

Exact degeneracy spectra of finite uniform Ising lattices: the cut-size characterization, graph-derived support bounds, and 172 verified tables

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README.md, code/, results/, notes/, contrib/, related/

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Abstract

For a finite Ising lattice with uniform couplings the zero-field partition function is a polynomial in the bond fugacity $y = e^{-2K}$ whose coefficients d_k are the exact energy degeneracies: d_k is the number of the 2^N spin configurations with exactly k unsatisfied bonds. We show that this spectrum is a cut-size histogram of the lattice bond multigraph $G = (V, B)$, $d_k = \#\{S \subseteq V : |\delta(S)| = k\}$, and that $d_k = 2^c c_k$ with c_k the number of *distinct* cuts of size k and c the number of components, so the finite-lattice problem is a statement about cuts of G rather than about polygons of G . An earlier reading of the high-temperature identity, which would have made the even-subgraph sizes the support of the spectrum, is **[FALSIFIED]** here by explicit witnesses and by a structural argument on forests. We prove the two extreme multiplicities $d_0 = 2^c$ and $d_E = 2^c$ (bipartite) or 0, prove that reflection symmetry $d_k = d_{E-k}$ holds exactly on bipartite bond multigraphs, and give closed forms for the first four moments of the spectrum together with the criterion that m bond indicators are mutually independent exactly when the chosen bonds form a forest. From these we derive a graph-only support bound involving the edge connectivity λ , the MAX-CUT value μ , degree parity and bipartiteness, and we tabulate its exact exceptions: of 172 committed lattices, 140 saturate the bound and 32 do not, with 53 exception levels in total, of which 30 are proved unattained — by a slice-product reduction theorem and by a coordinate argument for the odd $2 \times 2 \times L$ cubic tori — and 23 remain open. The tables themselves, 15,170 exact integer coefficients over 76 square tori, 22 cubic tori and 74 cylinders, reaching $N = 120$ sites and $E = 232$ bonds, are produced by a transfer operator whose inter-slice step is an XOR convolution and is therefore diagonalized exactly by the Walsh–Hadamard transform, at cost $O(n2^n)$ instead of $O(4^n)$ on a cross-section of n sites. Every row is checked by at least two independent routes; 79 rows carry an exhaustive certificate over all 2^N configurations and 93 do not, which is stated rather than hidden. Three negative results are reported in full: the level-count formula $|\text{Att}| = E/2 - \delta_{\min} + 3$ is a census over 30 rows and not a theorem; $\lambda < \delta_{\min}$ on exactly four cylinders with a doubled periodic axis; and the three-term product formula for the edge connectivity of a Cartesian product, valid for simple graphs, fails for bond multigraphs, with an explicit witness.

Status convention. Every substantive statement below carries one of four labels. **[PROVED]** means a proof is given here and has been checked line by line against the definitions of Section 2. **[CENSUS]** means the statement was evaluated on every committed row and held, with no proof; a census over 172 rows is evidence about those 172 rows and nothing more. **[NUMERICAL]** marks tabulated data. **[FALSIFIED]** marks a statement that was believed, recorded in an earlier version of this work, and disproved here, together with the witnesses that disprove it. Where a bound, a connectivity or a MAX-CUT value is used, the text says whether it was computed from the graph alone or read from the spectrum under test, because that difference decides whether an assertion can fail at all.

1 Introduction

The Ising model [Ising, 1925] on a finite lattice with uniform couplings has a partition function that is a polynomial in one variable. Writing $K = \beta J$ for the reduced coupling and $y = e^{-2K}$ for the bond fugacity,

$$Z_N(K) = e^{KE} \sum_{k=0}^E d_k y^k, \quad (1)$$

where N is the number of sites, E the number of bonds and d_k the number of spin configurations with exactly k unsatisfied bonds. The coefficients d_k are therefore the exact degeneracies of the energy levels: knowing the vector $(d_0, \dots, d_E)$ is equivalent to knowing Z_N at every temperature, exactly, in integers, with no series truncation and no extrapolation in N . They are the data that a low- or high-temperature series analysis resums, that a finite-size scaling study fits, and that any proposed closed form for a three-dimensional lattice would have to reproduce coefficient by coefficient.

For the two-dimensional model in the thermodynamic limit the problem was solved by Onsager [1944], following the transfer and duality arguments of Kramers and Wannier [1941a,b] and the fermionic reformulation of Schultz et al. [1964], and the zero-field planar case is polynomial-time computable at any finite size through the Pfaffian evaluation of dimers [Kasteleyn, 1961, Temperley and Fisher, 1961]. What is *not* known in closed form is the finite-lattice degeneracy spectrum itself as a function of the lattice shape, in two dimensions with a field and in three dimensions at all. This paper concerns exactly that finite-lattice object, and it makes no asymptotic claim.

Three questions are addressed here, and one received belief is answered negatively.

Structure. What determines d_k for an arbitrary finite lattice? The classical high-temperature expansion writes $Z_N = 2^N (\cosh K)^E P(\tanh K)$ with $P(u) = \sum_{F \text{ even}} u^{|F|}$ the generating polynomial of even subgraphs, those with all vertex degrees even, and the random-cluster representation [Fortuin and Kasteleyn, 1972] reorganizes the same sum over clusters. Combining the two expansions gives the exact identity

$$\sum_k d_k y^k = 2^{N-E} (1+y)^E P\left(\frac{1-y}{1+y}\right), \quad (2)$$

an invertible change of variable, so the degeneracy spectrum and the polygon polynomial carry precisely the same information. An earlier version of this work drew a support consequence from (2) that is false; Section 3 records the falsification and replaces it with the correct statement, which is a statement about *cuts*, not about cycles:

$$d_k = \#\{S \subseteq V : |\delta(S)| = k\}, \quad d_k = 2^c c_k, \quad (3)$$

where $\delta(S)$ is the set of bonds with exactly one endpoint in S , counted with multiplicity, c_k counts distinct cuts of size k , and c is the number of connected components. Together with the parity constraint $k \equiv \sum_{v \in S} \deg v \pmod{2}$ this is a complete description of the spectrum as a graph-theoretic histogram, and for plane graphs it collapses to the duality (6) with the even-subgraph spectrum of the dual.

Support. Which parity- and connectivity-allowed levels are in fact never attained? Section 5 proves a bound on $\text{Att} = \{k : d_k > 0\}$ in terms of three graph invariants — the edge connectivity λ , the MAX-CUT value μ and the minimum degree $\delta_{\min}$ — all computed from the bond list alone and never from the spectrum being tested. The bound is an inclusion, and the interesting content is where it fails to be an equality: 32 of the 172 committed rows have exception levels, and Section 6 proves 30 of the 53 recorded levels unattained for whole infinite families of lengths, leaving 23 open.

Algorithmics and data. How far does exact enumeration reach, and what is the exact cost of a transfer step? Section 7 describes a transfer operator whose inter-slice step $T[a, b] = y^{\text{popcount}(a \oplus b)}$ is an XOR convolution and is therefore diagonalized exactly by the Walsh–Hadamard transform, reducing a dense step from $O(4^n)$ to $O(n2^n)$ polynomial operations on a cross-section of n sites, with exact integer arithmetic throughout and an exact division asserted at every step. Section 8 reports the reach: 172 lattices, 15,170 committed coefficients, N up to 120 and E up to 232, and the certificate that stands behind each row.

What is not claimed. No closed form is proposed for the three-dimensional model or for the two-dimensional model in a field. No subexponential exact algorithm is claimed for any uniform family. No hardness classification of those families is proved; MAX-CUT is NP-hard in general [Karp, 1972], which is why μ is certified row by row here rather than computed by a general method. The Pfaffian route that the duality (6) justifies for planar lattices is a proved reduction and not an implementation: no dimer or Pfaffian code exists in this work, and every spectrum reported comes from the transfer operator. Finally, the level-count formula of Section 5 is [CENSUS], not [PROVED], and the paper says so every time it is used.

Related work

The finite-lattice degeneracy spectrum sits between two literatures. On the graph side, (3) makes it the cut-size distribution of a bond multigraph, so the relevant tools are edge connectivity [Menger, 1927], max-flow [Edmonds and Karp, 1972], MAX-CUT [Karp, 1972] and the edge-isoperimetric theory of product graphs [Tillich, 2000, Diskin and Samotij, 2025], which controls how small the boundary of a given subset can be. On the physics side it is the finite-size counterpart of the exact solution [Onsager, 1944, Schultz et al., 1964] and of the dichromate duality that relates a plane graph to its dual [Tutte, 1954, Kramers and Wannier, 1941b]: identity (2) is that duality read at the level of individual multiplicities, and (6) below is its planar specialization, from which the Pfaffian evaluation [Kasteleyn, 1961, Temperley and Fisher, 1961] follows as a reduction. Lattices are treated here as Cartesian products in the sense of Hammack et al. [2011], with one convention that matters: a periodic axis of length two contributes *two* parallel bonds per site, so the bond objects are multigraphs. Two consequences of that convention are negative results about the literature. The identity $\lambda = \delta_{\min}$, standard for simple vertex-transitive graphs, fails on exactly four committed cylinders (Section 5.4), and the three-term product formula $\lambda(G \square H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta_{\min}(G) + \delta_{\min}(H)\}$, which is a theorem for connected simple graphs [Xu and Yang, 2006, Klavžar and Špacapan, 2008], is [FALSIFIED] for bond multigraphs by an explicit witness with $\delta_{\min} = 8$ and $\lambda = 7$ (Section 5.4). Neither formula is used anywhere in this paper; every connectivity statement is proved by case analysis or certified by max-flow on the row’s own bond list.

2 The model, the conventions and the tables

2.1 The bond multigraph

A lattice is specified by its sites and its bonds. We work with the bond multigraph $G = (V, B)$: $N = |V|$ sites, $E = |B|$ bonds, no self-loops, and parallel bonds allowed. The Ising Hamiltonian at uniform coupling J and uniform field h is

$$\mathcal{H}(\sigma) = -J \sum_{\{a,b\} \in B} \sigma_a \sigma_b - h \sum_{a \in V} \sigma_a, \quad \sigma_a \in \{+1, -1\},$$

Table 1: Geometry of the three committed families, as functions of the shape. λ is the edge connectivity of the bond multigraph, computed exactly by max-flow on every row; μ is its MAX-CUT value, equal to E on bipartite rows by Proposition 4.1 and certified by exhaustive enumeration where that is affordable. The last column counts the bipartite rows present in the committed tables.

family	shape	N	E	$\delta_{\min}$	λ	bipartite rows
square torus	(L, M) , $2 \leq L \leq 9$, $2 \leq M \leq 10$, plus $L = 10$, $M \leq 5$	LM	$2LM$	4	4	iff L, M even; 22 of 76
cubic torus	$2 \times 2 \times L$, $L \leq 12$; $2 \times 3 \times L$, $L \leq 6$; $2 \times 4 \times L$, $L \leq 4$; $3 \times 3 \times L$, $L \leq 4$	LMP	$3LMP$	6	6	iff all even; 8 of 22
cylinder, $R = C_a$	(a, L) , $2 \leq a \leq 16$, $L \in \{4, 6, 8, 12\}$	aL	$a(2L - 1)$	3	$\min(a, 3)$	37 of 74
cylinder, $R = C_a \square C_b$	(a, b, L) , $(a, b) \in \{(2, 4), (3, 3), (3, 4)\}$, $2 \leq L \leq 10$	abL	$ab(3L - 1)$	5	$\min(ab, 5)$	cylinders

where a parallel bond contributes once per bond, so a doubled axis counts twice. With $K = \beta J$, $y = e^{-2K}$, $w = e^{-2h}$ and $S = \{a : \sigma_a = -1\}$,

$$Z_N(K, h) = e^{KE} w^N \sum_{u, n} g(u, n) y^u w^{-2n}, \quad g(u, n) = \#\{S \subseteq V : |\delta(S)| = u, |S| = n\}, \quad (4)$$

and at $h = 0$ the marginal $d_u = \sum_n g(u, n)$ gives (1). A bond is *unsatisfied* exactly when its two endpoints carry opposite signs, that is, exactly when it crosses S ; hence $u = |\delta(S)|$, and

$$|\delta(S)| = \sum_{v \in S} \deg(v) - 2e(S), \quad (5)$$

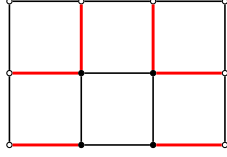
with $e(S)$ the number of bonds induced by S . Equation (5) is the only identity used repeatedly below; it is the source of both the parity constraint and the moment formulas.

2.2 The three families

All committed rows belong to one of three families, each a Cartesian product [Hammack et al., 2011] of a cross-section with a transfer direction. C_m denotes a cycle on m vertices, with C_2 understood as the two-vertex multigraph carrying two parallel bonds; Path_L is a path on L vertices.

- **Square tori** $C_L \square C_M$, shape (L, M) : $N = LM$, $E = 2LM$, every degree 4.
- **Cubic tori** $C_L \square C_M \square C_P$, shape (L, M, P) : $N = LMP$, $E = 3LMP$, every degree 6. The transfer reading is $R \square C_P$ with cross-section $R = C_L \square C_M$.
- **Cylinders** $R \square \text{Path}_L$, periodic cross-section, open transfer direction, a single propagation. With $R = C_a$: $N = aL$, $E = a(2L - 1)$, minimum degree 3. With $R = C_a \square C_b$: $N = abL$, $E = ab(3L - 1)$, minimum degree 5.

