

Exact level structure of finite uniform Ising lattices: the density of states is a cut-size spectrum

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Abstract

Rewriting the partition function of a finite Ising lattice as a sum of 2^N exponentials, one per spin configuration, turns the question “how many distinct energies does this lattice have?” into a counting question about the exponents. We show that the exponent attached to a configuration is exactly the size of a cut in the bond graph of the lattice: the number of unsatisfied bonds equals $|\delta(S)|$ for the set S of down spins. The density of states is therefore a cut-size spectrum, and four consequences follow immediately: a parity constraint on the attainable values, an empty band $1 \leq u \leq \lambda - 1$ below the edge connectivity λ , a multiplicity mirror $m(u) = m(|E| - u)$ that holds if and only if the bond graph is bipartite, and a four-branch formula for the number of levels. For even-sided tori the formula is proved rather than observed: an explicit construction with non-adjacent rows of down spins reaches every even cut size up to $|E|/2$, and the mirror supplies the rest. This settles the published $2 \times 6 \times 2$ example, whose bond graph is $C_4 \square C_6$ and whose 2^{24} terms collapse to exactly $4 \cdot 6 - 1 = 23$ distinct exponent values at zero field. The count is a property of the bond graph, not of the name of the lattice: six twenty-four-site specifications share that graph and one table, while the same box with doubled length-2 bonds or with open boundaries gives 33 or 41 levels. A census of 94 uniform lattices with up to 1000 sites supports the formula on 92; for the two exceptions every missing level is certified by an induced-edge obstruction. We also record where a folklore rule fails: $\lambda = \delta_{\min}$ holds on all 94 lattices of the census, 6 of which carry doubled bonds, but fails on 16 of the 204 small specifications tested, every one of them doubled and five vertex-transitive under explicitly certified automorphisms of the bond multiset.

1 Introduction

The Ising model [11] is one of the few many-body models that can be solved exactly, but exactness comes in two very different flavours. The classical solutions [4, 13] describe the thermodynamic limit of an infinite lattice; a *finite* lattice has instead a finite spectrum, and everything about its equilibrium statistics is contained in the exact density of states

$$g(b, M) = \#\{s \in \{\pm 1\}^V : b(s) = b, M(s) = M\}, \quad (1)$$

the number of configurations with b satisfied nearest-neighbour bonds and magnetisation M . For a uniform coupling J and a uniform field h ,

$$Z(\beta, J, h) = \sum_{b, M} g(b, M) e^{\beta J(2b - |E|)} e^{\beta h M}, \quad (2)$$

so the table g is a complete finite-lattice solution: every thermodynamic quantity at every temperature and field follows from it by summation. The same table is the object that series-expansion and

finite-size-scaling methods consume, and its zero-field extreme — the maximum number of unsatisfied bonds — is the maximum-cut problem that underlies the exact treatment of spin-glass ground states on planar lattices [2].

1.1 The question

Wei [15] rewrote the partition function of a finite lattice as a sum of 2^N exponentials, one per spin configuration, valid in any dimension and with or without a field. Writing $u(s) = |E| - b(s)$ for the number of *unsatisfied* bonds,

$$Z(\beta, J, h) = e^{\beta J |E|} \sum_{s \in \{\pm 1\}^V} \exp(-2\beta J u(s) + \beta h M(s)), \quad (3)$$

the sum has 2^N terms but, in general, far fewer distinct exponent values. That observation raises two questions, which the published problem statement poses explicitly [9]: how such a sum can be evaluated for large N , especially at nonzero field; and how the multiplicity of the repeated exponent values — the degeneracy of each energy level — can be determined. The worked example in Wei [15] is a $2 \times 6 \times 2$ lattice whose $2^{24} = 16\,777\,216$ terms are reported to collapse to 23 distinct exponent values; the bond and boundary convention that produces that count is not stated. The natural follow-up question is whether the number of distinct exponent values depends on the finite lattice at all, and if so on what.

1.2 What this paper shows

The central observation is that the exponent attached to a configuration is a cut size. If $S = \{i : s_i = -1\}$ is the set of down spins, a bond is unsatisfied exactly when it has one end in S , so

$$u(s) = |E| - b(s) = |\delta(S)|, \quad (4)$$

the number of edges of the bond graph G crossing the cut $(S, V \setminus S)$ (Theorem 1). Identity (4) is elementary, but it relocates the whole problem: the zero-field spectrum of a uniform Ising lattice *is* the cut-size spectrum of its bond graph, with multiplicities, and (1) becomes a count of subsets of prescribed size and prescribed cut. This is stronger than the classical reduction of spin-glass *ground states* to maximum cut [2], which concerns one level; here every level is a cut value and every multiplicity is a number of subsets.

From (4) the paper derives four structural results (Section 3), a level count (Section 4), a connectivity result with an explicit family of counterexamples (Section 5), a census of 94 uniform lattices with up to 1000 sites (Section 6), the resolution of the $2 \times 6 \times 2$ example (Section 7), a coincidence class of twenty-four-site lattices (Section 8) and closed forms in one dimension (Section 9). In summary:

1. **Parity and a connectivity gap.** Since $|\delta(S)| = \sum_{v \in S} \deg v - 2|E(S)|$, every level has the parity of some vertex-degree sum; in particular all cut sizes are even when G is Eulerian. No level lies strictly between 0 and the edge connectivity $\lambda(G)$, so the band $1 \leq u \leq \lambda - 1$ is always empty (Proposition 3).
2. **A mirror that characterises bipartiteness.** The zero-field multiplicities satisfy $m(u) = m(|E| - u)$ for every u if and only if G is bipartite, and equivalently the level set is symmetric under $u \leftrightarrow |E| - u$ (Theorem 4). Hence the exact ground-state bond count vanishes, $b_{\min} = 0$, precisely on bipartite bond graphs. The same involution refines to the joint table: flipping

one colour class exchanges the total and the staggered magnetisation while reflecting u , so the magnetisation-resolved table at u equals the staggered-magnetisation-resolved table at $|E| - u$; the naive joint mirror $g(b, M) = g(|E| - b, -M)$ is *false* on every bipartite lattice we examined, and we exhibit the failing entries exactly.

3. **A four-branch level count.** Under an explicit interior hypothesis, the number of distinct exponent values at zero field is one of four closed expressions in $|E|$, λ , $u_{\max}$, bipartiteness and parity of the degrees (Proposition 5).
4. **Even-sided tori, proved.** For $C_L \square C_M$ with $L, M \geq 4$ even the interior hypothesis is a theorem: an explicit construction with pairwise non-adjacent rows of down spins attains every even cut size from $\lambda = 4$ to $|E|/2$, and the mirror supplies the upper half, so the level set is $\{0\} \cup \{u \text{ even} : 4 \leq u \leq 2LM - 4\} \cup \{2LM\}$ and the level count is $LM - 1$ (Theorem 6).
5. **Wei's example.** The $2 \times 6 \times 2$ periodic lattice under the simple-bond convention has bond graph $K_2 \square C_6 \square K_2 \cong C_4 \square C_6$, so its level count is $4 \cdot 6 - 1 = 23$ — the published number, obtained from the formula with no enumeration, and with the convention now pinned. The same 24 sites with doubled length-2 bonds give 33 levels and with open boundaries give 41; all three are confirmed by exhaustive enumeration of 2^{24} configurations (Section 7). At nonzero field the natural count is the number of attainable (b, M) pairs, which is 208 on those same twenty-four sites.
6. **One graph, six names.** Six uniform specifications of 24 sites — the tori 4×6 and 6×4 , the periodic boxes $2 \times 6 \times 2$ and $2 \times 2 \times 6$, and two mixed-boundary variants — are one bond graph, $C_4 \square C_6$, certified by explicit vertex maps checked bond by bond, and they have byte-identical tables $g(b, M)$: all $49 \times 25 = 1225$ entries agree. A degeneracy table therefore does not identify the uniform family that produced it. Two certified negative controls show that the level count is not enough either: the doubled 12×2 torus shares N , $|E|$, the degree sequence, bipartiteness and the level count 23, yet has 247 attainable (b, M) pairs instead of 208 (Proposition 14).
7. **Where a folklore rule fails.** $\lambda = \delta_{\min}$ holds on all 94 lattices of the census and is proved here for open chains, grids and boxes and for two-dimensional tori $C_L \square C_M$, $L, M \geq 3$; it is classical for simple vertex-transitive graphs [12, 14]. It fails for bond *multigraphs*: among the 204 uniform specifications with $N \leq 12$, 16 have $\lambda < \delta_{\min}$, every one carrying parallel bonds, and five of them are vertex-transitive under explicitly certified automorphisms (Proposition 10). The classical theorem does not reach them, because they are not simple graphs.
8. **One dimension in closed form.** The complete bivariate table $g(b, M)$ of the ring C_n and the open chain P_n is given by run-counting formulas, valid at any field and any size, from which the level counts $\lfloor n/2 \rfloor + 1$ and n and the cut spectra $\{0, 2, \dots, 2\lfloor n/2 \rfloor\}$ and $\{0, 1, \dots, n - 1\}$ follow (Theorem 15). These formulas are not new — the same counting problem is classical in one dimension [1, 3, 5, 6] — and they are used here as an exact oracle sharing no code with the enumeration engines, and as the proof of the interior hypothesis in one dimension.
9. **Two exact exceptions, certified.** The interior hypothesis holds on 92 of the 94 censused lattices. It fails on exactly two, both 5-regular and both frustrated: the $2 \times 3 \times 4$ and $2 \times 3 \times 3$ boxes with one open axis. Both miss $u = \lambda + 1 = 6$ and $u = \lambda + 2 = 7$, and the larger also misses $u = 50, 51$ just below $u_{\max} = 52$. Every one of these absences is certified here by an induced-edge obstruction: writing $u = dn - 2e$ for a d -regular graph and an n -subset with e

internal edges, the required e lies outside the exact range of internal-edge counts attained at that size (Proposition 7). The near-maximum absences are a genuinely different phenomenon from the near-minimum ones, and we do not have a criterion that predicts them.

On Wei’s first question — evaluating the sum for large N — the paper gives a partial answer. The level *set* (not the multiplicities) is obtained by a boolean reachability recursion along the narrowest axis whose cost is linear in the swept length and exponential only in the width of that axis, with no ceiling on the size of the integers involved; it is what allows the census to reach $N = 1000$, where 2^{1000} configurations are summarised in a fraction of a second. Whether the multiplicities admit the same treatment is open (Section 11).

1.3 What is not claimed

Everything here is finite, exact and uniform: one coupling J , one field h , hypercubic bond graphs in one, two and three dimensions, open or periodic per axis. No subexponential-in- N algorithm uniform in lattice width is claimed, no closed form for the multiplicities in dimension two or three, no computational-hardness classification, and no general criterion for the interior hypothesis. Disordered couplings, antiferromagnetic signs on non-bipartite lattices and multi-valued couplings are outside the scope of every theorem. The statement that the coincidence class of the full table equals the isomorphism class is *verified* on all 412 uniform specifications with $N \leq 18$ and not proved in general. The asymptotic branch of the published problem — a closed form or a hardness answer for arbitrary size — is untouched.

