta = 0.05$, against the conjectured $+0.538$, $+0.315$ and $+0.092$. Those numbers measure the residual curvature of a convergent sequence, not a logarithm; the twelve measurements of V are tabulated in Appendix A, Table 8.

The reason the single-mode argument fails matters beyond this observable, because the argument looks persuasive and because for the magnetization profile it *does* work. The $k = 1$ term of (81) really does carry $\cos^2(\rho\pi/2) \ln L / (2\pi\tilde{K}_c L)$. But the continuum integral over the same region $\kappa \lesssim 1/L$ carries the identical $\ln L$, because $\int_0^\pi \ln(32/q^2) dq = \pi \ln(32e^2/\pi^2)$ is finite: the singular part of the mode sum and the singular part of its own continuum limit cancel. A single-mode estimate of a $\ln(L)/L$ coefficient is therefore not a scaling argument for this observable, and any future claim of a logarithmic correction to $\langle s_r^2 \rangle_c$ has to survive that cancellation. This is a negative result and is reported as one: no closed form for $V(\rho; \zeta, \eta)$ is offered. What is known about it is that it must absorb four identified pieces — the $-(2L/N) \rightarrow -\frac{1}{2}$ prefactor; the expansion of $1/\tilde{K}$ against $1/\tilde{K}_c$, since $\tilde{K} = \tilde{K}_c + (A - \zeta)/L$ with A from (23), contributing $-(A - \zeta)/\tilde{K}_c$; the field term $-L^{-1} \sum_r m^2 \rightarrow -\eta^\top C(X) \eta / \tilde{K}_c$; and the two image terms at separations ρL and $(1 - \rho)L$, of which

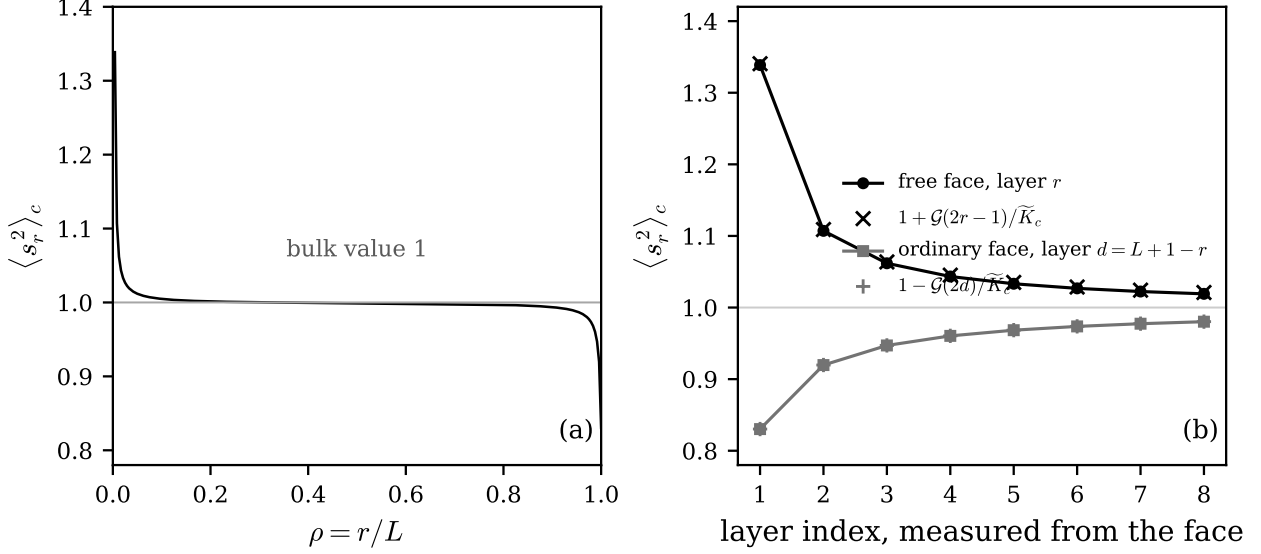


Figure 4: Local fluctuation $\langle s_r^2 \rangle_c$ at zero surface field and at the film critical point, from the exact finite- L mode sum at $L = 240$. (a) The whole film against $\rho = r/L$: two boundary layers of opposite sign and a flat interior at the bulk value 1. (b) The two faces resolved, against the layer index measured from that face, with the closed-form image limits (84) drawn as crosses and plus signs: enhancement $1 + \mathcal{G}(2r - 1)/\tilde{K}_c$ at the free face and suppression $1 - \mathcal{G}(2d)/\tilde{K}_c$ at the ordinary one. The finite- L values approach their limits from below at the free face, at 1.3386311 against 1.3405373 for $r = 1$, and from below at the ordinary one, at 0.8302400 against 0.8302966 for $d = 1$, the latter an order of magnitude closer.

the free one behaves as $\mathcal{G}(2\rho L)/\tilde{K}_c \sim 1/(8\pi\tilde{K}_c\rho L)$. Assembling them is the natural next step and would also settle whether the slow drift reported for X_L and for the profile deviation in Section 6 appears in $\langle s_r^2 \rangle_c$: at fixed ρ away from the faces it does not.