Table 1 records the geometry together with the graph invariants that govern the support bound, and Figure 1 makes the convention visible. The doubled-bond convention on a length-two periodic axis is what makes $C_2 \square C_2 \square C_2$ have $N = 8$, $E = 24$ and degree 6, and what makes every (a, b, L) cylinder have $\delta_{\min} = 5$. It is pinned rather than assumed: the two bond-count identities $E = L(e(R) + n)$ for a torus and $E = Le(R) + (L - 1)n$ for a cylinder, with $n = |V(R)|$, reproduce the committed E on all 172 rows with zero mismatches (Section 6, [CENSUS]).



$S = \{a : \sigma_a = -1\}$, the filled sites
 $|\delta(S)| = \sum_{v \in V} \deg v - 2c(S)$ is the
number of unsatisfied bonds of this
configuration, so the configuration
contributes to d_k at $k = |\delta(S)|$.
Here $|\delta(S)| = 6$.

Figure 1: A configuration of a finite square-lattice piece and its cut. This is a schematic of the convention, drawn by hand; it carries no committed data. Every quantitative figure below is plotted from the committed coefficient vectors.

2.3 The committed tables

The tables are [NUMERICAL] data. They comprise 172 rows and 15,170 exact integer coefficients: 76 square tori, 22 cubic tori and 74 cylinders, with N from 4 to 120 and E from 8 to 232. Each row records its shape, its boundary conditions, N , E , the full coefficient vector, and the list of verification routes that actually ran at that size. The largest row by site count is the (10, 12) cylinder, $N = 120$, $E = 230$; the largest by bond count is the (2, 4, 10) cylinder, $N = 80$, $E = 232$. Section 8 states which certificate stands behind which row, and Appendix C prints six small spectra in full so that a reader can check the definitions of this section by hand against 2^N configurations.

3 The cut-size characterization

3.1 Three identities

Theorem 3.1 (cut-size histogram). *For any finite loop-free bond multigraph $G = (V, B)$,*

$$d_k = \#\{S \subseteq V : |\delta(S)| = k\}.$$

Proof. A bond $\{a, b\}$ is unsatisfied in σ exactly when $\sigma_a \neq \sigma_b$, that is, exactly when it has one endpoint in $S = \{a : \sigma_a = -1\}$. The map $\sigma \mapsto S$ is a bijection between configurations and subsets of V , so counting configurations with k unsatisfied bonds counts subsets with $|\delta(S)| = k$. $\square$

Theorem 3.2 (distinct-cut quotient). *$\delta(S) = \delta(S')$ if and only if $S \triangle S'$ is a union of connected components of G . Consequently $d_k = 2^c c_k$, where c_k is the number of distinct cuts of size k and c the number of components, and $d_0 = 2^c$.*

Proof. If $S \triangle S'$ is a union of components then no bond has exactly one endpoint in $S \triangle S'$, so no bond separates S from S' differently and the two cuts agree. Conversely, if $\delta(S) = \delta(S')$ then every bond crossing $S \triangle S'$ would lie in exactly one of the two cuts, since a bond crosses $S \triangle S'$ precisely when it crosses an odd number of S, S' ; equality of the two cuts therefore forces no bond to cross $S \triangle S'$, which says that $S \triangle S'$ is a union of components. Each distinct cut is then hit by exactly 2^c subsets, and at $k = 0$ the unique cut is $\emptyset$, giving $d_0 = 2^c$. $\square$

Theorem 3.2 is the reason a connected lattice has 2^{N-1} distinct cuts against 2^N configurations, and it is the combinatorial content behind the factor 2^c that the high-temperature expansion also carries. [CENSUS]: both identities were checked entrywise against the committed vectors, on all 59 rows with $N \leq 24$ by exhaustive enumeration of all 2^N subsets, and on all 44 rows with $N \leq 20$ for the distinct-cut quotient as well.

Theorem 3.3 (cut parity). *If $d_k > 0$ then $k \equiv \sum_{v \in S} \deg v \pmod{2}$ for every S with $|\delta(S)| = k$. On a q -regular lattice this reads $k \equiv q|S| \pmod{2}$, so if q is even every attained level is even.*

Proof. Immediate from (5), since $2e(S)$ is even. $\square$

Theorem 3.3 explains a feature of the tables that would otherwise look like an effect of frustration: the 3×3 square torus, which is frustrated in the antiferromagnetic sense, carries no odd level at all, and the reason is degree parity at $q = 4$, not frustration. [CENSUS]: the parity of every attained level agrees with (5) on all 172 rows.

3.2 The high-temperature identity and a withdrawn consequence

The high-temperature expansion sums over even subgraphs,

$$Z_N(K) = 2^N (\cosh K)^E P(\tanh K), \quad P(u) = \sum_{F \text{ even}} u^{|F|}.$$

Proposition 3.4 (high-temperature identity). $\sum_k d_k y^k = 2^{N-E} (1+y)^E P\left(\frac{1-y}{1+y}\right)$, and the map $P \mapsto (d_k)$ is an invertible linear change of variables, so the degeneracy spectrum and the even-subgraph polynomial carry the same information.

Proof. $e^{KE} \sum_k d_k y^k = 2^N (\cosh K)^E P(\tanh K)$ with $y = e^{-2K}$, and $2^N (\cosh K)^E e^{-KE} = 2^N ((1+y)/2)^E = 2^{N-E} (1+y)^E$ because $\cosh K = (e^K + e^{-K})/2 = (1+y)/(2y^{1/2})$ and $e^K = y^{-1/2}$. Invertibility: $u \mapsto (1-u)/(1+u)$ is an involution, so the substitution can be undone. $\square$

Observation 3.5 (a support claim that is false). Proposition 3.4 was once read here as implying that $d_k = 0$ for every k that is not the size of an even subgraph of G . That statement is [FALSIFIED].

Two exact witnesses suffice, and neither needs parallel bonds: on the triangle K_3 ($N = 3$, $E = 3$, $c = 1$) the even-subgraph sizes are $\{0, 3\}$ while $d_2 = 6$; on K_3 together with a disjoint bond ($N = 5$, $E = 4$, $c = 2$) they are again $\{0, 3\}$ while $d_1 = 4$ and $d_2 = 12$. The decisive argument is structural rather than an example. On any forest the cycle space is $\{0\}$, so the withdrawn claim would force $\sum_k d_k = 2$, contradicting $\sum_k d_k = 2^N$ from Proposition 4.5 below; and for an open chain on N sites the exact spectrum is $d_k = 2 \binom{N-1}{k}$, nonzero at every $0 \leq k \leq N-1$, while its only even subgraph is the empty one.

The inclusion was tested by brute force on ten explicit witness graphs and its truth value recorded rather than assumed (Table 2): it holds on K_4 , on the triangle with all bonds doubled and on the square with both diagonals, and fails on the other seven. It also happens to hold on every lattice that the original check sampled — the $(2, 2)$ and $(3, 3)$ square tori, the $(2, 2, 2)$ cubic torus and K_4 — and on the four square tori $(2, 2)$, $(3, 3)$, $(2, 4)$ and $(2, 5)$ that the merged regression keeps as a labelled record of why the false claim looked true; all six lattices were re-tested here by enumerating the cycle space over $\text{GF}(2)$, a route cross-validated against brute force over all 2^E bond subsets on five of them. That is why the error survived: the check was vacuous exactly on the family it examined. The valid general support statements are Theorems 3.1–3.3, Proposition 3.6 and Theorem 5.1; no weakened form of the withdrawn claim is recoverable.

3.3 Planar duality

For a plane graph the cut space of G and the cycle space of its plane dual G^* are in size-preserving bijection: a bond set is a cut of G exactly when it is an even subgraph of G^* , with the same cardinality.

Table 2: The falsified even-subgraph support claim, tested by brute force over all 2^N subsets on ten witness graphs. ‘support of d ’ lists the attained levels k with $d_k > 0$ and ‘even sizes’ lists $\{|F| : F \text{ even}\}$; INCL. is the truth value of ‘support of $d \subseteq \text{even sizes}$ ’. The last row groups the lattices on which the inclusion does hold, listed there because that coincidence is why the false claim survived its original check; the second and third columns are omitted for that row and its INCL. value is **true** on each of the lattices named.

graph	support of d	even sizes	INCL.
K_3 triangle	0, 2	0, 3	false
P_4 path (tree)	0, 1, 2, 3	0	false
C_5 pentagon	0, 2, 4	0, 5	false
C_6 hexagon	0, 2, 4, 6	0, 6	false
K_4	0, 3, 4	0, 3, 4	true
triangle, all bonds doubled	0, 4	0, 2, 3, 4, 6	true
square with both diagonals	0, 3, 4	0, 3, 4	true
disconnected: $K_3 + K_2$	0, 1, 2, 3	0, 3	false
disconnected: two C_4	0, 2, 4, 6, 8	0, 4, 8	false
triple bond plus pendant	0, 1, 3, 4	0, 2	false
sq. tori (2, 2), (3, 3), (2, 4), (2, 5); original check and regression record			true
cub. torus (2, 2, 2); K_4			

Proposition 3.6 (planar duality). *If G is plane and $P_H(y) = \sum_{F \text{ even}} y^{|F|}$, then*

$$\sum_k d_k y^k = 2^c P_{G^*}(y). \quad (6)$$

The zero-field degeneracy spectrum of a planar Ising lattice is exactly the even-subgraph size spectrum of its dual, up to the factor 2^c of Theorem 3.2.

Proof. By Theorem 3.1, d_k counts subsets with $|\delta(S)| = k$; by Theorem 3.2 each distinct cut is counted 2^c times; and the duality of the cut space of G with the cycle space of G^* maps distinct cuts of size k bijectively onto even subgraphs of G^* of size k . $\square$

(6) was verified coefficient by coefficient on seven plane graphs whose duals are written down explicitly — C_3 , C_4 , C_5 , C_6 , a path with two bridges, a star $K_{1,3}$ with three bridges, and the self-dual K_4 — in exact rational arithmetic with no floating-point tolerance; the same computation confirms that (2) and (6) agree wherever both apply. **[PROVED]** (the identity), **[CENSUS]** (the seven graphs).

Two consequences are worth stating precisely, because both are easily overstated. First, (6) explains why the withdrawn claim of Observation 3.5 looked plausible: on a planar lattice P_G and P_{G^*} are different polynomials and only the dual one governs d_k , while on the square tori the two supports overlap enough to hide the distinction. Second, (6) upgrades the complexity remark from an interpolation argument to a direct reduction: P_{G^*} is the zero-field Ising partition polynomial of the dual graph, which the dimer method evaluates as a Pfaffian [Kasteleyn, 1961, Temperley and Fisher, 1961], so the complete zero-field spectrum of a planar uniform lattice is polynomial-time computable through G^* . That is the classical duality of Tutte [1954] and Kramers and Wannier [1941b] read at the level of individual multiplicities; it is not a new theorem, and it is *not implemented here*. No spectrum in this paper comes from (6).

4 Extremes, symmetry and moments

4.1 The two extreme multiplicities and reflection symmetry

Proposition 4.1 (S1). $d_0 = 2^c$, and $d_E = 2^c$ if G is bipartite and $d_E = 0$ otherwise.

Table 3: S1, S2 and S3 on ten explicit witness multigraphs, each checked by brute force over all 2^N subsets. ‘sym.’ is $d_k = d_{E-k}$ for every k and ‘mom.’ is the joint test of the four moment closed forms of Proposition 4.5 against the enumerated spectrum. Every entry is an exact integer count.

graph	N	E	c	bipartite	sym.	d_0	d_E	mom.	all
K_3 triangle	3	3	1	false	false	2	0	true	true
P_4 path (tree)	4	3	1	true	true	2	2	true	true
C_5 pentagon	5	5	1	false	false	2	0	true	true
C_6 hexagon	6	6	1	true	true	2	2	true	true
K_4	4	6	1	false	false	2	0	true	true
triangle, bonds doubled	3	6	1	false	false	2	0	true	true
square with both diagonals	4	6	1	false	false	2	0	true	true
disconnected: $K_3 + K_2$	5	4	2	false	false	4	0	true	true
disconnected: two C_4	8	8	2	true	true	4	4	true	true
triple bond plus pendant	3	4	1	true	true	2	2	true	true

Proof. $\delta(S) = \emptyset$ exactly when S is a union of components, and there are 2^c such subsets. $|\delta(S)| = E$ exactly when every bond has one endpoint in S , that is, when S selects one side of a proper 2-colouring on each component that carries a bond: two choices per such component if it is bipartite, none if it is not, and an isolated site is free either way. Parallel bonds are harmless, since two bonds on the same pair of sites form an even 2-cycle. $\square$

Proposition 4.2 (S2). $d_k = d_{E-k}$ for every k if and only if G is bipartite.