Section 2 fixes conventions; Sections 3–4 contain the structural theory; Sections 5–8 the census, the example, and the coincidence class; Section 9 the one-dimensional closed forms; Section 10 the verification record, including what was recomputed independently and what could not be; and Section 11 the limitations and open questions. Appendix A collects the proofs, Appendix B the one-dimensional derivations, Appendix C the data tables and Appendix D the enumeration protocol and the provenance record.

2 Lattices, bond graphs and conventions

2.1 From a lattice specification to a bond graph

A *uniform lattice specification* is a triple $(\mathbf{n}, \mathbf{c}, \kappa)$: a shape $\mathbf{n} = (n_1, \dots, n_d)$ with $d \in \{1, 2, 3\}$ and $n_a \geq 2$ except for one-dimensional chains, a boundary condition $\mathbf{c} = (c_1, \dots, c_d)$ with $c_a \in \{\text{o}, \text{p}\}$ (open or periodic) per axis, and a bond convention $\kappa \in \{\text{simple}, \text{doubled}\}$. Its sites are the tuples $V = \prod_a \{0, \dots, n_a - 1\}$, indexed lexicographically, and its bonds are the nearest-neighbour pairs along each axis, with the ends of a periodic axis identified. Site coordinates are written $i = (i_1, \dots, i_d)$ and $N = |V| = \prod_a n_a$.

The *bond graph* $G = (V, E)$ of a specification is the multigraph whose edges are those bonds. It is a multigraph for one reason only: when an axis has length two and is periodic, stepping forward and stepping backward from a site lead to the same neighbour, so the two bonds have the same endpoints. The simple convention keeps one of them; the doubled convention keeps both. The two conventions therefore give different physics on the same sites — different bond counts, different degrees, different spectra — and a specification is simple if and only if it has no parallel pair, which is a property of the bond multiset and not of the declared convention. Writing t_{ij} for the multiplicity of the site pair $\{i, j\}$ in the bond multiset, we write $Q = \sum_{\{i, j\}} t_{ij}(t_{ij} - 1)$ for the number of ordered pairs of distinct bonds that share both endpoints; $Q = 0$ means the bond graph

is a simple graph, and a doubled length-two periodic axis contributes 2 for each site pair it doubles. Where the distinction matters it is stated explicitly, and every table in this paper records it.

Apart from length-two periodic axes the bond graph is simple, and the familiar identifications hold: a one-dimensional periodic specification is the cycle C_n and an open one is the path P_n ; a two-dimensional periodic specification with $n_1, n_2 \geq 3$ is the torus $C_{n_1} \square C_{n_2}$; an open two-dimensional specification is the grid $P_{n_1} \square P_{n_2}$; and the Cartesian product $\square$ is associative, so a periodic box is $C_{n_1} \square C_{n_2} \square C_{n_3}$ [10]. We use K_2 for the single bond, so a length-two axis contributes K_2 under the simple convention and K_2 with a doubled edge under the doubled one; the identity $K_2 \square K_2 = C_4$ (Lemma 11) is the reason several twenty-four-site specifications turn out to be one graph.

2.2 Spins, bonds and the quantities that are counted

A configuration is $s \in \{\pm 1\}^V$; its *down-spin set* is $S = \{i \in V : s_i = -1\}$ and its magnetisation is

$$M(s) = \sum_{i \in V} s_i = N - 2|S|. \quad (5)$$

A bond $\{i, j\} \in E$ is *satisfied* when $s_i s_j = +1$ and *unsatisfied* when $s_i s_j = -1$; we write $b(s)$ and $u(s)$ for the two counts, so that $b(s) + u(s) = |E|$ and

$$\sum_{\{i, j\} \in E} s_i s_j = b(s) - u(s) = 2b(s) - |E|. \quad (6)$$

The energy of s is $-J \sum s_i s_j - hM(s)$, so at $h = 0$ the energy levels are in bijection with the values of u , and at $J > 0$ the ground states are the configurations with $u = 0$ if such configurations exist.

For a subset $S \subseteq V$ we use the standard notation [7]: $\delta(S)$ is the set of edges with exactly one endpoint in S , $E(S)$ the set of edges with both endpoints in S , and $\lambda(G) = \min\{|\delta(S)| : \emptyset \neq S \subsetneq V\}$ the edge connectivity, with $\delta_{\min}(G) = \min_v \deg v$ the minimum degree. We also write

$$u_{\max}(G) = \max_{S \subseteq V} |\delta(S)|, \quad b_{\min}(G) = |E| - u_{\max}(G), \quad (7)$$

so $u_{\max}$ is the maximum cut size and $b_{\min}$ the smallest number of satisfied bonds over all configurations; $b_{\min} = 0$ means the lattice has an unsatisfied-bond-free configuration, i.e. a ground state of the ferromagnet with every bond satisfied.

The exact tables studied here are

$$g(b, M) = \#\{s : b(s) = b, M(s) = M\}, \quad n(b) = \sum_M g(b, M), \quad (8)$$

$$m(u) = \#\{S \subseteq V : |\delta(S)| = u\}, \quad \mathcal{C}(G) = \{u : m(u) > 0\}, \quad (9)$$

and $\mathcal{C}(G)$ is the *level set* of the lattice, whose cardinality $|\mathcal{C}(G)|$ is the number of distinct zero-field energies. Two normalisations are worth stating once, because they are the checks that every table in this paper passes: $\sum_{b, M} g(b, M) = 2^N$, and the magnetisation marginal is graph-free,

$$\sum_b g(b, M) = \binom{N}{(N+M)/2}, \quad (10)$$

since it counts subsets of a given size. Finally, a subset S is an *interior witness* for u if $|\delta(S)| = u$ and $\emptyset \neq S \subsetneq V$, and a *boundary witness* otherwise; the two boundary subsets $\emptyset$ and V both give $u = 0$, so $m(0) = 2$ for every bond graph and $0 \in \mathcal{C}(G)$ always.

2.3 Two counts, and which one Wei's question asks for

Equation (3) has 2^N terms whose exponents are $-2\beta J u(s) + \beta h M(s)$. At $h = 0$ the number of distinct exponents is exactly $|\mathcal{C}(G)|$. At $h \neq 0$ and generic $\beta J, \beta h$ it is the number of attainable pairs (b, M) , which is larger: for the $2 \times 6 \times 2$ lattice of Section 7 it is 208 rather than 23. We report both counts throughout, and use “level count” only for $|\mathcal{C}(G)|$. Nothing in the theory below depends on this distinction, since both counts are read off the same table g .

3 The density of states is a cut-size spectrum

3.1 The identity

Theorem 1 (cut identity). *Let $G = (V, E)$ be the bond graph of a uniform lattice specification, counted with multiplicity. For every configuration $s \in \{\pm 1\}^V$ with down-spin set $S = \{i : s_i = -1\}$,*

$$u(s) = |\delta(S)|, \quad b(s) = |E| - |\delta(S)|, \quad \sum_{\{i,j\} \in E} s_i s_j = |E| - 2|\delta(S)|. \quad (11)$$

Proof. A bond $\{i, j\}$ is unsatisfied exactly when $s_i \neq s_j$, that is, exactly when it has one endpoint in S and one in $V \setminus S$; this is the definition of $\delta(S)$, and parallel copies of a bond are counted separately on both sides. The other two identities are (6) together with $b + u = |E|$. $\square$

The identity needs neither uniformity of the coupling nor the lattice structure: it holds for any graph and any ± 1 labelling. Uniformity enters later, where degrees and parity are used.

Corollary 2 (the tables are cut counts). *For every bond graph G and every b, M ,*

$$n(|E| - u) = m(u), \quad g(b, M) = \#\left\{S \subseteq V : |\delta(S)| = |E| - b, |S| = \frac{N-M}{2}\right\}, \quad (12)$$

and the partition function is a sum over subsets,

$$\begin{aligned} Z(\beta, J, h) &= e^{\beta J |E|} \sum_{S \subseteq V} \exp\left(-2\beta J |\delta(S)| + \beta h (N - 2|S|)\right) \\ &= e^{\beta J |E|} \sum_{u \in \mathcal{C}(G)} e^{-2\beta J u} \sum_M g(|E| - u, M) e^{\beta h M}. \end{aligned} \quad (13)$$

At $h = 0$ the number of distinct exponent values in (3) is $|\mathcal{C}(G)|$, the number of distinct cut sizes of the bond graph.

Proof. The map $s \mapsto S = \{i : s_i = -1\}$ is a bijection from $\{\pm 1\}^V$ to the power set of V , and (5) and (6) express $M(s)$ and $b(s)$ in terms of S alone, as $N - 2|S|$ and $|E| - |\delta(S)|$. Each identity is therefore Theorem 1 applied once per index pair, and summing over the pairs gives (13). $\square$

Corollary 2 is the precise sense in which the finite-lattice density of states is a cut-size spectrum, and it converts the degeneracy question of Wei [15] into a question about the cut sizes of a graph. Only the zero-field statement is a purely graph-theoretic one: at $h \neq 0$ the exponents also depend on $|S|$, and the relevant count is the number of attainable pairs $(|\delta(S)|, |S|)$.

3.2 Parity and the connectivity gap

Proposition 3 (parity and gap). *Let $G = (V, E)$ be a finite bond graph.*

(a) *For every $S \subseteq V$,*

$$|\delta(S)| = \sum_{v \in S} \deg v - 2|E(S)|, \quad \text{so} \quad |\delta(S)| \equiv \sum_{v \in S} \deg v \pmod{2}. \quad (14)$$

If every degree is even, every level is even. If G has vertices of both degree parities, both parities occur among the values $\sum_{v \in S} \deg v$ and (14) excludes nothing.

(b) *If G is connected and $N \geq 2$, then no u with $1 \leq u \leq \lambda(G) - 1$ is a level, while 0 and $\lambda(G)$ are levels. Equivalently, the bond values $b = |E| - \lambda + 1, \dots, |E| - 1$ are absent from the zero-field spectrum, and the first excitation above the ground state costs $2J\lambda$.*

Proof. (a) Every edge incident with a vertex of S is either internal to S , and then counted twice in $\sum_{v \in S} \deg v$, or crosses the cut, and then counted once. (b) For $\emptyset \neq S \subsetneq V$ the set $\delta(S)$ is a cut in the sense of the definition of $\lambda(G)$, so $|\delta(S)| \geq \lambda$; equality is attained because the minimum is over finitely many subsets. The only subsets that are not interior are $\emptyset$ and V , both giving $u = 0$. $\square$

Part (a) re-proves, in two lines, the halving of the level spectrum of periodic lattices: a periodic box in which every site has even degree has only even cut sizes, so its 2^N configurations occupy at most $|E|/2 + 1$ levels. Part (b) is the reason the spectra in Figure 1 start at $u = \lambda$ rather than at $u = 1$: on a four-regular torus no configuration has one, two or three unsatisfied bonds.

3.3 A mirror symmetry that characterises bipartiteness

Theorem 4 (mirror). *Let $G = (V, E)$ be a finite bond graph with c connected components, and let $m(u)$ and $\mathcal{C}(G)$ be as in (9). The following are equivalent:*

- (i) *G is bipartite;*
- (ii) *$m(u) = m(|E| - u)$ for every u ;*
- (iii) *$\mathcal{C}(G)$ is symmetric: $u \in \mathcal{C}(G)$ if and only if $|E| - u \in \mathcal{C}(G)$.*

Consequently $u_{\max} = |E|$ and $b_{\min} = 0$ if and only if G is bipartite, so the exact ground-state bond count is a complete frustration invariant for these families.