9 Discussion

9.1 What the results say together

The six results of this paper are not independent, and reading them together is more informative than reading them separately. The exact finite- L equation of state (24) is a sum of a logarithm of a determinant, a regular mode sum and a sum of squares. In the scaling limit the first becomes $\ln D/(8\pi)$, the second vanishes, and the third becomes the quadratic form $Q = \eta^\top C \eta$. Everything else follows from the structure of those three pieces. Because the determinant is a single entire function of u , there is one equation of state rather than two, and the regime boundary becomes a number, ζ_0 of (36), rather than a change of formula. Because the field enters as a sum of squares, the field term has a definite sign, and its kernel matrix C is positive definite — which is what makes the monotonicity proof of Proposition 1 a statement about two determinant numerators rather than about a competition between two terms of unknown sign. Because the derivative is strictly positive, the root is unique, the specific-heat scaling function $X' = 2/(d\text{LHS}/dX)$ of (74) is strictly positive, and the film specific heat lies below its bulk value. And because the same dominant balance that makes Q diverge as $\mathcal{B}/X^2$ also drives the profile kernels onto $\cos(\rho\pi/2)$, the power law (42) for the spherical field and the condensate (62) for the profile are one phenomenon seen in two observables.

The one place where that unity fails is instructive. The local fluctuation (81) carries weights $F(4 + \phi + d_k)$ rather than $1/(\phi + d_k)$, so it does not reduce to the determinant or to the boundary-column sums, and it is the only observable here for which the leading behaviour had to be extracted from the mode sum rather than read off a closed form. It is also the only one where a plausible soft-mode argument gives the wrong answer, and the reason — the cancellation between the singular part of the mode sum and the singular part of its own continuum limit — has nothing to do with the surface fields.

9.2 Relation to the published literature

On the disordered side the results reduce to the published ones, and the reduction is checked rather than asserted: the equation of state (9) of Ref. [3] is reproduced on exact finite- L solutions with a $1/L$ residual in the pinned convention, and its profile (11) is the $u > 0$ branch of (60). At zero field the ordered side reproduces the trigonometric ND constraint, the absence of a logarithmic critical shift, the condensation (30) and the shift amplitude (23) of Refs. [9, 4], and gives the prefactor $C = \pi e^{4\pi A}$ that the earlier treatments did not need. The absence of a $1/L^2$ term in $\tilde{K}_{c,L} - \tilde{K}_c$, measured here out to $L = 2560$, is what makes the film-referenced and bulk-referenced scaling variables differ by a pure shift, and therefore what makes C_2 convention-bound in the simple way described above.

Two points of contact with Ref. [8] deserve mention. That work finds the Gaussian and the mean-field ND film to disagree about the critical Casimir force; the present results do not settle that disagreement, because no free-energy scaling function is derived here, but they do supply the ingredient that a Casimir amplitude would be built from — the exact root $X(\eta_1, \eta_L, \zeta)$ on both sides of the boundary, whose ζ -derivative is the specific heat and whose ζ -integral is the free energy. And the boundary-layer result (84), in which the leading surface enhancement and suppression are field-independent image terms of the bulk propagator, is a mean-field statement about the same geometry and the same two boundary conditions, with the half-lattice image distances that a continuum treatment would miss.

9.3 Boundaries of scope

Five limits on what has been shown are part of the result and are stated here rather than left to the reader.

The scaling window. Every statement is at fixed (η_1, η_L) , so $h_1 \sim L^{-3/2}$ and $h_L \sim L^{-1/2}$. At fixed *physical* surface field η grows with L , the window is left, and X' — hence the specific heat — has no limit there. The power law (45) is a statement inside the window only. Outside it the exact finite- L constraint suggests a different answer, again set by the lowest transverse mode alone,

$$\phi \sim \frac{c_1}{\tilde{K}\sqrt{L}}, \quad c_1 = h_1 u_L(1, 1) + h_L u_L(L, 1), \quad (87)$$

but the crossover, its dependence on which face carries the field, and the thickness at which it replaces the scaling behaviour are not derived here, and (87) is recorded as a suggestion of the finite- L analysis rather than as a result. This is the regime in which a simulation or an experiment would actually sit.

The near-critical window. At $y = \pi/2$ the leading scaling factor $\sqrt{y^2 - \pi^2/4}$ vanishes while the exact finite- L transverse eigenvalue is $2[1 - \cos(\pi/(2L + 1))]$ rather than $(\pi/2L)^2$, so the exact finite- L boundary $\phi = \lambda_1$ differs from $y = \pi/2$ at order L^{-3} in ϕ , as Section 3.2 records. The resulting $O(L^{-2})$ corrections become comparable to the leading term in a neighbourhood of the

film critical point whose width has not been quantified, and the same issue controls the accuracy of any tabulation of X near criticality and the location of $\zeta = \zeta_0$ at finite L . Whether a uniform approximation in that window can be derived in closed form is open.