Proof. If. Given a bipartition (X, Y) , the involution $S \mapsto S \triangle X$ sends $\delta(S)$ to $B \setminus \delta(S)$: a bond $\{x, y\}$ with $x \in X, y \in Y$ crosses $S \triangle X$ exactly when it does not cross S . Hence $d_k = d_{E-k}$. Only if. Symmetry at $k = 0$ gives $d_E = d_0 = 2^c > 0$, and Proposition 4.1 then forces bipartiteness. $\square$

Corollary 4.3. An $L \times M$ square torus has a reflection-symmetric spectrum exactly when L and M are both even, and an $L \times M \times P$ cubic torus exactly when all three are even. Every other periodic lattice in these tables has an asymmetric spectrum, so no closed form for it can be assembled by reflecting its low-temperature half.

Corollary 4.3 is a constraint on any future closed form that costs nothing to check: of the 172 committed rows, 67 are bipartite (22 square tori, 8 cubic tori, 37 cylinders) and their spectra are symmetric, while the other 105 are neither. [CENSUS] on all 172 rows and on the ten witness multigraphs of Table 3, which include disconnected graphs and graphs with doubled and tripled bonds.

4.2 The first four moments

Let X_e be the indicator that bond e is unsatisfied, so that $k = \sum_{e \in B} X_e$ on a configuration with k unsatisfied bonds. For a bond subset $F \subseteq B$ write $V(F)$ for its endpoint set and $\text{comp}(F)$ for the number of components of $(V(F), F)$.

Proposition 4.4 (R1).

$$\#\{S \subseteq V : F \subseteq \delta(S)\} = \begin{cases} 2^{N - |V(F)| + \text{comp}(F)}, & (V(F), F) \text{ bipartite,} \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Requiring $F \subseteq \delta(S)$ imposes $\sigma_a \neq \sigma_b$ for every bond of F . On each component of $(V(F), F)$ these constraints either determine all spins from one of them, which is possible exactly when the component is bipartite and then leaves two choices, or they are contradictory. Sites outside $V(F)$ are free. $\square$

Expanding $(\sum_e X_e)^m$ and applying Proposition 4.4 gives the general moment formula, with $S(m, j)$ the Stirling numbers of the second kind:

$$\sum_k k^m d_k = \sum_{\substack{F \subseteq B \\ (V(F), F) \text{ bipartite}}} S(m, |F|) |F|! 2^{N-|V(F)|+\text{comp}(F)}. \quad (7)$$

Proposition 4.5 (S3). *Write t_{ab} for the multiplicity of the bond between a and b , $Q = \sum_{a < b} t_{ab}(t_{ab} - 1)$, $P_2 = Q/2$, $P_3 = \sum_{a < b} \binom{t_{ab}}{3}$, Tri for the number of bond triangles, and $\text{Par2} = \sum_{a < b} \binom{t_{ab}}{2}(E - t_{ab})$. Then*

$$\sum_k d_k = 2^N, \quad (8)$$

$$\sum_k k d_k = E 2^{N-1}, \quad (9)$$

$$\sum_k k^2 d_k = (E(E+1) + Q) 2^{N-2}, \quad \text{so} \quad \text{Var}(k) = \frac{E+Q}{4}, \quad (10)$$

$$\sum_k k^3 d_k = E 2^{N-1} + 6A_2 + 6A_3, \quad (11)$$

with

$$A_2 = P_2 2^{N-1} + \left(\binom{E}{2} - P_2 \right) 2^{N-2},$$

$$A_3 = \left(\binom{E}{3} - \text{Tri} - \text{Par2} - P_3 \right) 2^{N-3} + \text{Par2} 2^{N-2} + P_3 2^{N-1}.$$

Proof. (8) counts configurations, and (9) is (7) at $m = 1$, where every single bond contributes 2^{N-1} . For (10) and (11) note that the exponent $N - |V(F)| + \text{comp}(F)$ equals $N - |F| + \text{cyc}(F)$, with $\text{cyc}(F) = |F| - |V(F)| + \text{comp}(F)$ the cyclomatic number, and that F contributes only when it is bipartite. At $|F| = 2$ a parallel pair has $\text{cyc} = 1$ and contributes 2^{N-1} , while any other pair, sharing a site or not, is a forest and contributes 2^{N-2} ; there are P_2 parallel pairs, which gives (10) once $S(2, 1) = S(2, 2) = 1$ is used and the mean $E/2$ is subtracted. At $|F| = 3$ the cases are exhausted: a triangle is an odd cycle and contributes nothing; three parallel bonds have $\text{cyc} = 2$ and contribute 2^{N-1} ; a parallel pair plus one further bond has $\text{cyc} = 1$ and contributes 2^{N-2} ; every other triple is a forest and contributes 2^{N-3} . With $S(3, 1) = 1$, $S(3, 2) = 3$, $S(3, 3) = 1$ this is (11). Appendix A carries the enumeration in full. $\square$

All four forms are exact on every committed row, in integers, with no tolerance: [CENSUS], 172 of 172 rows, and on the ten witness multigraphs of Table 3 by brute force over all 2^N configurations.

The terms Q , P_3 and Par2 vanish on a simple graph and none of them vanishes on a lattice with a length-two periodic axis, which is what pins the multigraph convention down. On the 2×2 square torus, all of whose axes are doubled, $E = 8$, $Q = 8$ and $\sum_k k^2 d_k = 320$, whereas the naive simple-graph reading $E(E+1)2^{N-2}$ would give 288; the two differ by exactly $Q2^{N-2} = 32$. Any candidate closed form that silently assumes simplicity fails on the smallest torus in these tables.

4.3 Independence of the bond indicators

Theorem 4.6 (forest criterion). *Let $F \subseteq B$ with $|F| = m$. The indicators $\{X_e : e \in F\}$ are m -wise independent under the uniform measure on configurations if and only if $(V(F), F)$ is a forest. The largest attainable m is therefore the size of a largest forest of G , namely $N - c$.*

Proof. By Proposition 4.4, for any $H \subseteq F$ the joint probability is $\Pr(X_e = 1 \ \forall e \in H) = 2^{\text{comp}(H) - |V(H)|}$ when $(V(H), H)$ is bipartite and 0 otherwise, while independence requires $2^{-|H|}$. Since $|V(H)| - \text{comp}(H) = |H| - \text{cyc}(H)$ for every graph, the two agree exactly when $\text{cyc}(H) = 0$; this must hold for every $H \subseteq F$, so F is a forest. A largest forest of G has $N - c$ bonds and is itself a forest, hence m -wise independent at $m = N - c$, and no larger F exists. $\square$

Observation 4.7 (an odd-cycle criterion that is false). Theorem 4.6 was recorded here once as “the chosen m bonds contain no odd cycle”. That is **[FALSIFIED]**: an even cycle is bipartite, so its joint probability is $2^{1-|H|}$, twice the independent value. On C_4 the four bonds contain no odd cycle, yet only the two alternating configurations satisfy all four constraints, giving $\Pr = 2/16 = 1/8$ against the $(1/2)^4 = 1/16$ that independence requires, and Proposition 4.4 returns $2^{4-4+1} = 2$ rather than $2^{4-4} = 1$. A pair of *parallel* bonds is an even 2-cycle and already fails at $m = 2$, which is exactly the $Q \neq 0$ term of (10): the error sat beside this work’s own correct statement that pairwise independence fails exactly when two bonds are parallel.

Two readings of the criterion are worth separating, because the tables contain both. Pairwise independence holds on every simple graph, precisely because simplicity excludes 2-cycles. And 3-wise independence fails on K_3 and on the 3×3 torus, both of which contain triangles, but holds on C_5 because any three bonds of C_5 form a forest — not because C_5 merely lacks triangles. The largest attainable m is $4 = N - c$ on C_5 and $L - 1$ on C_L .

The criterion is measured rather than argued. Over ten bond multigraphs and 6679 bond sets, every one of the 2^N configurations is enumerated explicitly to obtain the joint probabilities, Proposition 4.4 is evaluated independently of that enumeration, and the two are required to agree on every bond set: they do, with 0 disagreements on the eight of the ten graphs with $E \leq 12$, for which all 2^E bond subsets are affordable. Scoring the two candidate predicates against the measurement gives 0 mispredictions for the forest criterion and 207 for the odd-cycle criterion, the first disagreement being the digon of two parallel bonds at $m = 2$, and the disagreement set is exactly the bipartite non-forests, as Theorem 4.6 predicts. A largest forest is confirmed independent on all ten graphs at $m = N - c$, including $m = 9$ on the Petersen graph. The wider census behind this measurement — 25 multigraphs, 53929 bond sets, 0 forest mispredictions against 9826 for the odd-cycle criterion, and all 3359 simple cycles of length at most 8 failing independence at even and odd length alike — is reported in the accompanying data. **[PROVED]** (the criterion), **[CENSUS]** (the measurement). No moment formula, table or spectrum changes as a result: (7), (10) and (11) were correct as recorded.

5 Support of the spectrum

5.1 A graph-derived bound

Three graph invariants are used: the minimum degree $\delta_{\min} = \min_v \deg v$; the edge connectivity $\lambda = \min\{|\delta(A)| : \emptyset \neq A \neq V\}$; and the MAX-CUT value $\mu = \max\{|\delta(A)| : A \subseteq V\}$. All three are computed from the bond list alone and never from the spectrum under test: λ exactly by max-flow, since $\lambda(G) = \min_v \text{maxflow}(\text{root}, v)$ for any fixed root, with bond multiplicities as capacities so that parallel bonds are handled correctly [Edmonds and Karp, 1972, Menger, 1927]; and μ by Proposition 4.1 on bipartite rows and by exhaustive enumeration of all 2^N subsets where that is affordable, μ being NP-hard in general [Karp, 1972].

Theorem 5.1 (S4).

$$\text{Att} \subseteq \{0\} \cup (\{E\} \text{ if } G \text{ is bipartite}) \cup \{k : \lambda \leq k \leq U, k \equiv p \pmod{2}\},$$

where $U = E - \lambda$ if G is bipartite and $U = \mu$ otherwise, $p = 0$ when every vertex degree is even, and p is unconstrained otherwise.

Proof. $k = 0$ is attained by $S = \emptyset$, and $k = E$ is attained exactly when G is bipartite, by Proposition 4.1. Every nonzero attained k is at least λ and at most μ by definition of the two. If every degree is even then (5) makes every cut even, which is Theorem 3.3. If G is bipartite then Proposition 4.2 gives $k \in \text{Att}$ if and only if $E - k \in \text{Att}$, so $k \in \text{Att}$ with $k < E$ forces $E - k \geq \lambda$, that is, $k \leq E - \lambda$; and $\mu = E$ in that case, so the two upper bounds agree. No other restriction is used. $\square$

The content of Theorem 5.1 is the size of what it excludes. On a lattice with all degrees even, the parity-allowed range $\{0, 2, \dots, U\}$ loses exactly the $(\lambda - 2)/2$ values $\{2, 4, \dots, \lambda - 2\}$ at the bottom and, when the lattice is bipartite, their $(\lambda - 2)/2$ reflections $\{E - \lambda + 2, \dots, E - 2\}$ at the top. Nothing else follows from connectivity and parity, so the exceptions that remain are the whole subject of Section 6. The count is in λ , not in $\delta_{\min}$; the distinction is not academic, as Section 5.4 shows.

Where μ is not certified — a non-bipartite row too large to enumerate — the top of the bound falls back to the largest attained level, capped at $E - 1$, which Proposition 4.1 proves there because $d_E = 0$. That edge is then weak, and the census labels it as spectrum-derived rather than presenting it as a test. On the committed tables this concerns 66 of the 172 rows in the published census, which enumerates to $N \leq 27$, and 71 of the 172 in the independent re-derivation reported here, which enumerates to $N \leq 24$; all 67 bipartite rows are certified in both, by Proposition 4.1 rather than by enumeration.

5.2 Provenance is part of the result

An earlier version of this census read λ and μ from the very support it then bounded. That made the assertion $\text{Att} \subseteq \text{bound}$ a strict tautology on the 37 rows that are neither Eulerian nor bipartite, a parity test on the 68 Eulerian non-bipartite rows, and unable to report any exception at either edge of any row. The defect was demonstrated by injecting a level below the true connectivity and by deleting the connectivity level itself, both of which the old census passed, and it was reproduced independently by four readers. The census is now graph-derived, and five corruptions run as a standing regression: an injected level below λ , a deleted d_λ , an odd level injected into an Eulerian row, a deleted d_E on a bipartite row, and an injection into the formerly tautological class. Every one must be rejected and the uncorrupted control row must stay clean, or the suite fails; all seven cases pass.

Two all-rows assertions carry the content that the inclusion alone cannot.

Observation 5.2 (S4a, S4b). **S4a:** λ from max-flow equals the smallest nonzero attained cut $\min\{k > 0 : d_k > 0\}$. **S4b:** μ equals the largest attained cut, on every row where μ is graph-certified. Both hold on every committed row to which they apply: S4a on 172 of 172; S4b on 106 of 106 in the published census, which certifies μ up to $N \leq 27$, and on 101 of 101 in the independent re-derivation reported here, which certifies up to $N \leq 24$, with 0 mismatches in either. **[CENSUS]**

5.3 The saturation census

[CENSUS] Over the 172 committed rows the inclusion of Theorem 5.1 holds everywhere; 140 rows attain the whole allowed set and 32 do not. The 32 are listed in Appendix B with their exact exception sets, and every one of them has a three-entry shape: a cubic torus or a cylinder over a two-dimensional cross-section. Every square torus and every cylinder over a one-dimensional cross-section in these tables saturates the bound. Where the lattice is bipartite the exceptions come in reflected pairs k and $E - k$, as Proposition 4.2 requires. By family the census is 76 saturated of 76 square tori, 11 of 22 cubic tori and 53 of 74 cylinders.