Proof. (i) $\Rightarrow$ (ii). Fix a bipartition $V = A \sqcup A'$. Every edge joins A to A' , so for any $S \subseteq V$ the symmetric difference $S \triangle A$ flips the status of every edge: an edge crosses $\delta(S)$ exactly when it does not cross $\delta(S \triangle A)$. Hence $|\delta(S \triangle A)| = |E| - |\delta(S)|$, and $S \mapsto S \triangle A$ is an involution on the subsets of V that pairs the level u with the level $|E| - u$ bijectively.

(ii) $\Rightarrow$ (i). The subsets with $|\delta(S)| = 0$ are exactly the unions of connected components, so $m(0) = 2^c > 0$; by (ii), $m(|E|) = 2^c > 0$ as well. A subset with $|\delta(S)| = |E|$ has every edge crossing it, which is a bipartition.

(ii) $\Leftrightarrow$ (iii) is immediate, since a bijection of the level set is what (ii) provides and (iii) is its support. The final claim is $u_{\max} = |E| \iff |E| \in \mathcal{C}(G) \iff 0 \in \mathcal{C}(G)$ under the symmetry, and $0 \in \mathcal{C}(G)$ always. $\square$

Two refinements are needed to state the magnetisation-resolved version correctly, and both were measured before being believed.

A symmetry that always holds. For *any* bond graph, the global flip $s \mapsto -s$ maps S to $V \setminus S$, preserves $|\delta(S)|$ and negates M ; therefore

$$g(b, M) = g(b, -M) \quad \text{for every } b, M, \quad (15)$$

bipartite or not. This was verified entry by entry on every lattice enumerated exhaustively in this work.

A naive mirror that fails. It is tempting to combine Theorem 4 with (15) and assert $g(b, M) = g(|E| - b, -M)$ on bipartite lattices. That statement is false. The colour-class flip of the proof above maps u to $|E| - u$ but does not map M to $-M$: writing $M_A = \sum_{i \in A} s_i$, $M_{A'} = \sum_{i \in A'} s_i$ and the staggered magnetisation $M_s = M_A - M_{A'}$, it sends

$$(u, M, M_s) \mapsto (|E| - u, -M_s, -M). \quad (16)$$

Hence the exact joint statement is that the triple histogram $\mathcal{G}(u, M, M_s)$ satisfies $\mathcal{G}(u, M, M_s) = \mathcal{G}(|E| - u, -M_s, -M)$, and summing over M_s and using (15) gives the marginal consequence

$$\tilde{g}(u, M) = \tilde{h}(|E| - u, M), \quad (17)$$

$$\tilde{g}(u, M) = \#\{s : u(s) = u, M(s) = M\}, \quad \tilde{h}(u, M_s) = \#\{s : u(s) = u, M_s(s) = M_s\}.$$

In words: on a bipartite lattice the magnetisation-resolved table at level u equals the *staggered*-magnetisation-resolved table at level $|E| - u$. Equations (16) and (17) were checked entry by entry on the 4×4 and 6×6 tori, the $2 \times 2 \times 2$ periodic box and the open 3×4 grid, while the naive mirror was checked and fails on all four (Section 10; the exact counterexample and the full verification are in Appendix A.6). Physically, (17) says that the mirror exchanges the ferromagnetic and the antiferromagnetic order parameter: the colour-class flip sends M to M_s , not to $-M$, so the naive form would require the two order parameters to coincide up to sign configuration by configuration, which fails already for the two uniform states, whose staggered magnetisation is 0 while $M = \pm N$. On the 4×4 torus, 104 of the 1089 entries of the (u, M) table differ from their naive mirror image: the level $u = 4$ carries 16 configurations at $M = \pm 14$ and none at $M = \pm 2$, while the level $u = 28$ carries 16 at $M = \pm 2$ and none at $M = \pm 14$.

4 How many levels does a finite lattice have?

4.1 A four-branch count

Call a value u *parity-admissible* for G if $u \equiv \sum_{v \in S} \deg v \pmod{2}$ for some $S \subseteq V$; by Proposition 3(a) every level is parity-admissible. The admissible set is the even integers when every degree is even, and all integers otherwise: if some vertex has odd degree, $\{v\}$ gives an odd value, and either an even-degree vertex or a two-element subset gives an even one. Write

$$\mathcal{P}(G) = \begin{cases} \{u : u \equiv 0 \pmod{2}\}, & \text{every degree even,} \\ \mathbb{Z}, & \text{otherwise,} \end{cases} \quad t(G) = \begin{cases} |E| - \lambda, & G \text{ bipartite,} \\ u_{\max}, & \text{otherwise.} \end{cases} \quad (18)$$

The upper endpoint for a bipartite graph is forced by the mirror: if u is a level so is $|E| - u$, hence the largest level below $|E|$ is $|E| - \lambda$. The hypothesis that makes the count closed is that nothing else is missing.

(H) *every parity-admissible u with $\lambda \leq u \leq t(G)$ is a level of G , realised by an interior witness.*

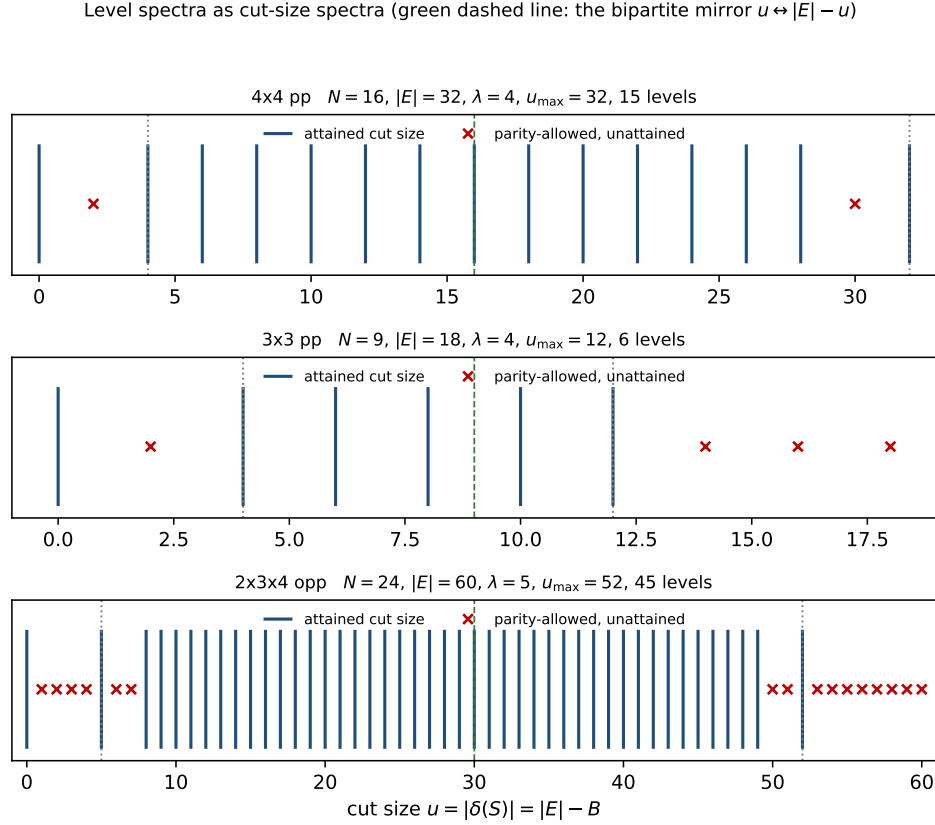


Figure 1: Level spectra as cut-size spectra, for three lattices of the census. Vertical bars are the attained cut sizes $u = |\delta(S)| = |E| - b$; crosses are values that the parity constraint of Proposition 3 allows but no subset attains; the dotted lines mark λ , $u_{\max}$ and $|E|/2$, the last being the mirror centre of Theorem 4. Top: the 4×4 torus, bipartite and Eulerian, with every even value from $\lambda = 4$ to $|E| = 32$ attained as Theorem 6 requires. Middle: the 3×3 torus, Eulerian but frustrated, with $u_{\max} = 12 < |E| = 18$ and no mirror. Bottom: the $2 \times 3 \times 4$ box with boundary conditions opp, the largest of the two exceptions of Proposition 7, with the interior holes at $u = 6, 7$ and the near-maximum holes at $u = 50, 51$ visible as crosses inside the parity-allowed range of that lattice.

Proposition 5 (level count). *Let G be a connected bond graph with $N \geq 2$, $|E|$ bonds, edge connectivity λ and maximum cut $u_{\max}$. If (H) holds then*

$$\mathcal{C}(G) = \{0\} \cup \{u \in \mathcal{P}(G) : \lambda \leq u \leq t(G)\} \cup \{|E|\} \text{ if } G \text{ is bipartite}, \quad (19)$$

and the number of distinct zero-field exponent values is

$$|\mathcal{C}(G)| = \begin{cases} |E|/2 - \lambda + 3, & G \text{ bipartite, all degrees even,} \\ |E| - 2\lambda + 3, & G \text{ bipartite, degrees not all even,} \\ (u_{\max} - \lambda)/2 + 2, & G \text{ non-bipartite, all degrees even,} \\ u_{\max} - \lambda + 2, & G \text{ non-bipartite, degrees not all even.} \end{cases} \quad (20)$$

In the second and fourth branches “degrees not all even” is exactly the condition $\mathcal{P}(G) = \mathbb{Z}$: it covers graphs whose degrees genuinely vary, such as an open grid, and also regular graphs of odd degree, such as the 5-regular boxes of Section 4.3 and the 3-regular torus $K_2 \square C_M$. The census tables use the same classification, where it is called *mixed degree*; it is a statement about parity and not about regularity, and the two are independent.

Proof. Every level is 0 or at least λ (Proposition 3), is parity-admissible, and is at most $u_{\max}$. If G is bipartite, the mirror of Theorem 4 sends a level $u > |E| - \lambda$ to a level $|E| - u < \lambda$, which must be 0; so $u = |E|$, and $\mathcal{C}(G) \subseteq \{0, |E|\} \cup (\mathcal{P}(G) \cap [\lambda, |E| - \lambda])$. If G is non-bipartite then $|E| \notin \mathcal{C}(G)$ and $\mathcal{C}(G) \subseteq \{0\} \cup (\mathcal{P}(G) \cap [\lambda, u_{\max}])$. Both inclusions are the first display, and (H) turns them into equalities. Counting gives (20): in the bipartite even-degree case λ and $|E|$ are both even, so the interval $\mathcal{P}(G) \cap [\lambda, |E| - \lambda]$ has $|E|/2 - \lambda + 1$ elements and the two extra levels 0, $|E|$ add 2; in the bipartite mixed case the interval has $|E| - 2\lambda + 1$ elements; in the non-bipartite even-degree case λ and $u_{\max}$ are even and the interval has $(u_{\max} - \lambda)/2 + 1$ elements; in the non-bipartite mixed case it has $u_{\max} - \lambda + 1$. Each count gains one more level from $u = 0$. $\square$

Hypothesis (H) is not a formality. It is exactly the statement that the spectrum has no interior holes, and Section 4.3 exhibits two lattices where it fails. What makes Proposition 5 useful is that (H) can be proved family by family: it follows from the closed forms of Section 9 for every ring and every open chain, and from the construction below for every even-sided torus.