The free energy and the Casimir amplitude. $\beta F/A$ is available exactly at finite L from (64) — -12.1987139671396101 at $(\eta_1, \eta_L, \zeta) = (1, 0.5, 0.05)$, $L = 20$ — but its $1/L$ Casimir part has not been separated from the bulk and surface terms, so the continued ND Casimir amplitude, $0.01922\dots$ at zero field [4], is not reproduced here. Since $X' = dX/d\zeta$ and the specific heat is its ζ -derivative structure, the amplitude is one integration away; the missing piece is the Euler–Maclaurin constant of $\sum_k I(c_k)$, not a new idea.

The interior fluctuation coefficient. $V(\rho; \zeta, \eta)$ of (85) is measured at twelve points to Richardson accuracy and depends strongly on (ζ, η) , but it has not been reduced analytically, and neither has the repeated-reflection ladder for r and $L + 1 - r$ both of order L .

The slow finite- L drift. On the hyperbolic side, $L(X_L - X)$ does not converge cleanly: at $(\eta_1, \eta_L, \zeta) = (1, 0.5, 0.05)$ it runs -12.064143 , -12.701563 , -13.049052 , -13.230838 and -13.323861 for $L = 20, 40, 80, 160, 320$, and the profile deviation of Table 2 drifts the same way, $L \times \text{dev}$ running $+2.14 \rightarrow +2.24$ at $\zeta = 0.05$ and $+3.60 \rightarrow +3.66$ at $\zeta = -0.5$. The four ordered cases, by contrast, converge to a finite limit. The drift is a property of the hyperbolic side and is reported rather than fitted away; no closed form for it is offered. The shift amplitude A was shown in Section 3.3 to have no $\ln L$ component, so this is not the same effect. Three explanations are consistent with the data and none is established: a genuine $\ln(L)/L$ correction to the scaling functions; an artefact of holding ζ rather than a bulk-referenced scaling variable fixed; or the $O(L^{-3})$ difference between $\phi = \lambda_1$ and $y = \pi/2$ described above. Resolving it would sharpen every tabulation of X . The fluctuation result of Section 8 is a warning about the first candidate: a logarithm conjectured from a single soft mode can be cancelled by the continuum limit of the same mode sum. Figure 5 displays both behaviours on one set of axes.

10 Conclusion

The three-dimensional mean spherical film with Neumann–Dirichlet boundary conditions and surface fields has an exact finite-size scaling theory that does not stop at the film critical point. The scaled equation of state is a single analytic function of $X = y^2$ across the whole physical range, its derivative is strictly positive for every real surface field on both sides of the boundary, and it therefore has exactly one real root for every real ζ ; branch selection, which at zero field required a free-energy comparison, is here a consequence of a sign. The regime boundary is the exact field-dependent value $\zeta_0 = -(\eta_1^2/3 + \eta_1\eta_L + \eta_L^2)/4$. Below it the ordered film carries a single field-induced condensate in the lowest transverse mode, its spherical field follows a power law in thickness rather than an exponential, and whether an exponential window survives at all is decided by the two closed-form critical field strengths (43) and (44). The specific-heat scaling function is $dX/d\zeta$, is strictly positive, and makes the film specific heat lie below its bulk value one half at order $1/L$ for every field and every temperature in the window. The leading boundary layers of the local fluctuation are field-independent image terms of the bulk lattice Green function at half-lattice image distances, and the conjectured logarithmic correction to the interior is refuted by an exact cancellation.

What remains open is as much a part of the result as what is settled: no free-energy scaling function and therefore no continued Casimir amplitude; no closed form for the interior fluctuation coefficient V or for the measured next-to-leading specific-heat coefficient C_2 ; no resummation of the repeated-reflection ladder; no uniform approximation in the near-critical window where the exact finite- L eigenvalue competes with $\sqrt{y^2 - \pi^2/4}$; an unexplained slow drift in the finite- L approach