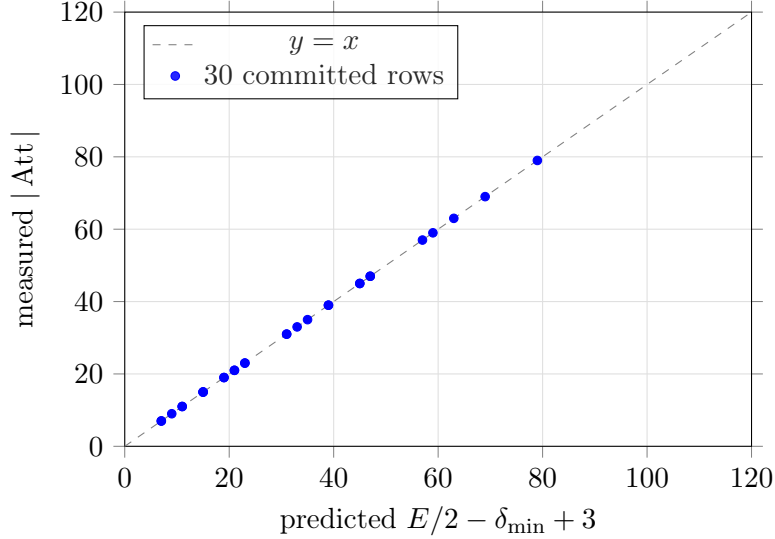


Figure 2: Observation 5.3 as a census. Each point is one committed row that is $\delta_{\min}$ -regular and bipartite with $\delta_{\min}$ even; the abscissa is the value (12) predicts from $(E, \delta_{\min})$ alone and the ordinate is the number of distinct levels present in the committed coefficient vector. All 30 points lie on the diagonal. The rows are 22 square tori and 8 cubic tori; no cylinder satisfies the hypothesis, because a cylinder is not regular. Nothing here is a proof: the same 30 points are equally consistent with a formula that fails at the first row not computed.

Observation 5.3 (S5, a level count that is not proved). If G is $\delta_{\min}$ -regular, bipartite and $\delta_{\min}$ is even, the number of distinct levels is

$$|\text{Att}| = \frac{E}{2} - \delta_{\min} + 3, \quad (12)$$

which for a four-regular bipartite torus reduces to $|\text{Att}| = N - 1$. This holds on every committed row to which it applies, 30 of 30 (Figure 2). It is **[CENSUS]**, not **[PROVED]**: (12) would follow from Theorem 5.1 if the bound were saturated, and saturation is exactly what is not proved. If it held generally, the distinct-level count of an arbitrarily large periodic lattice would be computable in constant time from $(E, \delta_{\min}, \text{bipartiteness})$ with no transfer at all, which is why the saturation failures of Appendix B are worth more than the successes.

5.4 $\lambda < \delta_{\min}$ on four cylinders, and a product formula that fails

The identity $\lambda = \delta_{\min}$ holds on 168 of the 172 committed rows and fails on exactly four, listed in Table 4. All four are cylinders whose periodic cross-section has length two, so the cross-section contributes parallel bonds while the separating cut uses only the single bonds of the open direction: $\delta_{\min} = 3$ but $\lambda = 2$. This is the same failure of the classical vertex-transitive identity that was found independently in the doubled-bond convention of a related finite-lattice tabulation, and it is why the identity is used nowhere here.

Lemma 5.4 (slice-cut lemma). *Let R be a connected cross-section on n sites with $\lambda(R) = \delta_{\min}(R) \geq 1$ and $n \geq \delta_{\min}(R) + 1$, and let $G = R \square \text{Path}_L$ with $L \geq 2$. Then*

$$\lambda(G) = \min(n, \delta_{\min}(R) + 1),$$

so G fails $\lambda = \delta_{\min}$ exactly when $\delta_{\min}(R) \geq n$.

Table 4: The four committed rows with $\lambda < \delta_{\min}$. λ is computed by max-flow on the bond multigraph and independently by exhaustive enumeration of all 2^N subsets, which is affordable at these sizes; both routes agree, and both are graph-derived. The last column is the prediction of Lemma 5.4.

shape	ends	N	E	$\delta_{\min}$	λ	Lemma 5.4
(2, 4) cylinder	periodic $\times$ open	8	14	3	2	$\min(2, 3) = 2$
(2, 6) cylinder	periodic $\times$ open	12	22	3	2	$\min(2, 3) = 2$
(2, 8) cylinder	periodic $\times$ open	16	30	3	2	$\min(2, 3) = 2$
(2, 12) cylinder	periodic $\times$ open	24	46	3	2	$\min(2, 3) = 2$

Proof sketch. A cut separating a prefix of whole slices from the rest has size $|A|$ for a full slice A , and the smallest such cut is n . Any proper non-empty $A \subset V(G)$ either meets some slice properly, in which case it pays at least $\lambda(R)$ within that slice and at least one between-slice bond per site of the intersection, or is a union of whole slices, in which case it pays at least n . Hence $\lambda(G) = \min(n, \delta_{\min}(R) + 1)$, and since $\delta_{\min}(G) = \delta_{\min}(R) + 1$ for $L \geq 2$, the failure $\lambda < \delta_{\min}$ is exactly $\delta_{\min}(R) \geq n$. $\square$

Lemma 5.4 is **[PROVED]**, and its hypothesis class was censused directly rather than assumed, in the recorded verification that accompanies the tables: 405 cylinders over designed cross-sections with 0 violations of the $\lambda(R) = \delta_{\min}(R)$ dichotomy, 84973 cylinders in a hunt for necessity counterexamples with 0 refutations of the lemma, and the predicted $\min(n, \delta_{\min}(R) + 1)$ correct on 74 of 74 committed cylinder rows with $\lambda(R) = \delta_{\min}(R)$ while 4 of the 74 have $\lambda < \delta_{\min}$. A separate census of 6342 factor pairs, multigraph factors included, checks the companion product statement $\lambda(G \square H) = \delta_{\min}(G) + \delta_{\min}(H)$ on the class $\lambda = \delta_{\min} \geq 1$, $|V| \geq \delta_{\min} + 1$ in both factors, with 0 violations; both witnesses of Observation 5.5 lie outside that class, which is what the class restriction buys. The consequence of Lemma 5.4 for the tables is that the failure on the four rows of Table 4 is not an artefact of the four lengths those tables happen to contain: it holds for this family at *every* $L \geq 2$, and the source census finds $\lambda < \delta_{\min}$ on the $(2, L)$ cylinders at every L from 2 to 14, of which the committed tables contain $L = 4, 6, 8, 12$. The corollary for simple lattices is positive and worth stating, because it isolates the doubled-bond convention as the only place the classical identity breaks here: every simple uniform hypercubic lattice with arbitrary mixed boundary conditions satisfies $\lambda = \delta_{\min} = \#\{\text{open axes}\} + 2\#\{\text{periodic axes}\}$, and the census finds $\lambda < \delta_{\min}$ on 0 of 406 simple products and 0 of 664 mixed-boundary simple hypercubic lattices. Two limits are recorded with it. The condition $\delta_{\min}(R) \geq n$ is sufficient but not necessary, and 771 cylinders fail without it; and the “only if” half of the doubled-bond failure criterion is **[CENSUS]** over 350 lattices rather than **[PROVED]**. The connectivities behind this paragraph were computed by three independent routes on 86982 graphs — Dinic max-flow, an Edmonds–Karp implementation, and exhaustive enumeration of all 2^N subsets on every graph with $N \leq 16$ — with maximum discrepancy 0 over 204567 assertions. Those four censuses are source-reported; the two facts this paper re-derives itself are the witness of Observation 5.5 and the four rows of Table 4, each confirmed here by max-flow and by exhaustive enumeration.

Observation 5.5 (a product formula that fails for multigraphs). For connected *simple* graphs the three-term identity

$$\lambda(G \square H) = \min\{\lambda(G)|V(H)|, \lambda(H)|V(G)|, \delta_{\min}(G) + \delta_{\min}(H)\} \quad (13)$$

is a theorem [Xu and Yang, 2006, Klavžar and Špacapan, 2008]. It is **[FALSIFIED]** for bond multigraphs. Take G on four sites with bonds 0–1 twice, 0–2 six times, 0–3 once and 1–3 five times, so $\delta_{\min}(G) = 6$ and $\lambda(G) = 3$, and H on three sites with bonds 0–2 twice and 1–2 twice, so $\delta_{\min}(H) = \lambda(H) = 2$. Then $\delta_{\min}(G \square H) = 8$ while $\lambda(G \square H) = 7$, and (13) predicts $\min\{3 \cdot 3, 2 \cdot 4, 6 + 2\} = 8$.

The witness was re-derived here: $\delta_{\min} = 8$, $\lambda = 7$ both by max-flow and by exhaustive enumeration of all 2^{12} subsets, and the predicted value 8. The mechanism is a product-shaped cut $A \times B$ with $|\delta_{G \square H}(A \times B)| = |\delta_G(A)||B| + |\delta_H(B)||A|$, a shape the three terms of (13) do not include; the formula fails exactly when one factor's gap $\delta_{\min} - \lambda$ exceeds the other factor's minimum degree, which parallel bonds allow and simplicity forbids. A search over 87556 factor pairs finds no counterexample at maximum bond multiplicity 4 and one at 5. This is a negative result about the literature rather than about the model, and it is reported because it dictates a working rule: no argument in this paper routes a multigraph connectivity through a published product formula. Every λ in the tables is computed on the row's own bond list, and the claims of this work are proved by case analysis rather than by (13), so none of them is affected.

6 Exception levels

The 32 non-saturated rows of Appendix B carry 53 exception levels between them, and their pattern is not random. The $2 \times 2 \times L$ tori with L odd miss exactly one level, $\mu - 2$; the $3 \times 3 \times L$ tori miss exactly $k = 8$ and the $3 \times 3 \times L$ cylinders exactly $k = 7$, independently of L ; the $2 \times 4 \times L$ and $3 \times 4 \times L$ cylinders miss $k = 7$ together with its reflection $E - 7$ when the lattice is bipartite and together with $\{\mu - 2, \mu - 1\}$ when it is not. This section proves 30 of the 53 levels unattained for whole families of lengths, and states precisely what the two proofs cannot reach.

6.1 A slice-product reduction

Every committed row is a slice product $R \square C_L$ or $R \square \text{Path}_L$ over a connected r -regular loop-free cross-section R on n sites with edge connectivity ρ : a two-dimensional row (a, L) has $R = C_a$, a three-dimensional row (a, b, L) has $R = C_a \square C_b$, periodic transfer ends give $T_L = R \square C_L$ and open ends give $P_L = R \square \text{Path}_L$. Write $S_c = \{x : (x, c) \in S\}$ for the slice of S at level c , $e(A)$ for the number of bonds of R induced by A , and

$$V_1 = \{(r+1)|A| - 2e(A) : A \subseteq V(R)\}, \quad V_2 = \{(r+2)|A| - 2e(A) : A \subseteq V(R)\}, \quad (14)$$

the cut sizes of a single slice at an end of a cylinder and of a single slice with two neighbouring slices respectively.

Theorem 6.1 (low-cut window). *Let $n \geq 2$ and set*

$$\begin{aligned} K &= \min(2\rho, n-1) && \text{for } T_L \text{ and } P_L \text{ with } L \geq 3, \\ K &= \min(2\rho, 2n-1) && \text{for } T_2, \\ K &= \min(2\rho-1, n-1) && \text{for } P_2. \end{aligned}$$

Then for every L in the stated range the attained levels in the window are exactly

<i>lattice</i>	$\{k \in (0, K] : d_k > 0\}$	<i>equals</i>
$T_L, L \geq 3$	$V_2 \cap (0, K]$	
T_2	$(V_2 \cup \{2\rho\}) \cap (0, K]$	
$P_L, L \geq 3$	$(V_1 \cup V_2) \cap (0, K]$	
P_2	$V_1 \cap (0, K]$	

In particular the low-level support of the whole infinite family is independent of L and is obtained from one enumeration of the 2^n subsets of R .

Proof outline. The slice decomposition is $|\delta(S)| = \sum_c w_c + B$ with $w_c = |\delta_R(S_c)|$ and B the between-slice total: $B = \sum_c |S_c \triangle S_{c+1}|$ on T_L with $L \geq 3$, $B = 2|S_0 \triangle S_1|$ on T_2 , where the transfer direction contributes two parallel bonds per site, and $B = \sum_{c < L-1} |S_c \triangle S_{c+1}|$ on P_L .

Call a slice *proper* if $S_c \notin \{\emptyset, V(R)\}$ and let m be the number of proper slices; an improper slice has $w_c = 0$ and a proper one $w_c \geq \rho$, so $\rho m \leq \sum_c w_c \leq k$.

Every listed value is attained. A value of V_2 is attained by $S = A \times \{c\}$ at any slice with two neighbours, a value of V_1 by an end slice of P_L , and 2ρ on T_2 by $S = (A \times \{0\}) \cup (A \times \{1\})$ with $|\delta_R(A)| = \rho$, whose between-slice term vanishes because the two slices are equal.