4.2 Even-sided tori: bars reach half the spectrum

Theorem 6 (complete spectrum of an even-sided torus). *Let $T = C_L \square C_M$ with $L, M \geq 4$ even, so $N = LM$, $|E| = 2LM$, every degree is 4, $\lambda = 4$, and T is bipartite. Then every even u with $4 \leq u \leq 2LM - 4$ is a level of T , no odd u is, and*

$$\mathcal{C}(T) = \{0\} \cup \{u \text{ even} : 4 \leq u \leq 2LM - 4\} \cup \{2LM\}, \quad |\mathcal{C}(T)| = LM - 1 = \frac{|E|}{2} - \lambda + 3 = N - 1. \quad (21)$$

In particular hypothesis (H) holds for T , and Proposition 5 applies with no unproved assumption.

Proof sketch. Write sites as (x, y) with $x \in \mathbb{Z}_L$, $y \in \mathbb{Z}_M$. Select the $M/2$ rows $y = 0, 2, \dots, M - 2$; since $M \geq 4$ no two of them are adjacent, so no bond joins two selected rows. A *bar* of length k , $1 \leq k \leq L - 1$, is the set $\{(x, 0) : 0 \leq x \leq k - 1\}$. Its cut consists of the two horizontal bonds at its ends — distinct because $k < L$ — and the $2k$ vertical bonds incident with its sites, so

$$|\delta(\text{bar of length } k)| = 2k + 2. \quad (22)$$

Because no bond joins two selected rows, an edge crosses the union of bars placed in distinct selected rows exactly when it crosses one of them, so cuts of such unions add. A union of j bars of lengths $k_1, \dots, k_j$ therefore has cut $2(k_1 + \dots + k_j + j)$, and since each k_i ranges freely over $\{1, \dots, L-1\}$ the sum $k_1 + \dots + k_j$ ranges over every integer from j to $j(L-1)$. The attainable values are thus $2t$ with $t \in [2j, jL]$ for some $j \in \{1, \dots, M/2\}$; these intervals are contiguous because $2(j+1) \leq jL+1$ whenever $L \geq 3$, and their union is $[2, ML/2]$. Hence every even u with $4 \leq u \leq ML = 2LM/2 = |E|/2$ is attained by an explicit interior witness. The mirror of Theorem 4 maps that set onto the even values in $[|E|/2, |E|-4]$, Proposition 3 excludes the odd values and $u = 1, 2, 3$, and Theorem 4 supplies $u = |E|$. The count is $1 + ((2LM-4-4)/2+1) + 1 = LM-1$. Full details, including the additivity argument and the exact witness for each u , are in Appendix A.2. $\square$

The construction is not merely existential: for each of the fourteen tori on which it was run, the single-bar cuts were measured against the bond multiset and found to be $2k+2$, additivity was re-measured on up to forty random unions of bars, and the resulting even values up to $|E|/2$ were compared with the exact level set. Coverage is complete precisely when M is even — the ten tori with an even second side — and incomplete for the four with an odd second side, where the construction reaches only $2\lfloor M/2 \rfloor L$ and the remaining even values (22, 24 for the 5×5 torus, 14 for 3×5 , 58, 60 for 4×15 and 8 for 3×3) are supplied by the enumeration rather than by a witness. For the 10×10 torus, with $N = 100$, the construction is the only available evidence: it certifies every even level from 4 to 100 by an explicit subset, and the mirror the rest, for a lattice whose 2^{100} configurations no enumeration in this work touches.

The same argument applied to odd-sided tori gives nothing, because there the mirror is unavailable and $u_{\max} < |E|$; those lattices are covered by the census rather than by a theorem. Whether (H) holds for every torus $C_L \square C_M$ with $L, M \geq 3$ is open.

4.3 Two exact exceptions, and why they are exact

Proposition 7 (the exception class). *Hypothesis (H) holds for 92 of the 94 lattices of the census of Section 6. It fails for exactly two, both 5-regular, both non-bipartite, both with one open axis:*

shape	boundaries	N	$ E $	λ	$u_{\max}$	$ \mathcal{C} $	(20)
$2 \times 3 \times 4$	opp	24	60	5	52	45	49
$2 \times 3 \times 3$	opp	18	45	5	33	28	30

The missing levels are $\{6, 7, 50, 51\}$ and $\{6, 7\}$ respectively. For each of them, and for every subset size n , the identity $u = dn - 2e$ between the cut size u , the number e of internal edges and the common degree $d = 5$ forces e to a single value; exhaustive enumeration of all 2^N subsets gives, for each n , the exact set of attainable e , and the required value lies outside it. For $u = 6$ and $u = 7$ in both lattices, and for $u = 51$ in the $2 \times 3 \times 4$ box, the required e lies outside the whole interval $[e_{\min}(n), e_{\max}(n)]$ of internal-edge counts attained at size n ; for $u = 50$ it lies inside that interval at $n = 12$ — where $e_{\min} = 4$ and $e_{\max} = 24$ — but no 12-subset has exactly $e = 5$ internal edges, although $e = 4$ and $e = 6$ both occur. The gaps are therefore theorems about these two graphs and not artefacts of sampling.

Proof. The level sets are exact: both lattices are within the $N \leq 24$ enumeration cap, so every one of the 2^{24} and 2^{18} subsets was enumerated and its cut computed. The per-size data are the tables of $e_{\min}(n)$ and $e_{\max}(n)$ in Appendix C, obtained from the same enumeration; substituting $u = 5n - 2e$ and checking the parity of $5n - u$ at each n gives the certificate described in the statement. As a control, the same test was applied to every attained level of both lattices and excluded none. $\square$

The two kinds of gap have different statuses. The near-minimum gaps $u = 6, 7$ are instances of a general mechanism: a cut of size $\lambda + 1$ or $\lambda + 2$ needs a subset that is almost a minimum cut but carries more internal edges than any subset of that size has, and the per-size bound $u \geq dn - 2e_{\max}(n)$ detects this. The near-maximum gaps $u = 50, 51$ just below $u_{\max} = 52$ need the dual bound $u \leq dn - 2e_{\min}(n)$, and for $u = 50$ even that is not enough: the obstruction is the absence of 12-subsets with exactly five internal edges, a fact about this graph with no structural explanation that we can offer. We therefore record the following as a conjecture rather than a result.

Conjecture 8 (interior gaplessness). Let G be the bond graph of a uniform lattice specification. Then (H) holds for G unless some subset size n violates the induced-edge bounds, that is, unless for some parity-admissible $u \in [\lambda, t(G)]$ the required internal-edge count $e = (dn - u)/2$ is unattainable at every size n . Equivalently, interior holes in a uniform lattice spectrum are always explained by induced-edge sparsity or density at some fixed subset size.

Conjecture 8 is consistent with all 94 censused lattices and with the two exceptions, for which it is verified by exhaustive enumeration rather than deduced. It is not a criterion: computing $e_{\min}(n)$ and $e_{\max}(n)$ for all n is not cheaper than enumerating the subsets, and the $u = 50$ gap shows that the interval bounds alone do not suffice.

5 The first level: edge connectivity of the bond graph

Proposition 3(b) makes the edge connectivity λ of the bond graph the location of the first excited level, and Proposition 5 makes it one of the four inputs to the level count. It is therefore worth knowing when λ equals the minimum degree $\delta_{\min}$, which is the value one reads off a lattice picture.

Proposition 9 (simple uniform lattices). Let $G = G_1 \square \cdots \square G_d$ be a product of paths and cycles, $G_a = P_{n_a}$ with $n_a \geq 2$ or $G_a = C_{n_a}$ with $n_a \geq 3$, and put $c_a = \delta_{\min}(G_a)$, that is $c_a = 1$ for a path and $c_a = 2$ for a cycle. Then

$$\lambda(G) = \delta_{\min}(G) = \sum_{a=1}^d c_a. \quad (23)$$

This covers every simple bond graph of the census of Section 6: open chains, rings, grids, cylinders, tori and boxes, with any mixture of open and periodic axes, since a periodic axis of length two under the simple convention contributes the path $P_2 = K_2$.

The proof is an induction on d with a fibre argument, and is given in Appendix A. Two special cases are worth stating separately because they are the ones used in Theorem 6: a torus $C_L \square C_M$ with $L, M \geq 3$ has $\lambda = 4$, and an open grid $P_L \square P_M$ has $\lambda = 2$, the corner degree.

The equality fails as soon as the bond graph is allowed to have parallel bonds, which is exactly what the doubled convention produces on a periodic axis of length two.

Proposition 10 (a vertex-transitive bond multigraph with $\lambda < \delta_{\min}$). Let G be the bond multigraph of the doubled two-dimensional specification with $n = (2, 2)$ and boundary conditions op: four sites, six bonds, two site pairs of multiplicity two, so $Q = 4$. Then $\delta_{\min}(G) = 3$, $\lambda(G) = 2$, and G is vertex-transitive under the product translations $i \mapsto i + t$, $t \in \mathbb{Z}_2 \times \mathbb{Z}_2$, which permute the bond multiset and act transitively on the sites. The same three conclusions hold for the doubled specifications with $n = (2, 2)$ and po, and with $n = (2, 2, 2)$ and opp, pop, ppo, the last three having $(\delta_{\min}, \lambda) = (5, 4)$. Consequently $\lambda = \delta_{\min}$ fails even for vertex-transitive finite bond graphs, and Proposition 9 does not extend to multigraphs.

Proof. Every site of G carries two parallel bonds along the doubled periodic axis and one open bond, so $\delta_{\min} = 3$. A column of two sites is separated by exactly the two open bonds, giving a cut of size 2; enumeration of all 2^4 subsets attains 2 and never 1, so G has no bridge and $\lambda = 2$. Each translation $i \mapsto i + t$ maps the bond multiset onto itself bond by bond, and the orbit of any single site is the whole vertex set, which is the certified vertex-transitivity. The four remaining specifications are handled identically, with the translations of $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2^3$; their degrees, connectivities and cuts are listed in Table 7 and exhibited in Appendix A.3. $\square$

The census behind the proposition is exhaustive rather than sampled. The 204 uniform lattice specifications with $N \leq 12$ split into 102 simple ones, all with $Q = 0$, and 102 doubled ones; enumeration of all 2^N vertex subsets finds $\lambda = \delta_{\min}$ in 188 cases and $\lambda < \delta_{\min}$ in 16. All 102 simple specifications satisfy the equality, and all 16 violations are doubled, so no simple specification violates it; the two routes used to compute λ — a subset enumeration returning the exact minimum and a transfer computation returning only attainability — agreed on all 204. The complete list of violations is Table 7, which marks the five for which the axis-flip translations were certified to act transitively; for the other eleven the same test was run and produced a smaller orbit, which is not evidence that they fail to be vertex-transitive, only that this family of automorphisms does not certify it.