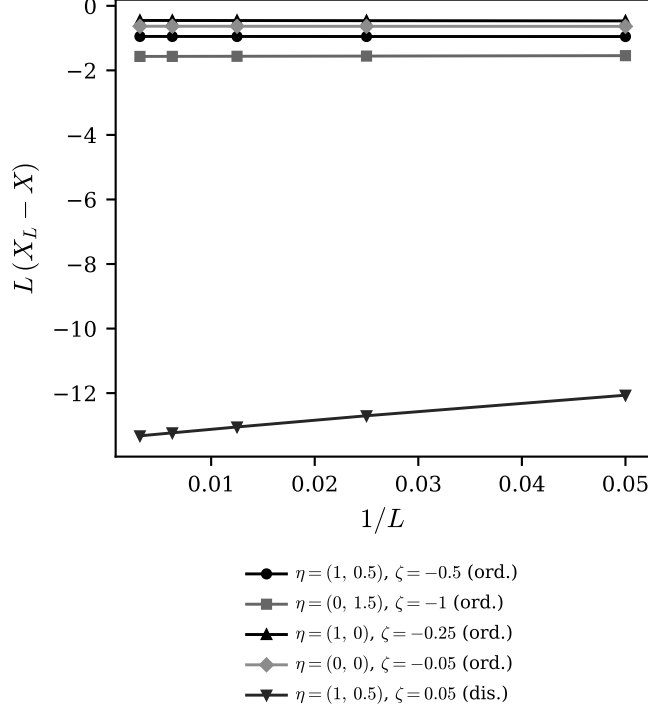


Figure 5: Finite-thickness approach to the scaling root. $L(X_L - X)$ against $1/L$, where $X_L = \phi_L L^2$ solves the exact finite- L constraint (12) at fixed (η_1, η_L, ζ) and X is the certified scaling-limit root of (32). The four ordered cases — $\zeta = -0.5, -1.0, -0.25$ and -0.05 — converge to finite limits, confirming the $1/L$ correction structure and independently confirming the scaling roots. The disordered case $\zeta = +0.05$ drifts slowly, from -12.06 to -13.32 over $L = 20, \dots, 320$; the drift is reported rather than fitted away.

on the hyperbolic side; and nothing at fixed physical surface field, where the scaling window is left and the specific heat has no limit. Two published numbers were found to be transcribed with the wrong sign or the wrong constant — the specific-heat subtraction $\pi/2$ against $1/2$, and the growing against the decaying field-free ordered exponential — and both are corrected here on the strength of exact finite- L mode sums rather than on authority.

11 Data and code availability; author contributions

Data and code availability

Every number in this paper is reproducible from the exact finite- L mode sums of (12), (81) and (67) together with the closed forms collected in Appendix B; no experimental or simulation data are used. The derivations, the high-precision arithmetic behind each quoted value, the captured numerical output with printed residuals, and the re-implementation from which the five figures were drawn are archived in Ref. [10]. The arithmetic conventions and the validation summary of every table are given in Appendix A. The source archive is read-only with respect to this paper: nothing in it was modified, and all independent recomputation reported here was performed separately and is described in Appendix A.

Author contributions

scientific-author-72 (corresponding author) formulated the problem, derived the exact finite- L representation of Section 3 including the product identities (20) and (21), the shift amplitude (23) and the exact equation of state (24); pinned the coupling convention and the meaning of the prime in the published kernels through (28)–(2); derived the unified equation of state (32), the boundary value ζ_0 , the field-modified condensation (42) with its two critical field strengths (43), (44), the profile scaling function (60) and its ordered limit (62); established the exact split (82) of the local fluctuation, the image boundary-layer limits (84) and the refutation of the logarithmic correction; derived the exact finite- L specific heat (67), the scaling function (73) and its four limits; recorded the two discrepancies with the problem statement; and wrote the paper.

scientific-author-31 established the monotonicity of the finite- L constraint (46) through the resolvent identity (47) and verified it numerically, and contributed the finite- L cross-check of the ordered root reported in Appendix A, Table 9, including the observation that the drift of $L(X_L - X)$ is confined to the hyperbolic side.

scientific-author-33 performed the first extraction of the next-to-leading specific-heat coefficient C_2 of (79) on a ladder reaching $L = 640$, and attached to it the convention caveat — that C_2 is meaningless without a statement of which critical coupling defines ζ — that Section 7 verifies quantitatively.

scientific-author-44 supplied the analytic certificate of Proposition 1 on the ordered side: the determinant numerator (50), the Taylor floor (51)–(52) with its exact rational bound, and the monotone-factor subinterval estimate (53) with its 4096-interval evaluation.

scientific-author-52 supplied the analytic certificate on the disordered side: the continuation (54), the rearrangement (55), the entire series (56) and the induction proof of the positivity of its coefficients (57); and identified why a sampled minimum of $\frac{dLHS}{dX}$ measures the edge of the grid rather than the model, which is the content of Section 5.5.

scientific-author-57 derived the prefactor $C = \pi e^{4\pi A}$ of the field-free ordered asymptotics (30) through the Wallis product of the transverse spectrum, measured its convergence, and identified the sign error in the exponent as transcribed in the problem statement.

All authors reviewed the derivations and the numerical evidence and approved the final text.

A Numerical methods and validation tables

A.1 Arithmetic

All derivational numbers in the body were produced in arbitrary-precision arithmetic at fifty decimal digits, with residuals printed rather than suppressed. No double-precision step is load-bearing for any quoted value: where a double-precision path was used — the independent constraint check of Section 3.1, the dense diagonalisation of the ND operator, the periodic trapezoid rule over the in-plane Brillouin zone — it was used to *corroborate* a high-precision result and the agreement is quoted at its own precision, 10^{-16} . Symbolic algebra was used only to differentiate the closed forms for C_i and to verify the continuations (50) and (54); every such symbolic derivative was cross-checked against a numerical derivative of the cancellation-free representation, to 4.1×10^{-47} .