Nothing else is attained. Fix S with $0 < k = |\delta(S)| \leq K$. *Step 1:* $k \leq 2\rho$ forces $m \leq 2$, and $k \leq 2\rho - 1$ forces $m \leq 1$ on P_2 . *Step 2:* $m = 0$ is impossible, since if the slices are not all equal then some adjacent pair differs and contributes n to B , or $2n$ on T_2 , and on T_L with $L \geq 3$ at least two pairs differ; hence $k \geq n$ on P_L and $k \geq 2n$ on T_L , each of which exceeds the corresponding K by the caps on K alone, with no bound on r or ρ relative to n . *Step 3:* $m = 2$ forces $k = 2\rho$, both proper slices minimal and $B = 0$; but $B = 0$ makes all slices equal, so $m \in \{0, L\}$, a contradiction for $L \geq 3$, consistent only on T_2 where it yields exactly the extra value 2ρ , and excluded outright on P_2 by $K \leq 2\rho - 1$. *Step 4:* $m = 1$ leaves one proper slice $A = S_{c_0}$. If the remaining slices contain both $\emptyset$ and $V(R)$ then $B \geq n$ and $k \geq \rho + n > K$, a contradiction; so they are all $\emptyset$, or all $V(R)$, in which case $\delta(S) = \delta(A' \times \{c_0\})$ with $A' = V(R) \setminus A$. Either way $k = (r + d)|A| - 2e(A)$ with d the number of neighbouring slices of c_0 , which is 2 on T_L , both 1 and 2 on P_L with $L \geq 3$, and 1 on P_2 ; that is exactly the four clauses. $\square$

Two scope notes matter for reading the tables. The hypothesis $n \geq 2$ is needed for ρ to be defined at all, and no committed row is one-dimensional. Regularity is inessential: r enters only through $|\delta_R(A)| = r|A| - 2e(A)$, so the same proof carries verbatim for any connected loop-free cross-section once V_1 and V_2 are stated intrinsically as $\{|\delta_R(A)| + d|A|\}$; that form was verified exhaustively on eight non-regular, non-lattice cross-sections, including a doubled K_3 for which even $\rho > n$; that enumeration is recorded in the accompanying verification and was not re-run here.

Two defects in the first written version of this proof were found and repaired, and neither changes the theorem. Step 2 justified $2\rho < 2n$ by $\rho \leq r \leq n - 1$; $\rho \leq r$ is correct but $r \leq n - 1$ is false for bond multigraphs, failing on two cross-sections the tables actually use, C_2 with $n = r = 2$ and $C_2 \square C_2$ with $n = r = 4$. The inequality is not load-bearing, since the caps $K \leq n - 1$ and $K \leq 2n - 1$ do the work directly, which was confirmed exhaustively on cross-sections with $r \geq n$. A corollary claimed agreement with the absence of any $(a, b, 2)$ torus row from the exception list, but the $(2, 3, 2)$ cubic torus is in that list with exception [30]; its window is $K = \min(8, 11) = 8$, so the theorem is uncontradicted and only the bookkeeping was wrong.

Observation 6.2 (verification of Theorem 6.1). The right-hand sides of all four clauses were computed from each row's cross-section alone, by enumerating the 2^n subsets of R , and compared with the committed coefficient vector: the predicted set equals $\{k \in (0, K] : d_k > 0\}$ on 172 of 172 rows, 125 of them non-vacuously, and all 172 bond-count identities $E = L(e(R) + n)$ for a torus and $E = Le(R) + (L - 1)n$ for a cylinder reproduce the committed E . The T_2 clause is quantitative as well as qualitative: it predicts $d_8 = 18$ for the $(3, 3, 2)$ cubic torus, confirmed here by enumerating all 2^{18} subsets, and all 18 witnesses are of the form $A \times \{0, 1\}$ with $|\delta_R(A)| = 4$. The recorded verification additionally carries two families beyond the tables, $R \square C_L$ and $R \square \text{Path}_L$ for $R = C_3 \square C_3$ and $R = C_2 \square C_4$, to every length $L = 2, \dots, 40$ against an exact truncated transfer, with agreement at every length; that extension is reported from the accompanying capture and was not re-run here. **[CENSUS]** on the committed rows; the lengths beyond the tables are predictions of a **[PROVED]** theorem.

The mechanism the theorem supplies is worth stating in words, because it is what turns tabulated absences into statements about every L : a cut of size at most 2ρ can occupy at most one proper slice, so it is forced to be slice-local, and no amount of length can create a new small cut. That is why $d_8 = 0$ on the $3 \times 3 \times L$ tori and $d_7 = 0$ on the $3 \times 3 \times L$, $2 \times 4 \times L$ and $3 \times 4 \times L$ cylinders are now facts about every $L \geq 2$ rather than observations about the lengths these tables happen to contain.

6.2 The odd $2 \times 2 \times L$ cubic tori

The window K of Theorem 6.1 is bounded by the cross-section, while 28 of the 53 recorded exception levels grow with L ; the reduction cannot reach them. For one family a different argument does.

Theorem 6.3 (odd $2 \times 2 \times L$). *Let G be the $2 \times 2 \times L$ cubic torus with L odd, so $N = 4L$, $\delta_{\min} = 6$ and $E = 12L$ under the doubled-bond convention, and write $m = E - |\delta(S)|$ for the number of monochromatic, that is satisfied, bonds of a configuration. Then $m \geq 4$; the transverse count satisfies $m_{xy} \equiv 0 \pmod{4}$; $m = 6$ is impossible; and $m = 4$ is attained. Hence $\mu = E - 4$, the level $E - 6 = \mu - 2$ is unattained, the levels $E - 5$ and $E - 3$ are excluded by parity, E itself is excluded because the lattice is not bipartite when L is odd, and the exception set of the family is exactly $\{\mu - 2\}$ for every odd $L \geq 3$.*

Proof outline. Decompose $m = m_z + m_{xy}$ into the bonds along the four column z -cycles and the bonds within the layers. Each column is an odd cycle and therefore carries an odd, hence nonzero, number of monochromatic bonds, giving $m_z \geq 4$ and $m \geq 4$. The four signs around a layer 4-cycle multiply to $+1$, and each $+1$ contributes two parallel monochromatic bonds, so $m_{xy} \equiv 0 \pmod{4}$. If $m = 6$ then $m_z = 6$ and $m_{xy} = 0$; but $m_{xy} = 0$ makes every layer a checkerboard, which makes $m_z = 4r$ for an integer r , contradicting $m_z = 6$. A checkerboard with one repeated step on the odd cycle gives $m = 4$, so $\min m = 4$ and $\mu = E - 4$. The levels $E - 5$ and $E - 3$ are odd and excluded by (5) at constant degree 6, and E is excluded by Proposition 4.1 since an odd L makes the lattice non-bipartite. $\square$

Theorem 6.3 is a coordinate argument with no slice-product reduction in it, which is why it reaches levels far above K . It was verified against the committed spectra by a third algorithm that appears nowhere else in this work: an exact cyclic transfer over the $2^4 = 16$ states of one 2×2 layer, with polynomial weights in m .

Observation 6.4 (verification of Theorem 6.3). The layer transfer reproduces all 11 committed $2 \times 2 \times L$ rows, $L = 2$ to 12, coefficient by coefficient: 935 coefficients, 0 mismatches, and the configuration total exactly 2^{4L} on every row — including the $(2, 2, 12)$ row at 2^{48} configurations and the three odd rows whose top edge the support census could not graph-certify. Their $\mu = E - 4$ is therefore certified by an exact enumeration equivalent to exhaustive, rather than inferred from the spectrum. On all five committed odd- L rows the largest attained level is exactly $E - 4$, $d_{E-6} = 0$, every attained level is even and $d_E = 0$, which reproduces the entries [30], [54], [78], [102] and [126] of Appendix B. The exception set is exactly $\{\mu - 2\}$ on twelve odd lengths $L = 3, 5, 7, 9, 11, 13, 15, 21, 31, 41, 51, 61$, seven of which no committed table contains, with $m = 6$ counted 0 on every one. The decomposition $m = m_z + m_{xy}$ was itself checked configuration by configuration over all 2^{12} at $L = 3$ and all 2^{20} at $L = 5$, with zero violations of $m_z \geq 4$, of m_z even, of $m_{xy} \equiv 0 \pmod{4}$ or of the unattainability of $m = 6$, and with $\min m = 4$ and $\#\{m = 4\}$ equal to the committed d_{E-4} in both cases, namely 6 and 10. The oddness hypothesis is necessary rather than an artefact: every even L tested, 2, 4, 6, 8, 12, 20, 40, 60, is bipartite, attains $m = 0$ exactly twice and attains $m = 6$ exactly $8L$ times — the very level the odd case forbids. [CENSUS] on the committed rows, [PROVED] for the family.

6.3 What is proved and what is not

Observation 6.5 (the residue). Across the 32 non-saturated rows there are 53 exception levels. Theorem 6.1 proves 25 of them unattained, all at $k \leq K$; Theorem 6.3 proves 5 more, the $\mu - 2$ level of each odd- L $2 \times 2 \times L$ row, all far above K . So 30 of the 53 are proved unattained and 23 remain untouched, because K is bounded by the cross-section while those levels grow with L . By rows, 14 are fully explained, 14 in part — the low level proved, the reflected partner $E - k$ or the top-end pair $\{\mu - 2, \mu - 1\}$ not — and 4 not at all: the tori $(2, 3, 2)$, $(2, 3, 4)$, $(2, 3, 6)$

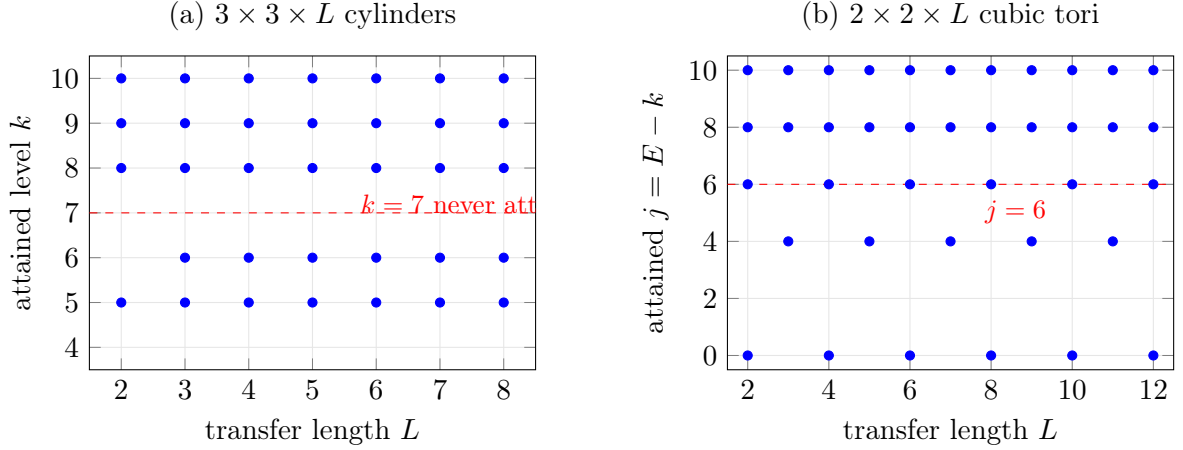


Figure 3: Two exception patterns, plotted from the committed coefficient vectors. (a) The attained levels $0 < k \leq 10$ of the nine $3 \times 3 \times L$ cylinders, $L = 2, \dots, 8$: the level $k = 7$ is absent at every length, which Theorem 6.1 proves for all $L \geq 2$ from the cross-section alone, here $n = 9$, $\rho = 4$, $K = 8$. At $L = 2$ the level 6 is absent as well, the P_2 clause, whose window is $K = \min(2\rho - 1, n - 1) = 7$. (b) The attained values of $j = E - k \leq 10$, the number of monochromatic bonds, for the eleven $2 \times 2 \times L$ cubic tori, $L = 2, \dots, 12$: $j = 6$ is absent at every odd L and present at every even L , which is Theorem 6.3 and its even- L control, where $\#\{j = 6\} = 8L$. Odd lengths also lack $j = 0$ and $j = 2$, since $m \geq 4$ there, while even lengths attain $j = 0$ exactly twice, being bipartite.

and $(2, 4, 3)$, whose only exception is $\mu - 2$ and whose cross-section is larger than $C_2 \square C_2$. All four counts were re-derived here from the committed vectors and the recomputed windows, and agree with the published census cell by cell.

Observation 5.3 is unchanged by both theorems, and it is worth being explicit about why: (12) is a claim about $\delta_{\min}$ -regular *bipartite* lattices and counts their levels at the top end as well as the bottom, while the rows Theorem 6.3 settles are not bipartite and Theorem 6.1 reaches only $k \leq K$. Nothing about (12) moved, and it remains [CENSUS].

What remains open is sharper and smaller than before. Can the one-proper-slice reduction be pushed above 2ρ , or is the obstruction of Step 3 — that at $k = 2\rho$ the two-proper-slice case opens, and beyond it the number of proper slices grows without control — essential? The residue above K is the $2 \times 3 \times L$ and $2 \times 4 \times L$ tori, whose exception is again $\mu - 2$ but whose cross-section is larger, together with the reflected partners $E - k$ of the bipartite rows, which Proposition 4.2 reduces to the same question. And does a counterexample to the observed pattern exist above K anywhere, since the constant-time level count (12) is exactly the statement that it does not?