The classical theorem that a connected vertex-transitive *simple* graph has $\lambda = \delta_{\min}$ [12, 14] is not contradicted: the five certified examples are vertex-transitive multigraphs, and simplicity is exactly the hypothesis that fails. Nothing in this paper uses that theorem; Proposition 9 proves the equality for the families that appear here, and the exhaustive census measures it everywhere else. The physical content is that the doubled convention changes the spectral gap: on the same four sites the first excitation costs $2J\lambda = 4J$ rather than $6J$, and Proposition 3(b) says the levels $u = 1$ and, in the simple reading, $u = 3$ are then absent for different reasons.

All 94 lattices of the census satisfy $\lambda = \delta_{\min}$, as do all 102 simple specifications with $N \leq 12$ and 86 of the 102 doubled ones. Six lattices of the census are declared doubled, and none of the six violates the equality: in five of them every axis is periodic, so the parallel bonds raise the minimum cut as much as they raise the minimum degree, and in the sixth — the $2 \times 6 \times 2$ specification with boundary conditions opo — both length-two axes are open, so the doubled convention leaves the bond multiset untouched and $Q = 0$. That the census contains no violation is therefore a property of the chosen families, where a doubled axis always comes with periodicity in every other axis, and not a theorem; the violations of Proposition 10 all combine a doubled length-two axis with at least one open axis.

6 Exact enumeration and the census of 94 lattices

6.1 Four enumeration routes

Every number in this paper is an exact integer, and every table was produced by at least one of four routes that share only the lattice constructor and differ in the object they carry.

1. **Exhaustive configuration enumeration.** All 2^N configurations are enumerated, $b(s)$ and $M(s)$ are accumulated, and $g(b, M)$ is returned. Exact and assumption-free; capped at $N \leq 24$ by the 2^N cost.
2. **Integer dictionary transfer.** The lattice is swept along its narrowest axis; the state is a slice configuration together with the running pair (b, M) , carried in a dictionary of exact

arbitrary-precision integers. Cost is linear in the swept length and exponential in the slice width; capped at cross-sections of 2^6 slice configurations.

3. **Dense integer transfer.** The same recursion with the table stored as a dense array and the transfers performed as vectorised block additions, which is much faster but confines multiplicities to 64-bit integers: capped at $N \leq 62$ by magnitude and at cross-sections of 2^6 periodic or 2^{12} open configurations by width.
4. **Boolean reachability.** The state is a slice configuration and the value is a bitmask of the cut sizes reachable so far, combined by bitwise OR. It answers only “is u a level?” and never “how many configurations have it?”, and in exchange it has *no magnitude ceiling*: the bitmask is independent of N . Capped at cross-sections of 2^8 periodic or 2^{10} open configurations.

Route 4 is the reason the census reaches $N = 1000$, where 2^{1000} configurations are summarised into a level set in a fraction of a second, and it is the sense in which the first of the two questions in Wei [15] — how many distinct exponent values there are — is answered for lattices far beyond enumeration: the cost is linear in the swept length and exponential only in the narrowest cross-section. It does not answer the second question, the multiplicities, and no route here produces multiplicities for $N > 62$.

Routes 1–3 also return the sublattice-refined table $g(b, m_A, m_B)$, where m_A counts down spins on one colour class of the bipartition when one exists. Since the sweep axis is chosen as the narrowest one, reversing the declared order of the axes forces a different sweep and must reproduce an identical table; this is used as a convention-independence check on every lattice where it applies.

6.2 The census

Ninety-four uniform specifications were enumerated: 27 one-dimensional rings and chains with n up to 1000, 44 tori, grids and cylinders with complete tables up to 62 sites and exact level sets up to 1000 sites, and 23 three-dimensional boxes with up to 27 sites, together with the doubled-bond variants of the length-two axes. Eighty-one have complete multiplicity tables; thirteen, all beyond the 64-bit magnitude ceiling or the width cap, have exact level sets from route 4 alone. Engine coverage of the 81 is 50 lattices confirmed by routes 1, 2 and 3 together, 2 by routes 2 and 3 (the two at $N = 25$, just above the exhaustive cap) and 29 by route 3 alone. Of the 13, twelve are beyond the magnitude ceiling — the largest is the $n = 1000$ ring, whose multiplicities do not fit in any fixed-width integer — and one, the 7×7 torus at $N = 49$, is beyond the width cap only. That 29 of the 81 tables rest on a single multiplicity route is a limitation of the census and is recorded as such; one of them, the 6×6 torus, is additionally confirmed entry by entry by an external row-transfer computation (Section 10).

Structurally the census splits into the four classes that Proposition 5 distinguishes: 30 bipartite lattices with all degrees even, 39 bipartite whose degrees are not all even, 18 non-bipartite with all degrees even and 7 non-bipartite whose degrees are not all even. The complete tables are Tables 3–5 of Appendix C, and Figure 2 plots the exact level count against the value predicted by (20) for all 94, marking the two exceptions of Proposition 7.

Three identities were checked on every lattice for which a multiplicity table exists, and each held without exception:

$$\sum_{b,M} g(b, M) = 2^N, \quad \sum_b g(b, M) = \binom{N}{(N+M)/2}, \quad \mathcal{C}_{\text{route 1-3}} = \mathcal{C}_{\text{route 4}}, \quad (24)$$

the first two being (10) and the normalisation of Section 2, the third an agreement between two structurally different enumerations. In addition $\lambda = \delta_{\min}$ on all 94 — proved for the 88 simple ones by Proposition 9 and measured for the 6 doubled ones, which are not among the violations of Proposition 10 — the mirror of Theorem 4 holds exactly on every bipartite lattice and fails on every non-bipartite one, and $b_{\min} = 0$ exactly on the bipartite ones. The level count (20) holds on 92 of the 94.

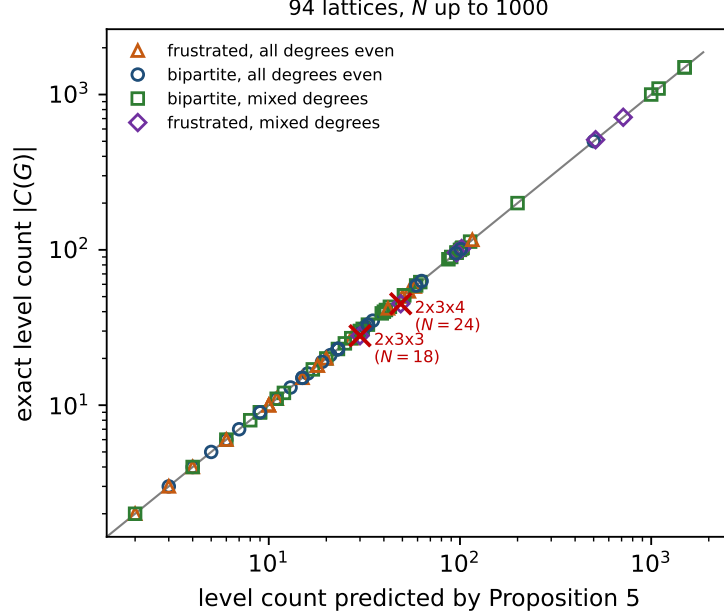


Figure 2: Exact level count $|C(G)|$ against the count predicted by Proposition 5, for all 94 lattices of the census, on logarithmic axes and grouped by the four structural classes. Points on the diagonal satisfy hypothesis (H); the two marked exceptions are the $2 \times 3 \times 4$ and $2 \times 3 \times 3$ boxes with boundary conditions opp (Proposition 7). The 13 lattices beyond the multiplicity ceiling contribute their level sets from route 4 and are indistinguishable in this plot from the enumerated ones.

The census is not a random sample. Sizes were chosen to sit just inside and just outside each engine cap, so that every route is exercised at its limit and every structural class is represented; the thirteen level-set-only entries exist precisely to test the formula where no multiplicity table can be built. Nothing in Section 4 depends on the census: the theorems are proved, and the census is where they are tested and where the two failures were found.

7 The $2 \times 6 \times 2$ example and the three conventions

The worked example of Wei [15] is a $2 \times 6 \times 2$ lattice whose $2^{24} = 16\,777\,216$ terms are reported to collapse to 23 distinct exponent values. The bond convention and the boundary conditions that produce that number are not stated, and the count is not reproducible from the description alone: the same twenty-four sites support three natural bond graphs, and the three conventions give three different answers to the same degeneracy question.

Lemma 11 (a length-two periodic axis is a path or a digon). *Under the simple convention an axis of length two contributes the single bond $K_2 = P_2$; under the doubled convention it contributes K_2*

with a parallel edge. Consequently the periodic simple 2×2 specification has bond graph $K_2 \square K_2$, which is isomorphic to C_4 under the vertex map $(0,0) \mapsto 0$, $(1,0) \mapsto 1$, $(0,1) \mapsto 3$, $(1,1) \mapsto 2$, and the map preserves all four bonds. It follows that the periodic simple specifications $2 \times 6 \times 2$, $2 \times 2 \times 6$, 4×6 and 6×4 all have bond graph $C_4 \square C_6$.

Proof. $K_2 \square K_2$ has four vertices of degree two and four bonds, and the map above is a bijection sending its bond multiset onto the bond multiset of C_4 ; this was checked bond by bond, and all eight bond-preserving bijections — the four rotations of C_4 composed with the map and their reflections — were enumerated. The product is associative and commutative up to isomorphism [10], so $K_2 \square C_6 \square K_2 = (K_2 \square K_2) \square C_6 = C_4 \square C_6$, and $C_4 \square C_6$ is the bond graph of the 4×6 torus. For the mixed specifications the same conclusion holds because an *open* length-two axis also contributes K_2 : an axis of length two has one bond per pair of sites whether or not it is periodic, so the doubled convention has nothing to double. Explicit vertex maps from each of the six specifications onto the 4×6 torus, each re-checked on all 48 bonds, are recorded in Section 8. $\square$

Corollary 12 (the published count is $LM - 1$). *The periodic simple $2 \times 6 \times 2$ lattice has bond graph $C_4 \square C_6$, so by Theorem 6*

$$\mathcal{C} = \{0\} \cup \{u \text{ even} : 4 \leq u \leq 44\} \cup \{48\}, \quad |\mathcal{C}| = 4 \cdot 6 - 1 = 23, \quad (25)$$

and the number of distinct exponent values at zero field in (3) is exactly 23. The count is obtained from the formula without enumerating a single configuration; exhaustive enumeration of all 2^{24} configurations confirms it, and gives 208 attainable pairs (b, M) , which is the corresponding count at nonzero field.

Corollary 12 identifies the convention behind the published number: it is the periodic simple-bond convention, under which the lattice is 4-regular with $|E| = 48$ and $\lambda = 4$. It also explains why the number is small. For an even-sided torus the level count is $N - 1$, so a 24-site lattice whose bond graph happens to be $C_4 \square C_6$ has 23 levels for a reason that has nothing to do with its being written as a three-dimensional box: the same count belongs to the 4×6 and 6×4 tori, to the periodic $2 \times 2 \times 6$ box, and to two mixed-boundary variants (Section 8).