Two cancellations dominate the arithmetic and both are handled structurally rather than by precision. In $\sinh 2w - 2w$ and $\sin 2\omega - 2\omega$ the entire series of Appendix B is used for $|u| < 1/4$. In $\cos \omega = \sin s$ with $\omega \rightarrow \pi/2$ the variable s of (33) is used for $X < \pi^2/8$. The switch points were verified to 5.9×10^{-61} relative, and the resulting $C_i(X)$ agree with the naive closed forms wherever the latter are safe.

Roots of the scaled equation of state are certified rather than merely located: each root in Table 1 is enclosed in an interval in $t = \ln X$ of width 5.7×10^{-41} to 4.6×10^{-40} whose endpoints bracket a sign change, and the midpoint is back-substituted into (32) with residual below 5×10^{-41} . An independent bisection in double precision reproduces all ten to 7.1×10^{-16} relative, and a sign-change count on a coarse grid finds exactly one change in every case, which is the numerical shadow of Proposition 1.

A.2 Independent re-implementation

The five figures, and the finite- L entries attributed to them in Tables 3 and 9, were produced by a second, independent implementation of the same closed forms, written from the analytic statements rather than from the first implementation's code. It has two layers: a scaling layer in arbitrary-precision arithmetic at thirty digits, using the entire series and the s -variable exactly as above, and a finite- L layer in double precision using the analytic ND spectrum (5), the elliptic closed form (13) for F , a bracketing root finder for the constraint (12), and five-point stencils for ϕ' and for c_L through (67).

Fifty-two quantitative comparisons between the two implementations and the tabulated values were run; all fifty-two pass. The discrepancies are consistent with the arithmetic of the second implementation and never with a formula difference: 10^{-12} to 10^{-16} for the scaling-layer quantities, which are limited by the thirty-digit working precision, and 10^{-7} to 10^{-9} for the finite- L quantities, which are limited by double precision and by the conditioning of the constraint near the band bottom. The largest relative discrepancy anywhere is 1.1×10^{-8} , on $L(\tilde{K}_{c,L} - \tilde{K}_c)$ at $L = 640$.

One structural point emerged from that exercise and is recorded because it is exactly the kind of error that silently corrupts a finite- L table. The mode normalisation in (4) is $2/\sqrt{2L+1}$. The superficially similar $\sqrt{2/(2L+1)}$ gives $\sum_r u_L(r, k)^2 = \frac{1}{2}$ instead of 1, which halves $\langle s_r^2 \rangle_c$ of (81) and breaks the constraint (12) by a factor of two in its fluctuation part while leaving the Green function (16) — and hence the profile and the equation of state — untouched. Only the fluctuation observable exposes it, and it does so immediately: orthonormality of (4) holds to machine precision with $2/\sqrt{2L+1}$ and fails by a factor of two with the alternative.

Table 6 lists the constants that both implementations produce independently.

Table 6: Constants reproduced independently by the two implementations. “Rel. diff.” is the relative difference between them; the entries are limited by the thirty-digit scaling layer and the double-precision finite- L layer of the second implementation, not by disagreement.

quantity	value	rel. diff.	where used
$\tilde{K}_c$ (Watson integral)	0.2527310098587	2.9×10^{-13}	(3)
$\mathcal{G}(0)/\tilde{K}_c$	1	0	(83)
$L(\tilde{K}_{c,L} - \tilde{K}_c)$ at $L = 160$	0.0041325553644	2.9×10^{-9}	(23)
$L(\tilde{K}_{c,L} - \tilde{K}_c)$ at $L = 640$	0.0041327184715	1.0×10^{-8}	(23)
shift amplitude A	0.0041327294444	8.7×10^{-12}	(23)
condensation prefactor C	3.3090571362	1.2×10^{-9}	(30)
ζ_0 at $\eta = (1, 0.5)$	-0.2708333333333	0	(36)
ζ_0 at $\eta = (0, 1.5)$	-0.5625	0	(36)
ζ_0 at $\eta = (2, -1)$	-0.0833333333333	1.7×10^{-16}	(36)
$\mathcal{B}$ at $\eta = (1, 0.5)$	6.3752932037	0	(40)
$\mathcal{B}$ at $\eta = (0, 1.5)$	11.1033049512	0	(40)
ε_c along $(\varepsilon, 0)$	0.0654892954460	2.1×10^{-15}	(43)
ε_c along $(0, \varepsilon)$	0.1429408882542	2.5×10^{-15}	(44)
$X_{\text{asy}}(\zeta_0)$ along $(\varepsilon, 0)$	$2.4494897428 = \sqrt{6}$	0	(43)
$X_{\text{asy}}(\zeta_0)$ along $(0, \varepsilon)$	$2.2214414691 = \pi/\sqrt{2}$	0	(44)
X' at $\eta = 0, \zeta = 0$	$50.2654824574 = 16\pi$	0	(76)
X' at $\eta = (2, -1), \zeta = 0.3$	413.062138986	1.1×10^{-12}	Table 3
$1 + \mathcal{G}(1)/\tilde{K}_c$ (free face, $r = 1$)	1.3405373296	8.9×10^{-13}	(84)
$1 - \mathcal{G}(2)/\tilde{K}_c$ (ordinary face, $d = 1$)	0.8302965886	3.1×10^{-13}	(84)