7 Exact transfer with a diagonalized step

7.1 Decomposition

A uniform periodic lattice is propagated along one axis. Its bond multigraph decomposes into one in-plane bond multiset, shared by every cross-section, and a uniform inter-section multiplicity q assigned to *directed* steps $p \rightarrow p + 1 \pmod{\text{layers}}$, so that a transfer direction of length two is not double counted. The decomposition is validated by an explicit bond-accounting assertion,

$$\text{layers} \cdot |\text{plane bonds}| + \text{layers} \cdot q \cdot n = E,$$

and lattices that do not satisfy it are refused rather than silently mis-transformed. The assertion reproduces the committed E on every row ([CENSUS], 172 of 172).

7.2 The XOR step and its exact diagonalization

Label the 2^n spin states of a cross-section by integers $a \in \{0, \dots, 2^n - 1\}$ and let the transfer between adjacent cross-sections be

$$T[a, b] = y^{\text{popcount}(a \oplus b)}, \quad (15)$$

the weight contributed by the between-slice bonds alone; the in-plane bonds act by a diagonal operator. (15) depends on a and b only through $a \oplus b$, so it is an XOR convolution and the Walsh–Hadamard transform diagonalizes it exactly:

$$\widehat{T}(s) = \sum_d y^{\text{popcount}(d)} (-1)^{s \cdot d} = (1 + y)^{n - |s|} (1 - y)^{|s|}, \quad \text{or} \quad (1 + y^q)^{n - |s|} (1 - y^q)^{|s|} \quad (16)$$

with inter-section multiplicity q , where $|s| = \text{popcount}(s)$. One step therefore costs two transforms and a pointwise product, $O(n2^n)$ polynomial operations, instead of the $O(4^n)$ of a dense matrix–vector product, with all arithmetic in exact integers and an exact division by 2^n at each step, asserted rather than assumed. Equation (16) is the standard factorization of the Hadamard transform of a product kernel, $\sum_d y^{\text{popcount}(d)} (-1)^{s \cdot d} = \prod_i (1 + (-1)^{s_i} y)$; it was re-checked algebraically here, and the division by 2^n is the one place where a floating-point implementation would lose exactness, which is why it is asserted.

Equation (16) is the mechanism that makes exact spectra reachable at cross-sections where a dense transfer matrix does not fit in memory. It is also the limit of the method: the cost is exponential in the cross-section and only polynomial in the length, so an $L \times L \times L$ cubic torus costs $O(L^2 2^{L^2})$ polynomial operations — $L = 4$ is 2^{16} states with a degree-48 polynomial, and $L = 5$ is already 2^{25} states.

7.3 What the admission guard cost

Rows are cross-checked during the sweep by a second, dense transfer-matrix implementation that shares no code with the Hadamard path. Whether that cross-check runs is decided by an admission guard, and the guard was wrong: it charged a torus one propagation where the code performs 2^n , which oversubscribed the run deadline and silently dropped the tail of every shape list. The corrected guard charges

$$\text{starts} \cdot \text{steps} \cdot 4^n \cdot (E + 1), \quad \text{starts} = \begin{cases} 2^n, & \text{torus,} \\ 1, & \text{cylinder,} \end{cases}$$

and every row records its own measured time and operation estimate. The consequences are reported rather than quietly repaired. Under the corrected guard no new row is admitted, because the true cost is never below the old estimate, but the deadline is no longer oversubscribed: 80 of the 177 shapes in the sweep specification are admitted, and the five square tori (10, 6) through (10, 10) that were never computed become reachable. Those five would still be skipped by the dense cross-check, so computing them would add size and not an independent check. The 51 committed rows marked as skipped by the dense route are the rows on which that cross-check did not run; they rest on the sum rules, on Propositions 4.1–4.5 and Theorem 5.1, on the exhaustive certificate where one reaches, and on the external enumerators of Section 8. The labels remain truthful because the rows were computed before the guard was corrected, and no committed coefficient changes as a result of the correction.

8 The tables and how they were certified

8.1 Reach

Table 5 states the reach of the implementation: one core, exact integers, and no floating-point arithmetic anywhere in a committed number. The sweep persists each row as soon as it is

Table 5: Reach of the committed tables, by family. The shape ranges are the shapes actually present, not the shapes attempted; the largest row of each family is given by its site and bond count. The three in-field lattices are the only ones for which the bivariate polynomial $g(u, n)$ is committed.

family	shapes present	rows	largest
square tori	$L \times M$ with $2 \leq L \leq 9$, $2 \leq M \leq 10$ complete in both indices, plus $10 \times 2, \dots, 10 \times 5$	76	9×10 : $N = 90$, $E = 180$
cubic tori	$2 \times 2 \times L$, $L = 2..12$; $2 \times 3 \times L$, $L = 2..6$; $2 \times 4 \times L$, $L = 2..4$; $3 \times 3 \times L$, $L = 2..4$	22	$2 \times 2 \times 12$: $N = 48$, $E = 144$
cylinders, $R = C_a$	$a = 2..16$, $L \in \{4, 6, 8, 12\}$	53	10×12 : $N = 120$, $E = 230$
cylinders, $R = C_a \square C_b$	$(a, b) \in \{(2, 4), (3, 3), (3, 4)\}$, $L = 2..10$	21	$2 \times 4 \times 10$: $N = 80$, $E = 232$
in-field $g(u, n)$	$(2, 2)$, $(3, 3)$, $(2, 2, 2)$ only	3	$N = 9$, $E = 18$

computed, so a run that exhausts its budget keeps every lattice it finished instead of losing the group.

Per-row wall clock in the committed captures runs from 0.0s at $N = 4$ to 488.46s on the $(10, 5)$ square torus at $N = 50$, and it is not monotone in N : the largest row, the $(10, 12)$ cylinder at $N = 120$, took 2.58s because its cross-section is only ten sites, while the $(9, 10)$ torus at $N = 90$ took 462.76s. The two external enumerators below took 34.1s and 770.5s on the machine that ran them. Those are source-reported timings from a different machine and are not comparable with each other or reproducible here.

8.2 The routes and what each cannot see

Table 6 lists the routes behind the tables and, for each, its blind spot. Every row is covered by at least two routes, and each row records which ones ran at its size.

Two routes are genuinely external: they were written by other authors, in a different language, by a different algorithm. The first compiles a Gray-code enumerator in C verbatim, digest-pinned before compilation, and runs it on all 74 committed rows with $N \leq 30$, the limit that file sets for itself: 8317963920 vertex subsets in 34.1s, reproducing every committed coefficient of all 74 rows exactly, with its own N , E and $\sum_k d_k = 2^N$ agreeing with the committed sizes. Ten of the 74 have $27 < N \leq 30$ and were therefore beyond this work’s own exhaustive certificate. The second, also compiled verbatim and digest-pinned, has no size cap of its own and builds its periodic bond list from the lattice dimensions given on its command line, so unlike the first it tests the lattice construction as well as the histogram; it enumerates spin configurations rather than vertex subsets, which is the same object under $\sigma \leftrightarrow S = \{v : \sigma_v = +1\}$, so for these rows it also confirms the physical reading of d_k without using the cut-size identity. It was run on every cubic-torus row with $30 < N \leq 36$ — five rows — plus $(2, 3, 3)$ as a convention self-test and $(4, 2, 4)$ as an axis permutation of $(2, 4, 4)$: 219043594240 configurations, 576 coefficients, 770.5s wall clock on one core, every committed coefficient reproduced exactly, both sum rules holding on the enumerator’s own output against the committed N and E , odd levels absent as Theorem 3.3 requires at constant degree 6, the support top equal to the committed largest attained level, $d_k = d_{E-k}$ on the two bipartite rows, and the permuted axes giving an identical spectrum. A standing negative control perturbs one committed coefficient and one committed N per row and requires each perturbation to be reported. Three of the five rows — $(2, 4, 4)$, $(2, 3, 6)$ and $(3, 3, 4)$ — had no external certificate of any kind before that run, and $(3, 3, 4)$ is also skipped by the dense route, so for that row the run is the first check that is neither this work’s own transfer route nor a sum rule.

Table 6: The verification routes and their reach over the 172 committed rows. ‘exhaustive’ means all 2^N configurations or vertex subsets were enumerated by that route. The right-hand column states what the route cannot detect, which is the part of the residual that matters.

route	rows	what it cannot see
sum rules (8), (9)	172	any redistribution of weight among levels that preserves the two sums; a wrong bond list that preserves N and E
Hadamard transfer (this work)	172	nothing on its own: it is the route under test
dense transfer, independent code	121	the 51 rows it was not admitted to, all of them large
exhaustive 2^N , own route, $N \leq 16$	30	the 142 rows with $N > 16$
exhaustive 2^N certificate, $N \leq 27$	64	the 108 rows with $N > 27$
external Gray-code enumerator in C, $N \leq 30$	74	the bond list, which this work builds and hands to it
second external enumerator, cubic $30 < N \leq 36$	5	every non-cubic row, and the three cubic rows beyond $N = 36$
cut characterization 3.1–3.3	59 exhaustive	the 113 rows above its enumeration limit
λ by max-flow, μ where certified	172 / 106	the spectrum: these give a bound, not an equality
injection regression	7 cases	nothing about the uncorrupted rows

8.3 The residual

Observation 8.1 (what is not certified). Of the 172 committed rows, 79 carry an exhaustive certificate over all 2^N configurations from at least one route and 93 do not (Figure 4). The 93 are 34 square tori, 56 cylinders and the three cubic tori $(2, 2, 10)$, $(2, 2, 11)$ and $(2, 2, 12)$, whose projected costs at the measured rate are about 1, 17 and 275 hours of exhaustive walk. Those three are $2 \times 2 \times L$ rows, which the layer transfer of Observation 6.4 reproduces exactly at every L by a third algorithm, so they are weaker evidence than the rest but not unevidenced; the projection is printed in the capture rather than left implicit. The second external route reaches only fully periodic simple-cubic shapes, and the largest cross-section it propagated is the 3×3 one, $n = 9$ sites; the largest fully periodic cross-section the corrected dense guard admits at all is $n = 6$. What neither external route tests is the bond list this work builds: a mis-constructed lattice would survive both, and that residual is covered instead by the bond-accounting assertion of Section 7 and by the independent re-derivation described next.

The re-derivation reported in this paper is a further route of the same kind and shares no code with any of the above. It rebuilds the three families from their geometric definitions; reproduces all 15,170 committed coefficients; certifies 59 rows by exhaustive cut histogram at $N \leq 24$ and 44 rows by the distinct-cut quotient at $N \leq 20$; graph-certifies μ on 101 rows; recomputes the 32 non-saturated rows and their 53 exception levels cell by cell against the published census, all ten columns of each; recomputes the windows K of Theorem 6.1 from each cross-section and the 25/28 split; verifies the predicted low support on all 172 rows; reproduces all 11 committed $2 \times 2 \times L$ rows by an independently written layer transfer; re-derives the witness of Observation 5.5 and the four rows of Table 4; checks (2) in exact rational arithmetic and (6) on explicitly constructed duals; and re-runs the independence measurement of Section 4.3, including the in-field tables of Section 8.4. It reports 0 failures over 2741 assertions, split 2204 for the tables and the support census, 329 for the theorem-level checks and 208 for the table-level cross-checks, in about 150s of wall clock on one core. Its own honest limit is that its exhaustive enumerations are budget-bounded, so its checks of (3) are complete only up to $N = 24$ and its

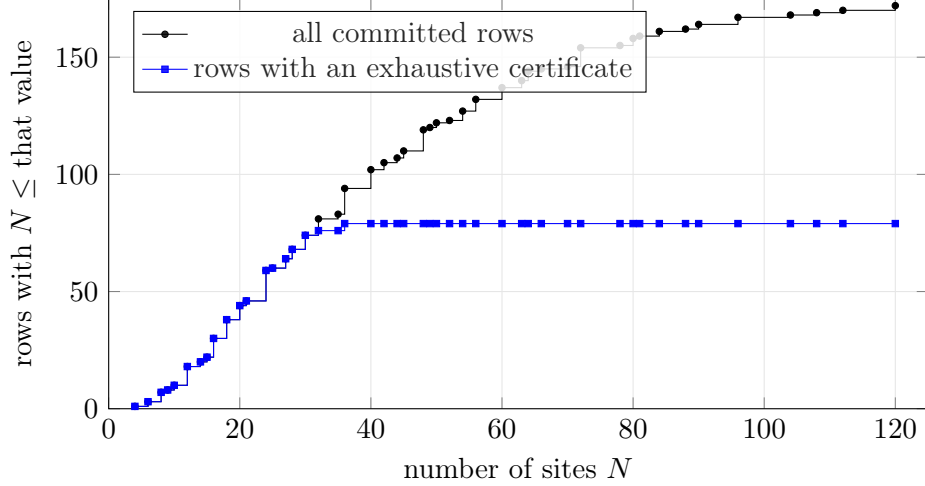


Figure 4: Where the evidence thins out. Both curves are cumulative counts over the 172 committed rows, against N ; the upper curve is every row and the lower one the rows for which some route enumerated all 2^N configurations or vertex subsets. The lower curve stops rising at $N = 36$ and ends at 79 rows, so the vertical gap at large N is the 93 rows of Observation 8.1. The steps are at $N = 16$ (this work’s own enumeration, 30 rows), $N = 27$ (the exhaustive certificate, 64), $N = 30$ (the first external enumerator, 74) and $N = 36$ (the second, 79).

check of (6) only where the relevant cycle space has dimension at most 23, while its checks of Theorem 3.3, Proposition 4.4, (10) and (11) are complete on all 172 rows.