7.1 The same sites, three bond graphs, three answers

Table 1: The three natural bond conventions on the same $2 \times 6 \times 2$ set of 24 sites. All three were enumerated exhaustively over $2^{24} = 16\,777\,216$ configurations, so every entry is exact. Q counts ordered pairs of distinct bonds sharing both endpoints; “levels” is $|\mathcal{C}|$, the number of distinct zero-field exponent values, and “pairs” is the number of attainable (b, M) pairs, the corresponding count at nonzero field. The predicted level count is (20) with the measured λ ; in the open row $\lambda = 3$ also follows from Proposition 9, since $P_2 \square P_6 \square P_2$ is a product of three paths.

bonds	graph	$ E $	Q	degrees	bip.	λ	levels	predicted	pairs
periodic, simple	$C_4 \square C_6$	48	0	4	yes	4	23	23	208
periodic, doubled	$2C_4 \square C_6$	72	48	6	yes	6	33	33	321
open, simple	$P_2 \square P_6 \square P_2$	44	0	3, 4	yes	3	41	41	453

Table 1 collects the three counts. Under the doubled convention each of the two length-two periodic axes is doubled, twelve site pairs per axis, so the bond multigraph is 6-regular with $|E| = 72$ and $Q = 48$; it is bipartite and Eulerian with $\lambda = 6$, and (20) gives $72/2 - 6 + 3 = 33$, which the

exhaustive enumeration confirms. Under open boundaries the bond graph is a mixed box with 44 bonds, degrees 3 (eight sites) and 4 (sixteen sites), bipartite but not Eulerian and $\lambda = 3$, giving $44 - 6 + 3 = 41$, again confirmed. In all three cases the level count is a function of $|E|$, λ , bipartiteness and degree parity — the dimension of the specification never enters.

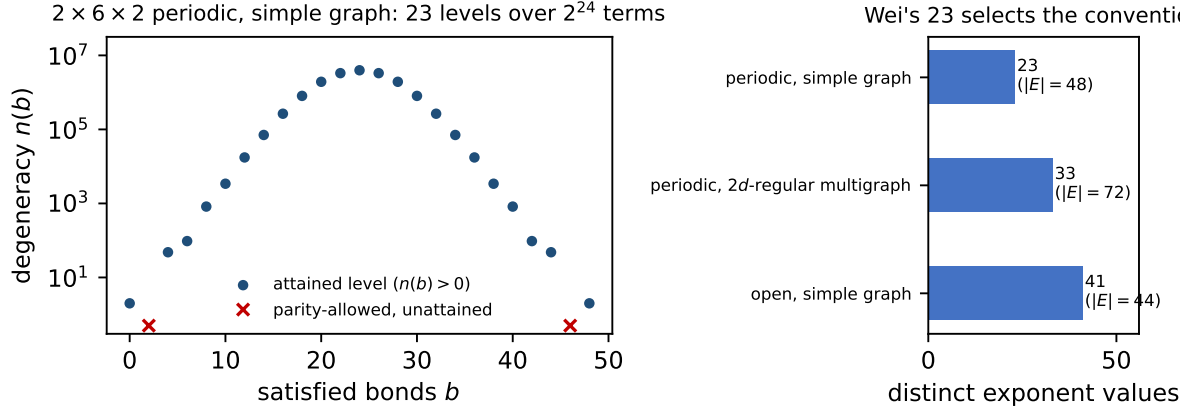


Figure 3: Left: the zero-field bond histogram $n(b)$ of the periodic simple $2 \times 6 \times 2$ lattice on a logarithmic scale, over all 2^{24} configurations. Only the 23 attained values are plotted as points; the crosses mark values that parity allows but no configuration attains, and by Proposition 3 they include the whole band adjacent to $b = |E|$ because $\lambda = 4$. The histogram is symmetric under $b \leftrightarrow |E| - b$ by Theorem 4. Right: the same 24 sites under the three bond conventions of Table 1.

The answer to the question as published — whether the number of distinct exponent values depends on the finite lattice — is therefore yes, but the dependence is not on the shape of the box. It is on the bond graph: on how many bonds the convention declares, on the edge connectivity that fixes the first level, on bipartiteness that fixes whether the top level is present, and on the parity of the degrees that fixes whether the spectrum is halved. Two specifications with the same bond graph have the same count no matter how different their names, and one specification can be given three counts by three conventions.

8 One bond graph, six lattices: what a degeneracy table can and cannot tell

Proposition 13 (a coincidence class of twenty-four-site lattices). *The six uniform specifications*

$$4 \times 6 \text{ and } 6 \times 4 \text{ (pp)}, \quad 2 \times 6 \times 2 \text{ and } 2 \times 2 \times 6 \text{ (ppp)}, \quad 2 \times 6 \times 2 \text{ (opp)}, \quad 2 \times 6 \times 2 \text{ (opo, doubled)} \quad (26)$$

have one bond graph, $C_4 \square C_6$, and one density of states. Each of the fifteen ordered pairs is isomorphic under an explicit vertex map that was re-checked bond by bond on all 48 bonds, and all six tables $g(b, M)$ agree in every one of their $49 \times 25 = 1225$ entries. Each table was produced independently by exhaustive enumeration of all 2^{24} configurations. All six have $N = 24$, $|E| = 48$, $Q = 0$, every degree 4, are bipartite, and have 23 levels and 208 attainable pairs (b, M) .

Proof. By Lemma 11 each of the six is a product $K_2 \square C_6 \square K_2$ in which the two K_2 factors come either from an open length-two axis, which has one bond in either convention, or from a periodic length-two axis under the simple convention, which also has one. The maps are the coordinate

identifications induced by $K_2 \square K_2 = C_4$, composed with the axis permutation that takes each specification to the 4×6 torus; each was verified to send the bond multiset of one specification bijectively onto that of the other. Table equality then follows from Theorem 1, since g counts subsets of a bond graph by cut size and subset size, both preserved by an isomorphism; it was nevertheless compared entry by entry, and each of the six tables was rebuilt from scratch by exhaustive enumeration. $\square$

The sixth specification deserves a remark, because it is the one that a reader would most likely mis-transcribe. It is *declared* with the doubled convention and is nevertheless a simple graph: its two length-two axes are open, and an open length-two axis has a single bond whether or not the convention would keep parallel copies. The quantity that decides simplicity is Q , the number of ordered pairs of distinct bonds sharing both endpoints, not the declared convention. The same specification with periodic length-two axes would have $Q = 48$ and 33 levels (Table 1).

8.1 What the coincidence does not imply

Isomorphic bond graphs necessarily have identical tables, so a coincidence class of tables is always a union of isomorphism classes. The converse — that a shared table implies a shared graph — is what makes a degeneracy table a complete invariant, and it is tested exhaustively up to eighteen sites in the closing subsection below, where it is measured rather than proved. Proposition 13 says something weaker and more useful in the other direction: *a table does not identify the uniform family that produced it*. The published count of 23 for a “three-dimensional $2 \times 6 \times 2$ lattice” is not a fact about three-dimensional lattices; it is a fact about $C_4 \square C_6$, and it belongs equally to a two-dimensional torus. Table 2 lists the fourteen specifications examined at $N = 16$ and $N = 24$ with their isomorphism classes, level counts and attainable-pair counts.

Table 2: The fourteen specifications at $N = 16$ and $N = 24$ studied for coincidences, with Q counting ordered pairs of distinct bonds that share both endpoints and the isomorphism class numbered as listed: class 3 is the six-member class $C_4 \square C_6$ of Proposition 13, class 6 the pair $3 \times 8 \cong 8 \times 3$, class 4 the doubled pair with $|E| = 72$, and classes 1, 2, 5 and 7 singletons. Levels and pairs are exact, from exhaustive enumeration of all 2^N configurations.

shape	boundaries	bonds	N	$ E $	Q	degree	bip.	class	levels	(b, M) pairs
2×8	<i>pp</i>	doubled	16	32	16	4	yes	5	15	103
4×4	<i>pp</i>	simple	16	32	0	4	yes	7	15	80
$2 \times 2 \times 4$	<i>ppp</i>	doubled	16	48	32	6	yes	2	21	125
12×2	<i>pp</i>	doubled	24	48	24	4	yes	1	23	247
$2 \times 2 \times 6$	<i>ppp</i>	simple	24	48	0	4	yes	3	23	208
$2 \times 6 \times 2$	<i>opo</i>	doubled	24	48	0	4	yes	3	23	208
$2 \times 6 \times 2$	<i>opp</i>	simple	24	48	0	4	yes	3	23	208
$2 \times 6 \times 2$	<i>ppp</i>	simple	24	48	0	4	yes	3	23	208
3×8	<i>pp</i>	simple	24	48	0	4	no	6	20	213
4×6	<i>pp</i>	simple	24	48	0	4	yes	3	23	208
6×4	<i>pp</i>	simple	24	48	0	4	yes	3	23	208
8×3	<i>pp</i>	simple	24	48	0	4	no	6	20	213
$2 \times 2 \times 6$	<i>ppp</i>	doubled	24	72	48	6	yes	4	33	321
$2 \times 6 \times 2$	<i>ppp</i>	doubled	24	72	48	6	yes	4	33	321

Proposition 14 (two certified controls). *The level count, even together with the obvious invariants, does not determine the table. The doubled 12×2 torus and the 4×6 torus both have $N = 24$, $|E| = 48$, every degree 4, are bipartite with $\lambda = 4$, and both have exactly 23 levels; their tables differ, the first having 247 attainable pairs (b, M) and the second 208, and their bond graphs are non-isomorphic, the first carrying $Q = 24$, that is twelve doubled site pairs, and the second none. The same phenomenon occurs at sixteen sites: the 4×4 torus and the doubled 2×8 torus share $N = 16$, $|E| = 32$, degree 4, bipartiteness and the level count 15, and have 80 and 103 attainable pairs, in that order.*

Proof. Both pairs were enumerated exhaustively. Non-isomorphism is certified by the parallel-pair count, which is invariant under any bijection of the vertex sets: no map can send a bond multiset with $Q = 24$ onto one with $Q = 0$. The difference in the number of attainable pairs is a difference of tables, and it is the pair count rather than the level count that detects it. $\square$

The table is as fine as isomorphism up to eighteen sites

All 412 uniform lattice specifications with $N \leq 18$ — shapes with up to three axes, every choice of open or periodic boundary per axis, and both bond conventions — were enumerated exhaustively, every table cross-checked entry by entry by the integer dictionary transfer route, and every pair compared for isomorphism by exact search over vertex bijections with bond-multiset verification. They fall into 98 isomorphism classes. Ninety-five groups share an identical full table $g(b, M)$ and each such group is a single isomorphism class: no non-isomorphic pair shares a full table, and none shares even the zero-field marginal $n(b)$. Ninety-one groups share an identical level set, and these do span several isomorphism classes, so the level set is strictly coarser than the table.

The status of this result is verification, not proof: it is an exhaustive statement about a finite family, and a single certified pair of non-isomorphic uniform lattices with identical tables, at any size, would refute the general conjecture that the two classes coincide. The evidence available here is that the coincidence groups and the isomorphism classes agree on all 412 specifications up to eighteen sites, and that the six-member coincidence of Proposition 13, which is the largest coincidence found at twenty-four sites, is explained by an isomorphism rather than by an accident.

9 One dimension in closed form

The one-dimensional families are the only ones for which the complete bivariate table is available in closed form at every size and every field. They serve three purposes here: an exact oracle that shares no code with any enumeration route, a proof that hypothesis (H) holds in one dimension, and a check that Proposition 5 specialises correctly.