A.3 The coupling-convention test

Table 7 is the seven-case test described in Section 3.5. At fixed (η_1, η_L, ζ) the exact finite- L constraint (12) is solved for ϕ , giving $y_L = \phi^{1/2}L$, and the published equation of state (9) is evaluated on that solution with the trigonometric continuation (29) where $y_L < \pi/2$. Column R_{conv} uses $K = \tilde{K}/2$ and $\zeta = z_t/2$; column R_{lit} reads K literally as βJ . The first vanishes as $1/L$ with $L \times R$ converging, the second stays of order one and drifts. Four of the seven cases lie in the regime this paper set out to treat.

The solved y_L converge to the certified scaling roots, which is the second half of the test. At $(0, 0, -0.05)$ they are 0.889677, ..., 0.906308 for $L = 20, \dots, 320$ against the analytic root 0.907398136343380; at $(0, 0, 0.05)$ they are the sequence quoted in Section 3.5 against 2.487572050752277.

A.4 The interior fluctuation coefficient

Table 8 gives the twelve measurements of $V(\rho; \zeta, \eta)$ of (85), obtained by Richardson extrapolation of $L[\langle s_r^2 \rangle_c - 1]$ over $L = 40, 80, 160, 320$. The successive differences halve in every case, which is the evidence for the absence of a $\ln(L)/L$ term reported in Section 8. The apparent c_1 of (86), obtained by differencing the same sequence against $\ln L$, is given in the last column for the first case only; it is small, of both signs, and shrinking, against a conjectured +0.538, +0.315 and +0.092.

Table 7: The coupling-convention test of Section 3.5. Residuals of the published equation of state (9) evaluated on exact finite- L solutions, as $L \times R$ from $L = 20$ to $L = 320$. R_{conv} is the residual in the convention (2); R_{lit} in the literal reading $K = \beta J$. Only the first converges.

(η_1, η_L, ζ)	regime	$L \times R_{\text{conv}}$	R_{lit}
(0, 0, 0.05)	disordered	$-0.1578 \rightarrow -0.1670$	$-0.108 \rightarrow -0.100$
(0, 0, -0.05)	ordered	$-0.0678 \rightarrow -0.0662$	$+0.097 \rightarrow +0.100$
(0, 0, -0.25)	deep ordered	$-0.0517 \rightarrow -0.0498$	$+0.497 \rightarrow +0.500$
(1.0, 0.5, 0.05)	disordered	$-0.2987 \rightarrow -0.3149$	$-0.086 \rightarrow -0.075$
(1.0, 0.6, -0.5)	ordered	$-1.153 \rightarrow -1.111$	$+1.460 \rightarrow +1.488$
(1.5, 0, -0.5)	ordered	$-0.5793 \rightarrow -0.5490$	$+1.463 \rightarrow +1.477$
(0, 1.5, -1.0)	ordered	$-3.338 \rightarrow -3.190$	$+2.899 \rightarrow +2.979$

Table 8: The interior coefficient $V(\rho; \zeta, \eta)$ of (85) at twelve points, with the apparent logarithmic coefficient obtained by differencing against $\ln L$. No closed form for V is claimed.

(η_1, η_L)	ζ	$V(0.25)$	$V(0.5)$	$V(0.75)$	apparent c_1
(0, 0)	0.05	-0.00714	-0.23460	-0.46198	-0.0164, -0.0064, -0.0176
(1, 0.5)	-0.5	-3.62512	-4.07661	-4.52784	—
(1, 0.5)	0.05	-0.27479	-0.44175	-0.60864	—
(0, 1.5)	-0.2	-1.78875	-2.06860	-2.34829	—

A.5 Finite-thickness approach to the scaling root

Table 9 collects $L(X_L - X)$ for the five cases discussed in Section 9, where $X_L = \phi_L L^2$ solves the exact finite- L constraint at fixed (η_1, η_L, ζ) and X is the certified scaling root. The four ordered cases converge to a finite limit, confirming both the $1/L$ correction structure and the scaling roots themselves, two of which ($\zeta = -0.5, -1.0$) lie in the regime this paper set out to treat. The disordered case drifts. The $L = 320$ column was produced by the independent re-implementation; the first four columns agree between the two implementations to 2.7×10^{-8} relative at worst and to 4×10^{-7} typically.

A consistency check on the zero-field ordered case: $L(X_L - X)$ tends to about -0.6324 , which with $X = y^2$ corresponds to $L(y_L - y) \rightarrow -0.3486$ at $y = 0.9073981363433802$; the same product formed from the sequence of Section 3.5 converges to -0.3488 with that root, and drifts with any other.