8.4 The in-field bivariate tables

With a uniform field the object is $g(u, n)$ of (4), and three mutually independent routes were required to agree cell by cell before any value was committed: the in-field Hadamard transfer, an exhaustive 2^N enumeration over a coordinate-built bond list, and the distinct-cut polynomial

$$\sum_{u,n} g(u, n) y^u z^n = \sum_{\text{distinct cuts } C} y^{|C|} (z^{|S_C|} + z^{N-|S_C|}), \quad (17)$$

which is Theorem 3.2 with the field fugacity attached. They agree on all three committed lattices, and the tables are printed in Appendix C. Four structural invariants hold on every one ([CENSUS]): the total $\sum_{u,n} g(u, n) = 2^N$; the distinct-cut count 2^{N-c} , namely 8, 256 and 128; the field sum rule $\sum_n n \sum_u g(u, n) = N2^{N-1}$, namely 32, 2304 and 1024; and the w -marginal $d_u = \sum_n g(u, n)$, which equals the committed zero-field row of the same lattice coefficient by coefficient. The tables were recomputed here cell by cell by exhaustive enumeration and by (17), including the spin-flip symmetry $g(u, n) = g(u, N - n)$ and the zero-field marginals. The three lattices are also the honest limit: no bivariate table exists beyond $N = 9$.

Observation 8.2 (a candidate answer, not a theorem). At zero field the data descends to the cut space, a 2^{N-c} -element vector space over $\text{GF}(2)$, because the identification $S \sim V \setminus S$ leaves $|\delta(S)|$ invariant; that quotient is exactly what makes (6), and hence the Pfaffian evaluation, available. A uniform field breaks the identification, since $|S|$ and $|V \setminus S|$ differ while $\delta(S) = \delta(V \setminus S)$, and (17) is where the breakage is visible: its two terms no longer coincide. The field therefore does not merely add a variable to the same object; it destroys the quotient that makes the zero-field spectrum a linear-algebraic quantity. This is recorded as a candidate answer to Open Question 9.1 and not as a separation result: it has not been turned into a theorem, and it is testable against committed data only once bivariate in-field tables exist at more than the three present sizes.

9 Discussion

The four questions this work was organised around now have answers of different strength, and it is worth stating the strength explicitly.

Open Question 9.1 (the in-field spectrum). Is the in-field degeneracy spectrum a strictly harder object than the zero-field one?

The zero-field spectrum is equivalent to the even-subgraph polynomial by Proposition 3.4 and is literally the dual graph’s even-subgraph spectrum by Proposition 3.6, hence Pfaffian-computable on planar lattices. With a uniform field the expansion becomes a sum over polygon configurations weighted by the enclosed area, and no Pfaffian-type evaluation is known. Observation 8.2 proposes a structural reason — the field destroys the $S \sim V \setminus S$ quotient — which is a candidate answer and not a separation theorem. Testing it needs bivariate tables at more than three sizes. *Status: open, with a candidate mechanism.*

Open Question 9.2 (the reach of the exact transfer). How far does the Hadamard-diagonalized exact transfer reach in three dimensions?

The committed cubic rows reach a fully periodic cross-section of $n = 9$ on $(3, 3, 2)$, $(3, 3, 3)$ and $(3, 3, 4)$; the largest cubic row is $2 \times 2 \times 12$ at $N = 48$, $E = 144$; and $4 \times 4 \times 4$, with 2^{16} cross-section states, was not attempted. Whether that is reachable with exact integer arithmetic, with modular arithmetic plus the Chinese remainder theorem, or not at all, is an engineering question with a definite answer that this work does not supply. The two cubic rows whose dense cross-check did not fit the budget, $(3, 3, 3)$ and $(3, 3, 4)$, are exactly the rows at the present boundary. *Status: open.*

Open Question 9.3 (the unattained levels). Which parity- and connectivity-allowed levels are never attained?

Theorem 5.1 gives the allowed set; Theorems 6.1 and 6.3 prove 30 of the 53 recorded exceptions unattained for whole families; 23 remain, concentrated in the $2 \times 3 \times L$ and $2 \times 4 \times L$ tori and in the reflected partners of the bipartite rows, which Proposition 4.2 reduces to the same question. A cross-section-free explanation of the low-level exceptions is no longer wanted, because Theorem 6.1 is one. *Status: partially answered; the residue is listed, not explained.*

Open Question 9.4 (constraints on a closed form). Do the exact small-lattice spectra constrain any candidate closed form?

Any proposed closed form for the three-dimensional partition function must reproduce the committed multiplicities exactly and must be consistent with Theorems 3.1–3.3, with Proposition 3.6 in the two-dimensional case, and with Propositions 4.1–4.5, which are closed-form constraints needing no table at all: four exact moments, $d_0 = 2^c$, and the criterion that the spectrum is reflection-symmetric exactly on bipartite bond multigraphs. The exact spectra of all 22 committed cubic tori are tabulated, and Appendix B records for every one of them the allowed levels that are never attained. Whether these are already strong enough to falsify the currently circulating candidate expressions has not been checked: no candidate expression has been transcribed into this work or tested against the tables. A proof or counterexample for the converse of the symmetry claim is no longer wanted either, since Proposition 4.2 proves both directions. *Status: open, with the constraints stated.*

Limitations

Four limits bound every claim above. First, the exhaustive certificates stop at $N = 36$, and 93 rows rest on weaker evidence; the reach of each route is tabulated rather than summarized, and

the per-row labels name the routes that actually ran. Second, the two external enumerators do not test the bond list, so a systematic error in the lattice construction would survive both; the bond-accounting assertion and the geometric re-derivation cover that risk, and they are assertions about this work’s own convention rather than independent confirmations of it. Third, (12) is a census on 30 rows; the paper uses it nowhere as a proof, and a reader should not either. Fourth, the timings quoted are those of the machines that produced the committed captures and of the machine that produced the re-derivation, and neither is a benchmark of the other.

Two further limits are worth naming because they concern what the tables are, not how well they are checked. The families are uniform and periodic or singly open: no disordered coupling, no nonuniform field, no other boundary geometry and no other lattice type appears anywhere in the data, and nothing here constrains those cases. And the in-field data stops at $N = 9$, so the candidate mechanism of Observation 8.2 cannot be tested against it at all.

10 Data and code availability

All data and code supporting this paper are in the source research repository [GeWu Journal Auto Author, 2026], at commit 5bca12abab3f1eecbbfa1b29cc8454ac32216c02. The committed tables are `results/spectra_square_tori.json`, `results/spectra_cubic_tori.json` and `results/spectra_cylinders.json`, with human-readable counterparts `results/spectra_square_tori.md`, `results/spectra_cubic_tori.md` and `results/spectra_cylinders.md`; the in-field tables are `results/infield_bivariate.json` and `results/infield_bivariate.md`. The support census is `results/support.md`, the exception-level verification `results/exception_levels.md`, the odd- $2 \times 2 \times L$ verification `results/odd_torus_2x2xL.md`, the independence measurement `results/independence_criterion.md`, the two external enumerator captures `results/external_c_spectrum.md` and `results/gray_code_n_gt_30.md`, the admission-guard analysis `results/sweep_guard_analysis.md` and the structural captures `results/symmetry.md`, `results/structure.md` and `results/verify.log`. The method is `code/ising_dos.py`, `code/sweep.py`, `code/symmetry.py`, `code/support.py`, `code/structure.py`, `code/independence_criterion.py`, `code/infield_bivariate.py`, `code/exception_levels.py`, `code/odd_torus_2x2xL.py`, `code/external_c_spectrum.py`, `code/gray_code_n_gt_30.py` and `code/check_contrib.py`. Material that is archived rather than merged, including the two external enumerators in C and the written proof of Theorem 6.3, is under `contrib/` and `related/`. Every quantitative figure of this paper is drawn from the committed coefficient vectors by a script that reads those files and nothing else, and no figure is smoothed, binned, fitted or extrapolated; the sole exception is Figure 1, a hand-drawn schematic of the cut convention that carries no data at all and says so in its caption. The eight figure data files are plain two- and three-column integer text, are distributed with the paper under `figures/`, and carry a header line naming the rows they came from.

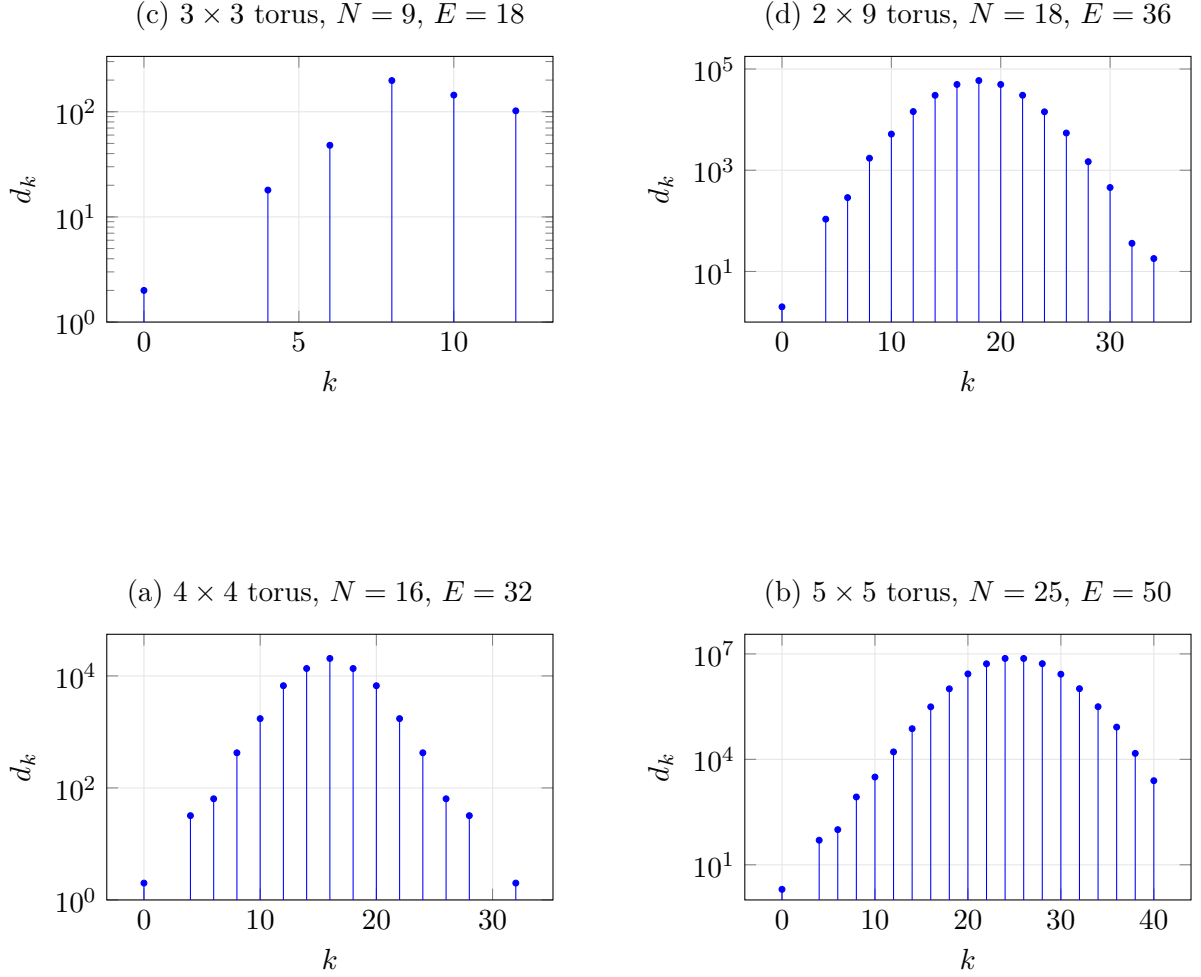


Figure 5: Four committed spectra, logarithmic ordinate, one stem per attained level k . Every height is a committed integer; nothing is smoothed, binned, fitted or extrapolated, and no level with $d_k = 0$ is drawn, so the gaps are the support. (a) The 4×4 torus is bipartite, so its spectrum is symmetric about $E/2 = 16$ by Proposition 4.2 and attains $d_E = 2^c = 2$ at the top. (b) The 5×5 torus is not bipartite: the spectrum is asymmetric, $d_E = 0$ by Proposition 4.1, and only even k occur because every degree is 4 (Theorem 3.3). (c) The frustrated 3×3 torus has no odd level at all, again by degree parity and not by frustration, and stops at $k = 12 = \mu < E = 18$. (d) The 2×9 torus is bipartite and symmetric; its doubled bonds make $Q \neq 0$, and the absence of $k = 2$ despite $\lambda = 4$ is the bottom edge of Theorem 5.1.

A The third-moment form in full

This appendix carries the enumeration behind (11), which Proposition 4.5 only summarizes.