Theorem 15 (closed forms for the ring and the chain). *Let $n \geq 3$ and let m be the number of down spins, so $M = n - 2m$. Write $\binom{a}{b} = 0$ outside $0 \leq b \leq a$, with the single exception $\binom{-1}{-1} = 1$.*

- (a) **Ring** C_n ($|E| = n$). *If j is the number of maximal runs of down spins around the ring, then $u = 2j$, $b = n - 2j$ and*

$$g(b, M) = \frac{n}{j} \binom{m-1}{j-1} \binom{n-m-1}{j-1}, \quad 1 \leq j \leq \min(m, n-m), \quad 1 \leq m \leq n-1, \quad (27)$$

together with the two uniform configurations $g(n, \pm n) = 1$. Summing over m gives the zero-field marginal

$$n(b) = \frac{n}{j} \binom{n-1}{2j-1} \quad \text{for } b = n - 2j, \quad 1 \leq j \leq \lfloor n/2 \rfloor, \quad n(n) = 2. \quad (28)$$

(b) **Open chain** P_n ($|E| = n - 1$). If t is the number of domain walls of the linear spin word, then $u = t$, $b = (n - 1) - t$ and

$$g(b, M) = \begin{cases} \binom{n-m-1}{k} \binom{m-1}{k-1} + \binom{m-1}{k} \binom{n-m-1}{k-1}, & t = 2k, \\ 2 \binom{n-m-1}{k-1} \binom{m-1}{k-1}, & t = 2k - 1. \end{cases} \quad (29)$$

Both formulas are exact integers for every admissible index, and both are valid at any J and any h through (2).

Corollary 16 (one-dimensional level structure). For $n \geq 3$,

$$\mathcal{C}(C_n) = \{0, 2, 4, \dots, 2\lfloor n/2 \rfloor\}, \quad |\mathcal{C}(C_n)| = \lfloor n/2 \rfloor + 1, \quad (30)$$

$$\mathcal{C}(P_n) = \{0, 1, 2, \dots, n - 1\}, \quad |\mathcal{C}(P_n)| = n. \quad (31)$$

Hence hypothesis (H) holds for every ring and every open chain, and Proposition 5 reproduces both counts: for even n the ring is bipartite and Eulerian with $\lambda = 2$, giving $n/2 - 2 + 3 = \lfloor n/2 \rfloor + 1$; for odd n it is frustrated and Eulerian with $\lambda = 2$ and $u_{\max} = n - 1$, giving $(n - 3)/2 + 2 = \lfloor n/2 \rfloor + 1$; the chain is bipartite with mixed degrees and $\lambda = 1$, giving $(n - 1) - 2 + 3 = n$.

Proof. Both statements are the level sets of Theorem 15, read off from the range of the run count j on the ring and of the wall count t on the chain; the derivations are in Appendix B. $\square$

Corollary 16 is the whole answer to the degeneracy question in one dimension: the ring partition function at zero field is a sum of $\lfloor n/2 \rfloor + 1$ distinct exponent values with the multiplicities (28), for every n , and at nonzero field a sum over the attainable pairs (b, M) counted by (27). The derivations of (27), (29) and (28), including the integrality of the factor n/j , are in Appendix B.

9.1 Relation to earlier one-dimensional results, and how these were checked

The counting problem behind Theorem 15 is classical. Run statistics of a circular binary sequence were obtained by Barton and David [3]; the distribution of the magnetisation of a finite Ising chain under several boundary conditions by Antal et al. [1]; domain statistics of a finite chain by Denisov and Hänggi [6]; and closed-form expressions for the fixed-magnetisation partition function of the one-dimensional model under periodic, antiperiodic and Dirichlet boundary conditions by Dantchev and Rudnick [5]. The formulas above are therefore not claimed as new. What is claimed is their use: as an oracle for the general theory of Sections 3–4, and as the proof of (H) in one dimension.

Three independent comparisons were run, in exact arithmetic throughout.

1. **Entry by entry against enumeration.** Both closed forms were compared with the exhaustive enumeration of all 2^n configurations and with the two transfer routes, entry by entry over the whole table. Every comparison gave a maximum absolute difference of zero: for rings $n \leq 14$ and chains $n \leq 14$ against exhaustive enumeration in an independent implementation, and for rings and chains up to $n = 62$ against both the dictionary route and the dense integer route.
2. **Partition functions against the transfer matrix.** The partition function assembled from the closed form through (2) was compared with the classical 2×2 transfer matrix [4] in exact rational arithmetic, at eight exact points $(e^{2\beta J}, e^{\beta h})$, for both one-dimensional families and 43 sizes up to $n = 400$: 688 comparisons, zero mismatches. An independent reimplementa-

repeated the same comparison over $n = 3, \dots, 40$ and obtained 608 further exact agreements. The transfer-matrix route shares no recursion, no data structure and no arithmetic domain with the closed form.

3. **Moments beyond every table.** At $n = 200, 400$ and 1000 no table fits in memory, so the closed forms were checked through their moments: $\sum g = 2^n$, the wall moment $\sum_{b,M} b g(b, M) = n 2^{n-1}$, the magnetisation marginal against (10), and the level count against $\lfloor n/2 \rfloor + 1$. All four held exactly, the largest integer involved being 2^{1000} , which has 302 digits.

10 Agreement between independent enumerations

Every claim in this paper is either proved or measured, and every measurement is an exact integer obtained by exhaustive enumeration or by an exact recursion. This section records what was cross-checked, by what, and what remains unchecked.

10.1 Cross-checks inside the census

The four routes of Section 6 share only the lattice constructor. Where their coverage overlaps they were compared entry by entry: 50 lattices by the exhaustive, dictionary-transfer and dense-integer routes together, two more by the latter two, and all 94 level sets by the boolean reachability route against the multiplicity routes wherever the latter apply. There were no disagreements. Three identities were checked on every table that exists — total weight 2^N , the binomial magnetisation marginal (10), and the equality of the spin formulation of “unsatisfied bond” with the set formulation $|\delta(S)|$ — and each held on all of them, configuration by configuration in the exhaustive range. Reversing the declared order of the axes forces a different sweep direction in the transfer routes and reproduced identical tables wherever it was applicable.

10.2 An independent reimplementaion

The whole record was re-derived by a second implementation written from the model description alone, sharing no code, no data structure and no enumeration order with the routes above. It rebuilt the lattices from the shape, boundary conditions and bond convention, enumerated the configurations and the vertex subsets separately, and re-tested each structural claim. Its results:

- on 37 specifications with $N \leq 18$: the cut identity of Theorem 1 held on every configuration; the parity constraint and the connectivity gap of Proposition 3 held on all of them; the mirror of Theorem 4 held exactly on the bipartite ones and failed exactly on the non-bipartite ones; $b_{\min} = 0$ occurred exactly on the bipartite ones; the level count (20) held on 36, the exception being the $2 \times 3 \times 3$ box of Proposition 7; and $\lambda = \delta_{\min}$ held on every simple bond graph and failed on four doubled ones;
- on 84 specifications with $N \leq 8$, enumerated exhaustively over all 2^N subsets: 65 simple bond graphs with no violation of $\lambda = \delta_{\min}$, and nine violations, every one carrying parallel bonds, with vertex-transitivity certified by explicit automorphisms of the bond multiset whose orbit of one site is the whole vertex set;
- the six-member coincidence class of Proposition 13, rebuilt by explicit coordinate maps from $K_2 \square K_2 = C_4$ and verified bond by bond, with all five independently enumerated tables byte-identical over their 1225 entries;

- the three conventions of Table 1, each enumerated over all 2^{24} configurations, reproducing 23, 33 and 41 levels and 208, 321 and 453 attainable pairs;
- the doubled 12×2 control, reproducing 23 levels and 247 pairs and the parallel-pair count $Q = 24$ that certifies non-isomorphism;
- the per-size induced-edge ranges behind Proposition 7, from which the certificates for $u = 6, 7, 50, 51$ were recomputed, together with the control that the same test excludes no level that is attained;
- the three candidate mirror symmetries of Section 3 on the 4×4 torus, the $2 \times 2 \times 2$ box, the open 3×4 grid and the frustrated 3×3 and 2×3 tori: the global flip (15) held everywhere, the colour-class flip (16) and its marginal consequence (17) held on the bipartite lattices, and the naive mirror $g(b, M) = g(|E| - b, -M)$ failed on every lattice tested, bipartite or not;
- the one-dimensional closed forms of Theorem 15 against exhaustive enumeration entry by entry, with maximum absolute difference zero, and against the 2×2 transfer matrix in exact rational arithmetic.

10.3 Two external recomputations

Two further checks were produced independently of this work by other authors, and were re-run or re-derived here.

An exact column transfer over vertex subsets. The first is a dynamic program whose state is a cross-section *subset* of vertices and whose value is either a bitmask of attainable cut sizes or a dictionary of exact counts, together with a Gray-code enumeration of all 2^N subsets as a third route. It shares no code and no enumeration structure with the routes used here, all of which work on spin configurations. It reproduced all 24 tables of the size ladder entry by entry with maximum absolute difference zero; the zero-field marginals and level counts of 35 lattices, including two at $N = 60$ that no route here confirms by exhaustive enumeration; and $\lambda = \delta_{\min}$ on all 35. Its only violation of the interior-support bound was the $2 \times 3 \times 4$ exception of Proposition 7, with the same four missing levels. The code was inspected before execution for network access and for file writes and found to have neither, and was then re-run against the tables reported here.

An exact row transfer over integers. The second is a dynamic program that sweeps one torus row at a time, keyed by the pair (bond sum, magnetisation) in arbitrary-precision integers. Its published integers were transcribed and re-derived from the routes used here rather than executed, so this is a value-level cross-check. It covers the complete zero-field histogram of the 6×6 torus: 35 levels, 482 attainable pairs, total weight $2^{36} = 68\,719\,476\,736$, all 35 multiplicities agreeing entry by entry with maximum absolute difference zero, the mirror $m(u) = m(|E| - u)$ holding on every level, and $u = 2$ absent exactly as $\lambda = 4$ requires. It also confirms the attainable-pair counts 22, 80, 208 and 482 for the 3×3 , 4×4 , 5×5 and 6×6 tori. This check matters because the 6×6 torus is the one census row whose multiplicity table came from a single route. Its convention differs from ours — it writes $u = (|E| - b')/2$ with $b' = \sum s_i s_j$ — but since $b' = 2b - |E|$ by (6) the two u are the same integer and the tables compare without rescaling. One annotation in that work is wrong and does not affect its data: it applies the bipartite-Eulerian branch $|E|/2 - \lambda + 3$ of Proposition 5 to the odd tori 3×3 and 5×5 , which are frustrated, where the correct branch is non-bipartite Eulerian and gives $(u_{\max} - \lambda)/2 + 2 = 6$ and 20 from $u_{\max} = 12$ and 40.

10.4 What was not verified

Twenty-nine of the 81 multiplicity tables rest on a single route, the dense integer transfer, because the dictionary route exceeded its width budget; one of these, the 6×6 torus, is now confirmed externally as described above, and the remaining 28 are confirmed only by the total-weight and magnetisation-marginal identities and by agreement of their level sets with the reachability route. Thirteen lattices have level sets but no multiplicities at all, since they lie beyond the 64-bit magnitude ceiling or the width cap; for them the level count (20) is the only structural statement, and for the one-dimensional ones it is additionally confirmed by the closed forms of Theorem 15, whose moment identities were checked at $n = 200, 400$ and 1000 . No lattice beyond $N = 1000$ was examined. Hypothesis (H) is unproved for every family other than rings, chains and even-sided tori, and Conjecture 8 is supported by two certified examples only. The statement that a shared table implies an isomorphic bond graph is verified on 412 specifications and not proved.