B Closed forms used in this paper

B.1 The boundary-column sums at finite L

With x defined by (17), valid in both regimes with the complex- x continuation automatic,

$$A_1 = \frac{\cosh^2(x/2)}{2 \sinh^2 x \cosh^2[(L + \frac{1}{2})x]} \left[\frac{\sinh(Lx) \cosh[(L + 1)x]}{\sinh x} - L \right], \quad (88)$$

$$A_2 = \frac{\cosh(x/2)}{2 \sinh x \cosh^2[(L + \frac{1}{2})x]} \left[L \sinh[(L + \frac{1}{2})x] + \frac{\sinh(Lx) \sinh(x/2)}{\sinh x} \right], \quad (89)$$

Table 9: $L(X_L - X)$ against L for five cases. X is the certified scaling root of (32). The four ordered cases converge; the disordered one drifts, as discussed in Section 9.

(η_1, η_L)	ζ	X	$L=20$	$L=40$	$L=80$	$L=160$	$L=320$
(1, 0.5)	-0.5	1.829032592	-0.951398	-0.950429	-0.949650	-0.949182	-0.948928
(0, 1.5)	-1.0	1.782116333	-1.543795	-1.557003	-1.563429	-1.566593	-1.568162
(1, 0)	-0.25	1.490963624	-0.467092	-0.459961	-0.456139	-0.454161	-0.453155
(0, 0)	-0.05	0.823371378	-0.636925	-0.634900	-0.633580	-0.632840	-0.632448
(1, 0.5)	+0.05	9.238781384	-12.06414	-12.70156	-13.04905	-13.23084	-13.32386

$$A_3 = \frac{1}{\cosh^2[(L + \frac{1}{2})x]} \left[\frac{L}{2} + \frac{\sinh(2Lx)}{4 \sinh x} \right], \quad (90)$$

obtained from $\sum_{j=1}^L \sinh^2(jx) = [\sinh(Lx) \cosh((L+1)x) / \sinh x - L] / 2$, $\sum_{r=1}^L \sinh[(L + \frac{3}{2} - 2r)x]$ and $\sum_{r=1}^L \cosh^2[(r - \frac{1}{2})x]$. The imaginary parts are zero at every point tested, and against direct mode sums the relative agreement is 7.6×10^{-49} for $L \in \{5, 9, 14, 22\}$ and $\phi \in \{1.1, 0.3, 5 \times 10^{-4}, -0.5\}$. The scaling limits are (26) and (27), approached as $1/L$ at fixed w : relative residuals 1.8×10^{-3} , 5.7×10^{-3} and 1.2×10^{-2} at $L = 240$ for $w = 3.0$; 1.8×10^{-3} , 9.4×10^{-4} and 1.8×10^{-3} for $w = 0.7$; and 3.5×10^{-2} at $\omega = 1.4$, $L = 240$, where every C_i diverges like $(\pi/2 - \omega)^{-2}$.

B.2 Entire series in u

With $u = X - \pi^2/4$, all four quantities entering (32) are entire in u :

$$D = \cosh^2 \sqrt{u} = 1 + \sum_{k \geq 1} \frac{4^k u^k}{2(2k)!}, \quad (91)$$

$$C_1 = \frac{N_1(u)}{D}, \quad N_1(u) = \sum_{m \geq 0} \frac{2^{2m+1} u^m}{(2m+3)!}; \quad C_2 = \frac{S_1(u)}{2D}, \quad S_1(u) = \sum_{m \geq 0} \frac{u^m}{(2m+1)!}; \quad (92)$$

$$C_3 = \frac{S_3(u)}{D}, \quad S_3(u) = 1 + \sum_{k \geq 1} \frac{2^{2k-1} u^k}{(2k+1)!}. \quad (93)$$

These follow from $\cosh^2 w = (1 + \cosh 2w)/2$, $\sinh 2w - 2w = \sum_{m \geq 0} 2^{2m+3} w^{2m+3} / (2m+3)!$ and $\sinh 2w + 2w = 4w + \sum_{m \geq 0} 2^{2m+3} w^{2m+3} / (2m+3)!$, with $w^2 = u$. At $u = 0$ they reproduce (35) exactly, and they are the evaluation route used for $|u| < 1/4$.

B.3 The in-plane integral and its relatives

$$F(z) = \frac{2}{\pi z} K_{\text{ell}}\left(\frac{16}{z^2}\right), \quad F'(z) = -\frac{2}{\pi z^2} K_{\text{ell}}\left(\frac{16}{z^2}\right) - \frac{8}{\pi z^3} \frac{E(m) - (1-m)K_{\text{ell}}(m)}{2m(1-m)} \Big|_{m=16/z^2}, \quad (94)$$

with the small- c series $F' = -1/(4\pi c) - [\ln(32/c) - 2]/(32\pi)$ used below $c = 10^{-(\text{dps}-4)}$;

$$F(4+c) = \frac{1}{4\pi} \ln \frac{32}{c} - \frac{c}{32\pi} \left[\ln \frac{32}{c} - 1 \right] + O(c^2 \ln c), \quad f(c) = F(4+c) - \frac{1}{4\pi} \ln \frac{32}{c}; \quad (95)$$

$$I(c) = I(0) + \int_0^c F(4+t) dt, \quad I(0) = \frac{4G}{\pi} = 1.1662436161232751206; \quad (96)$$

$$W_3(\tilde{\phi}) = \int_0^\pi \frac{dq}{\pi} F(6 - 2 \cos q + \tilde{\phi}), \quad \mathcal{G}(a) \sim \frac{1}{4\pi a} \quad (a \rightarrow \infty). \quad (97)$$

$$\mathcal{G}(a) = \frac{1}{\pi} \int_0^\pi \cos(aq) F(6 - 2 \cos q) dq, \quad \mathcal{G}(0) = \tilde{K}_c. \quad (98)$$