By (7) at $m = 3$, with $S(3, 1) = 1$, $S(3, 2) = 3$ and $S(3, 3) = 1$,

$$\sum_k k^3 d_k = \sum_{|F|=1} 2^{N-1} + 3 \cdot 2! \sum_{|F|=2} 2^{N-|V(F)|+\text{comp}(F)} + 1 \cdot 3! \sum_{|F|=3} 2^{N-|V(F)|+\text{comp}(F)},$$

where each sum runs only over the bipartite F . The first term is $E2^{N-1}$. For the other two it is enough to know the exponent, and the exponent is $N - |F| + \text{cyc}(F)$ with $\text{cyc}(F) = |F| - |V(F)| + \text{comp}(F)$, so the task is to list the bipartite bond sets of size two and three by cyclomatic number.

$|F| = 2$. A parallel pair, of which there are $P_2 = Q/2 = \frac{1}{2} \sum_{a < b} t_{ab}(t_{ab} - 1)$, has $|V(F)| = 2$, $\text{comp}(F) = 1$ and $\text{cyc} = 1$, and being a 2-cycle it is bipartite, so it contributes 2^{N-1} . Any other pair is a forest: two bonds sharing a site have $|V(F)| = 3$ and $\text{comp}(F) = 1$, and two disjoint bonds have $|V(F)| = 4$ and $\text{comp}(F) = 2$; both give $\text{cyc} = 0$ and contribute 2^{N-2} . There are $\binom{E}{2} - P_2$ of them. Hence

$$A_2 = \sum_{|F|=2} 2^{N-|V(F)|+\text{comp}(F)} = P_2 2^{N-1} + \left(\binom{E}{2} - P_2 \right) 2^{N-2}.$$

$|F| = 3$. A three-bond set with $\text{cyc} = 0$ is a forest and contributes 2^{N-3} . With $\text{cyc} \geq 1$ the cycle it contains has length at most three, so it is either a 2-cycle, that is, a parallel pair, or a triangle; there is no other possibility, and this is the step where a multigraph differs from a simple graph. Enumerating the four cases:

- *three parallel bonds*: $\text{cyc} = 2$, bipartite because every cycle is even, contributing 2^{N-1} ; there are $P_3 = \sum_{a < b} \binom{t_{ab}}{3}$ of them.
- *exactly two parallel bonds plus one further bond*: $\text{cyc} = 1$, bipartite, contributing 2^{N-2} ; there are $\text{Par2} = \sum_{a < b} \binom{t_{ab}}{2} (E - t_{ab})$ of them. The two counts are disjoint, because $\binom{t}{3}$ selects three bonds on one site pair while $\binom{t}{2} (E - t)$ selects two on one pair and one elsewhere.
- *a triangle*: three bonds on three distinct site pairs spanning three sites, of which there are Tri ; the cycle is odd, so Proposition 4.4 gives 0 and the term drops out.
- *everything else*: a forest, contributing 2^{N-3} ; there are $\binom{E}{3} - \text{Tri} - \text{Par2} - P_3$ of them.

Hence

$$A_3 = \left(\binom{E}{3} - \text{Tri} - \text{Par2} - P_3 \right) 2^{N-3} + \text{Par2} 2^{N-2} + P_3 2^{N-1},$$

and $\sum_k k^3 d_k = E2^{N-1} + 6A_2 + 6A_3$, which is (11).

The enumeration is complete at $|F| = 3$, and it would be misleading if read as a parity statement: at $|F| = 4$ the even cycle C_4 also contributes 2^{N-3} rather than 2^{N-4} , which is the content of Theorem 4.6 and the reason Observation 4.7 is a falsification rather than a curiosity.

All four moment forms were evaluated in exact integers on all 172 committed rows and on the ten witness multigraphs of Table 3, where they were also checked against brute-force enumeration of all 2^N configurations; there is no tolerance anywhere in the comparison. On a simple graph $Q = P_3 = \text{Par2} = 0$, and (10) and (11) reduce to $E(E+1)2^{N-2}$ and to a form involving only $\binom{E}{3} - \text{Tri}$; on a lattice with a doubled axis none of the three vanishes, and the smallest case is the 2×2 torus with $E = 8$, $Q = 8$ and $\sum_k k^2 d_k = 320$ against the naive 288.

B The non-saturated rows

Table 7 lists all 32 committed rows for which Att is strictly smaller than the allowed set of Theorem 5.1, with the recomputed window K of Theorem 6.1. Every quantity in it is graph-derived except μ on the rows where μ is not certified, and there the bound is spectrum-derived and capped at $E - 1$ as Section 5 describes; the exception levels themselves are read off the committed coefficient vectors. The rows are ordered by family and shape.

Table 7: The 32 committed rows whose support does not saturate the bound of Theorem 5.1. K is the window of Theorem 6.1 for that row’s cross-section, ‘bound’ is the number of allowed levels in the Theorem 5.1 set and $|\text{Att}|$ the number attained; their difference is the number of exception levels listed. A dagger marks a level proved unattained by Theorem 6.3 rather than by Theorem 6.1. ‘status’ is **all** when every exception level of the row is proved unattained, **part** when the low level is proved and its reflected partner $E - k$ or the top-end pair $\{\mu - 2, \mu - 1\}$ is not, and **none** when no proof reaches the row; the counts are 14, 14 and 4.

family	shape	bip.	N	E	λ	μ	$ \text{Att} $	bound	K	exception levels	status
cub. torus	$2 \times 2 \times 3$	no	12	36	6	32	14	15	3	$30^\dagger$	all
cub. torus	$2 \times 2 \times 5$	no	20	60	6	56	26	27	3	$54^\dagger$	all
cub. torus	$2 \times 2 \times 7$	no	28	84	6	80	38	39	3	$78^\dagger$	all
cub. torus	$2 \times 2 \times 9$	no	36	108	6	104	50	51	3	$102^\dagger$	all
cub. torus	$2 \times 2 \times 11$	no	44	132	6	128	62	63	3	$126^\dagger$	all
cub. torus	$2 \times 3 \times 2$	no	12	36	6	32	14	15	8	30	none
cub. torus	$2 \times 3 \times 4$	no	24	72	6	64	30	31	5	62	none
cub. torus	$2 \times 3 \times 6$	no	36	108	6	96	46	47	5	94	none
cub. torus	$2 \times 4 \times 3$	no	24	72	6	64	30	31	7	62	none
cub. torus	$3 \times 3 \times 3$	no	27	81	6	54	25	26	8	8	all
cub. torus	$3 \times 3 \times 4$	no	36	108	6	84	40	41	8	8	all
cylinder	$2 \times 4 \times 2$	yes	16	40	5	40	31	33	7	7, 33	part
cylinder	$2 \times 4 \times 3$	yes	24	64	5	64	55	57	7	7, 57	part
cylinder	$2 \times 4 \times 4$	yes	32	88	5	88	79	81	7	7, 81	part
cylinder	$2 \times 4 \times 5$	yes	40	112	5	112	103	105	7	7, 105	part
cylinder	$2 \times 4 \times 6$	yes	48	136	5	136	127	129	7	7, 129	part
cylinder	$2 \times 4 \times 7$	yes	56	160	5	160	151	153	7	7, 153	part
cylinder	$2 \times 4 \times 8$	yes	64	184	5	184	175	177	7	7, 177	part
cylinder	$2 \times 4 \times 9$	yes	72	208	5	208	199	201	7	7, 201	part
cylinder	$2 \times 4 \times 10$	yes	80	232	5	232	223	225	7	7, 225	part
cylinder	$3 \times 3 \times 2$	no	18	45	5	33	28	30	7	6, 7	all
cylinder	$3 \times 3 \times 3$	no	27	72	5	54	50	51	8	7	all
cylinder	$3 \times 3 \times 4$	no	36	99	5	75	71	72	8	7	all
cylinder	$3 \times 3 \times 5$	no	45	126	5	96	92	93	8	7	all
cylinder	$3 \times 3 \times 6$	no	54	153	5	117	113	114	8	7	all
cylinder	$3 \times 3 \times 7$	no	63	180	5	138	134	135	8	7	all
cylinder	$3 \times 3 \times 8$	no	72	207	5	159	155	156	8	7	all
cylinder	$3 \times 4 \times 2$	no	24	60	5	52	45	49	7	6, 7, 50, 51	part
cylinder	$3 \times 4 \times 3$	no	36	96	5	84	78	81	8	7, 82, 83	part
cylinder	$3 \times 4 \times 4$	no	48	132	5	116	110	113	8	7, 114, 115	part
cylinder	$3 \times 4 \times 5$	no	60	168	5	148	142	145	8	7, 146, 147	part
cylinder	$3 \times 4 \times 6$	no	72	204	5	180	174	177	8	7, 178, 179	part

C Selected exact data

Table 8 prints six committed spectra in full. They are the rows a reader can verify without this work’s code: each has $N \leq 12$, so all 2^N configurations can be listed by hand or in a few lines of any language, and the two moment sums in the last column are the check.

The in-field bivariate tables $g(u, n)$ of (4) follow, for the two committed lattices small enough to print: the (2, 2) square torus with $N = 4$, $E = 8$ and 45 cells, and the (3, 3) square torus with $N = 9$, $E = 18$ and 190 cells. Only the nonzero rows in u are printed; every omitted row is

identically zero, and columns run over $n = 0, \dots, N$. The row sums are the committed zero-field coefficients d_u , the column-weighted sums give the field sum rule $\sum_n n d'_n = N 2^{N-1}$, and the totals are 2^N : 16 and 512. The third committed lattice, the $(2, 2, 2)$ cubic torus with $N = 8$, $E = 24$ and 225 cells, is in the accompanying data. All three were recomputed here cell by cell, by exhaustive enumeration over a coordinate-built bond list and by the distinct-cut polynomial (17), and both routes reproduce every cell, every marginal and all four invariants of Section 8.4.

The $(2, 2)$ square torus, $N = 4$, $E = 8$. Distinct cuts $8 = 2^{N-c}$, total $16 = 2^4$, field sum $32 = 4 \cdot 2^3$.

Table 8: Six committed spectra in full, each small enough to check by hand against all 2^N configurations. Only the attained levels are listed; every omitted k in $[0, E]$ has $d_k = 0$. The last two columns are the total $\sum_k d_k$, which must equal 2^N by (8), and the first moment $\sum_k k d_k$, which must equal $E 2^{N-1}$ by (9).

lattice	ends	N	E	bip.	attained levels ($k : d_k$)	sums
sq. torus 2×2	periodic	4	8	yes	0 : 2, 4 : 12, 8 : 2	$2^4 = 16$, $8 \cdot 2^3 = 64$
sq. torus 2×3	periodic	6	12	no	0 : 2, 4 : 18, 6 : 26, 8 : 12, 10 : 6	$2^6 = 64$, $12 \cdot 2^5 = 384$
sq. torus 3×3	periodic	9	18	no	0 : 2, 4 : 18, 6 : 48, 8 : 198, 10 : 144, 12 : 102	$2^9 = 512$, $18 \cdot 2^8 = 4608$
cub. torus $2 \times 2 \times 2$	periodic	8	24	yes	0 : 2, 6 : 16, 8 : 30, 10 : 48, 12 : 64, 14 : 48, 16 : 30, 18 : 16, 24 : 2	$2^8 = 256$, $24 \cdot 2^7 = 3072$
cylinder $(2, 4)$, $R = C_2$	open	8	14	yes	0 : 2, 2 : 6, 3 : 8, 4 : 22, 5 : 24, 6 : 34, 7 : 64, 8 : 34, 9 : 24, 10 : 22, 11 : 8, 12 : 6, 14 : 2	$2^8 = 256$, $14 \cdot 2^7 = 1792$
cylinder $(3, 4)$, $R = C_3$	open	12	21	no	0 : 2, 3 : 18, 4 : 24, 5 : 36, 6 : 102, 7 : 192, 8 : 342, 9 : 566, 10 : 732, 11 : 738, 12 : 588, 13 : 384, 14 : 222, 15 : 108, 16 : 36, 17 : 6	$2^{12} = 4096$, $21 \cdot 2^{11} = 43008$

u	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	d_u
0	1	.	.	.	1	2
4	.	4	4	4	.	12
8	.	.	2	.	.	2

The $(3, 3)$ square torus, $N = 9$, $E = 18$. Distinct cuts $256 = 2^{N-c}$, total $512 = 2^9$, field sum $2304 = 9 \cdot 2^8$. Only even u occur, as Theorem 3.3 requires at constant degree 4, and none above $\mu = 12$, as Theorem 5.1 requires for a non-bipartite row. The allowed set is therefore $\{0, 4, 6, 8, 10, 12\}$ and the six printed rows are exactly that set, so this row saturates its bound; the levels $u = 14$ and $u = 16$, although even, are outside it because the top edge is μ and not $E - \lambda$ on a non-bipartite lattice. The spin-flip symmetry is visible in the table as $g(u, n) = g(u, N - n)$, and the row sums d_u are the committed zero-field coefficients 2, 18, 48, 198, 144, 102.

u	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	d_u
0	1	.	.	.	.	.	.	.	.	1	2
4	.	9	.	.	.	.	.	.	9	.	18
6	.	.	18	6	.	.	6	18	.	.	48
8	.	.	18	36	45	45	36	18	.	.	198
10	.	.	.	36	36	36	36	.	.	.	144
12	.	.	.	6	45	45	6	.	.	.	102

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