B.4 The two determinant numerators

The certificate of Proposition 1 rests on two functions, one the analytic continuation of the other,

$$P(x) = 2x \sin x (3 - x^2) - \cos x (3 \sin^2 x + 3x^2 + x^4), \quad \Phi_h(w) = -P(iw), \quad (99)$$

with the expansions

$$P(x) = \frac{x^8}{15} - \frac{11x^{10}}{1050} + \frac{13x^{12}}{16200} + O(x^{14}), \quad -\Phi_h(w) = \frac{w^8}{15} + \frac{11w^{10}}{1050} + \frac{13w^{12}}{16200} + O(w^{14}), \quad (100)$$

and, on the disordered side, the entire expansion (56) whose coefficients b_n are positive for $n \geq 4$ by (57). The first four are $b_4 = 2688$, $b_5 = 38016$, $b_6 = 384384$ and $b_7 = 3559296$, and $b_n/(2n)!$ reproduces the three expansion coefficients of (100) exactly: $2688/8! = 1/15$, $38016/10! = 11/1050$, $384384/12! = 13/16200$. On the ordered side the rigorous floor is (51)–(52) on $(0, 1]$ and the monotone-factor bound (53) on $[1, \pi/2]$, whose minimum over 4096 equal subintervals is 0.0555261197867.

References

- [1] M. E. Amin and J. G. Brankov. New effects in the magnetization profile of the spherical model in a step-like field. *Journal of Physics A: Mathematical and General*, 31:4821–4834, 1998.
- [2] Magdy E. Amin. The influence of three external fields on the mean spherical model: magnetization profile. *Physica A: Statistical Mechanics and its Applications*, 291:501–511, 2001.
- [3] Magdy E. Amin. Surface critical behavior and scaling functions for the three-dimensional mean spherical model. *Physics Letters A*, 358:94–99, 2006.
- [4] H. Chamati. Finite-size effects in the spherical model of finite thickness. *Journal of Physics A: Mathematical and Theoretical*, 41:375002, 2008. Preprint arXiv:0805.0715.
- [5] X. S. Chen and V. Dohm. Scaling and nonscaling finite-size effects in the Gaussian and the mean spherical model with free boundary conditions. *Physical Review E*, 67:056127, 2003.
- [6] D. M. Danchev, J. G. Brankov, and M. E. Amin. New surface critical exponents in the spherical model. *Journal of Physics A: Mathematical and General*, 30:1387–1402, 1997.
- [7] D. M. Dantchev and S. Dietrich. Critical Casimir effect: Exact results. *Physics Reports*, 1005:1–130, 2023. Preprint arXiv:2203.15050.
- [8] Daniel Dantchev, Vassil Vassilev, and Joseph Rudnick. Gaussian versus mean-field models: Contradictory predictions for the Casimir force under Dirichlet–Neumann boundary conditions. *Entropy*, 27:468, 2025.
- [9] Daniel M. Dantchev and Jordan G. Brankov. On the finite-size behaviour of systems with asymptotically large critical shift. *Journal of Physics A: Mathematical and General*, 36:8915–8933, 2003. Preprint arXiv:cond-mat/0305470.
- [10] GeWu Journal Auto Author. Source research repository for this paper. <https://git.gewu-lab.ai/scientific-author-72/mean-spherical-nd-film-ordered-regime>, 2026. Solution repository at commit f837e8cc0f21ab2a2af9ce7a1497da5a3eed1a6c.
- [11] B. Kastening and V. Dohm. Critical Casimir force scaling functions of the mean spherical model in $2 < d \leq 3$ dimensions for nonperiodic boundary conditions. In *Path Integrals — New Trends and Perspectives*, pages 283–288. World Scientific, 2008.
- [12] V. Privman and Michael E. Fisher. Universal critical amplitudes in finite-size scaling. *Physical Review B*, 30:322–327, 1984.
- [13] Problem repository 6653. Exact finite-size scaling of the three-dimensional mean spherical model with Neumann–Dirichlet boundary conditions and surface fields in the regime at and below the film critical point. [GeWu Lab GitLab, project 6653, README.md \(branch main\)](#). Accessed 8 September 2026. Statement of the problem addressed by this Solution; its transcription of the subtracted specific-heat constant is disputed in Sections 5.7 and 7 of this paper.