Finally, a limitation of the bibliography rather than of the computations. Every entry resolves: eleven DOIs were checked through Crossref with matching titles, authors, venues, volumes, pages and years, a twelfth — the arXiv identifier of the preprint entry — through the arXiv API with an exact title match, and the remaining three, one book and two web records, carry complete bibliographic data with an ISBN or a URL. Content beyond that record was available for only three of the fifteen during the preparation of this paper, and in all three cases it was the abstract and not the full text, so the other twelve could be checked at the level of their bibliographic record alone. Those twelve are used for attribution and historical statements alone; no technical claim in this paper rests on them. In particular the one-dimensional formulas of Theorem 15 are proved here and are *not* asserted to be equation-by-equation identical to any specific numbered formula in the earlier literature, and the connectivity results are proved and measured here rather than deduced from the classical vertex-transitive theorem [12, 14].

The data, the enumeration code and the figures of this paper are archived with the source repository [8]. Every integer quoted here is exact; no floating-point value enters any of them, and the only non-integer quantities reported anywhere are wall-clock timings.

11 Discussion

11.1 What the identity buys

Equation (4) is a one-line observation, and the results of Sections 3–4 are what it buys. Once the exponents of the 2^N -term rewriting are recognised as cut sizes, four pieces of structure come with them for free: a parity constraint that halves the spectrum of every Eulerian bond graph, an empty band below the edge connectivity that fixes the location of the first excited level and hence the finite-lattice spectral gap $2J\lambda$, a mirror symmetry that is present exactly when the lattice is unfrustrated, and a level count that needs four integers — $|E|$, λ , $u_{\max}$ and the parity of the degrees — plus one hypothesis. None of these requires knowing anything about the multiplicities, which is why they survive to $N = 1000$, where no multiplicity table can be built.

The practical consequence for reporting finite-lattice data is that a lattice specification is not determined by its shape and boundary conditions. The bond convention on length-two periodic axes changes $|E|$, the degree sequence, λ and the whole level set, and it does so visibly: 23, 33 and 41 levels on the same twenty-four sites (Table 1). A count of distinct exponent values is meaningless without it. Symmetrically, the count is not a dimension: six specifications of dimension two and three, with three different boundary-condition patterns and one declared as doubled, are a single bond graph and have a single table.

The reduction also sharpens the relation to the classical treatment of spin-glass ground states as maximum-cut problems [2]. That reduction concerns one level, the extreme one; here every level is a cut value and every multiplicity is a number of subsets of given size and given cut, so the whole density of states is a cut-size distribution. We make no complexity claim in the opposite direction: the identity is exact and elementary, and it says nothing by itself about how hard the multiplicities are to compute, which for general graphs is a different question from the one asked here.

11.2 The mirror, refined

Theorem 4 characterises bipartiteness by an exact symmetry of the zero-field histogram, and the refinement (17) says what the symmetry does to the field dependence: on a bipartite lattice, flipping one colour class exchanges the total and the staggered magnetisation. The magnetisation-resolved table of the ferromagnet at level u is therefore already the staggered-magnetisation-resolved table at level $|E| - u$, and the antiferromagnetic spectrum of a bipartite lattice carries no information that the ferromagnetic one does not. That the naive joint mirror $g(b, M) = g(|E| - b, -M)$ fails — measurably, on every bipartite lattice tested — is worth stating because it is the statement one would write down first, and because the failure is exactly the difference between the two order parameters. Whether an analogous exact symmetry characterises bipartiteness once couplings take several distinct uniform values, or once antiferromagnetic signs are placed on a non-bipartite lattice, is open: the proof given here uses unweighted bond counts and a single coupling, and neither is incidental to it.

11.3 Where the folklore fails

Proposition 10 is a cautionary result rather than a difficult one. The rule that edge connectivity equals minimum degree is true for connected vertex-transitive simple graphs [12, 14], and it is tempting to apply it to any uniform lattice, since uniform lattices look vertex-transitive. Five of the sixteen violations found here are vertex-transitive under certified automorphisms, and all sixteen carry parallel bonds; the hypothesis that fails is simplicity, and the doubled bond convention violates it on every length-two periodic axis. The physical effect is not cosmetic: on four sites the first excitation costs $4J$ rather than $6J$, and the band of absent levels below it is different. Since the doubled convention is a legitimate way to read a periodic axis of length two — it is what a literal transcription of the nearest-neighbour sum produces — the distinction is worth keeping explicit in any tabulation of finite-lattice spectra.

11.4 Open questions

Four questions are left open by this work.

Multiplicities uniformly in width. The reachability route answers the “how many distinct exponent values” question in time linear in the swept length and exponential only in the narrowest cross-section, with no magnitude ceiling, and that is how the census reaches $N = 1000$. Whether the multiplicities admit the same treatment is open. The transition between slices depends on the previous and current slice configurations only through their Hamming distance, which suggests a transform over that dependence as the natural next step; it is not attempted here. A cheaper intermediate target is to widen the dictionary route so that the 29 tables currently resting on a single route acquire a second confirmation.

When is the interior hypothesis true? It is proved for rings, chains and even-sided tori, holds on 92 of 94 censused lattices, and fails on two, where every missing level is certified by an induced-edge obstruction. Conjecture 8 is the natural generalisation and is supported by exactly those two

examples. Two sub-questions are separate and harder: whether the bars of Section 4.2 can be extended to odd-sided tori, where the mirror is unavailable and $u_{\max} < |E|$; and whether the near-maximum holes, which need the exact set of attainable internal-edge counts rather than its endpoints, obey any pattern at all.

Is the coincidence class of the table the isomorphism class? Isomorphic bond graphs share every multiplicity, so only the converse is open. It holds on all 412 specifications with $N \leq 18$ and the largest coincidence found at twenty-four sites is an isomorphism, but one certified pair of non-isomorphic uniform lattices with identical tables would refute it. The negative controls of Proposition 14 show which invariants cannot settle the question: $(N, |E|, \text{degrees}, \text{bipartiteness}, |\mathcal{C}|)$ does not determine the table, so a counterexample would have to be found among lattices sharing all five.

For which uniform families is $\lambda = \delta_{\min}$ provable? Proposition 9 proves it for every product of paths and cycles, which covers the 88 simple bond graphs of the census and the 102 simple specifications with $N \leq 12$; it was measured on all 204 of the latter and on all 94 censused lattices, the six doubled ones included. What remains is a uniform statement for simple hypercubic families with mixed boundary conditions at sizes beyond the enumeration cap, where the measurement stops being available.

11.5 Limitations

The scope is finite, exact and uniform: one coupling, one field, hypercubic bond graphs in one, two and three dimensions. Disordered couplings, antiferromagnetic signs on non-bipartite lattices and multi-valued couplings are outside every theorem. No subexponential-in- N algorithm uniform in lattice width is claimed, no closed form for the multiplicities in dimension two or three, no complexity classification, and nothing about the thermodynamic limit: the asymptotic branch of the published problem [9] is untouched. The census is a targeted sample of 94 specifications chosen to exercise every enumeration route at its cap, not a random sample of lattices, and thirteen of its entries carry level sets without multiplicities. Where a claim could not be established it is labelled as a conjecture, as a verification over a finite family, or as a limitation; the inventory of unchecked items is Section 10.

12 Conclusion

For a finite uniform Ising lattice the number of unsatisfied bonds of a configuration is the size of the cut defined by its down spins, so the exact density of states is a cut-size spectrum of the bond graph and the degeneracy question of Wei [15] becomes a graph-theoretic one. Four consequences are exact and need no enumeration: a parity constraint on the levels, an empty band from $u = 1$ to $u = \lambda - 1$ that fixes the finite-lattice spectral gap, a multiplicity mirror $m(u) = m(|E| - u)$ present exactly on unfrustrated lattices and refined to an exchange of total and staggered magnetisation, and a four-branch level count valid under an explicit interior hypothesis. That hypothesis was proved for rings, open chains and even-sided tori, the last by an explicit construction with non-adjacent rows of down spins whose cuts add; for even-sided tori the level count is $N - 1$, which settles the published $2 \times 6 \times 2$ example at 23 and identifies the convention that produces it. The count is an invariant of the bond graph and not of the lattice name: six twenty-four-site specifications are one graph, $C_4 \square C_6$, with one table of 1225 entries, while the same sites under a doubled or an open convention give 33 and 41 levels. A census of 94 uniform lattices with up to 1000 sites supports the level count on 92, and for the two exceptions every missing level is certified by an induced-edge obstruction computed from all 2^N subsets. Two negative results complete the picture: the equality

$\lambda = \delta_{\min}$ fails on sixteen doubled-bond specifications, five of them vertex-transitive under certified automorphisms, so the classical simple-graph theorem does not reach the multigraphs that a literal transcription of a periodic length-two axis produces; and a degeneracy table does not identify the uniform family that produced it, although within eighteen sites no two non-isomorphic specifications share one.

A Proofs of the constructive and connectivity results

A.1 Which specifications are regular

Propositions 9 and 10 and the census tables both distinguish regular specifications, those whose degrees are all equal, so $\delta_{\min}$ is the common degree d . The rule is exact:

A uniform lattice specification is regular if and only if every axis is either periodic or of length two. The common degree is then $d = \sum_a c_a$, where an axis contributes $c_a = 2$ if it is periodic with $n_a \geq 3$, $c_a = 2$ if it is periodic of length two under the doubled convention, and $c_a = 1$ if it has length two and is open or periodic under the simple convention.

The reason is that an axis contributes to the degree of a site the number of bonds along that axis incident with it. A periodic axis of length $n_a \geq 3$ gives two, the forward and backward neighbours being distinct. An axis of length two has a single site pair, so it gives one bond under the simple convention, whether open or periodic, and two under the doubled periodic convention. An open axis of length $n_a \geq 3$ gives two at interior sites and one at each of the two end sites, and this is the only source of degree variation in the whole family. Degrees add over the axes of a Cartesian product.

The consequences are the degree claims used in the body: the ring C_n with $n \geq 3$ is 2-regular, while the $n = 2$ ring is a single bond under the simple convention, 1-regular, and a digon under the doubled one, 2-regular; the chain P_n with $n \geq 3$ has degrees $1, 2, \dots, 2, 1$; a torus $C_L \square C_M$ with $L, M \geq 3$ is 4-regular, the simple torus $K_2 \square C_M$ is 3-regular and the doubled one 4-regular; a grid or a cylinder is regular only when every open axis has length two, so $P_2 \square C_6 \square P_2$ is 4-regular, which is why the six specifications of Proposition 13 are regular although two of them have open axes; a periodic box with every $n_a \geq 3$ is 6-regular; and the two exceptions of Proposition 7 are 5-regular, being an open axis of length two plus two periodic axes.

The rule was checked against the degree multiset recomputed from the bond multiset, not against the declared specification: it reproduces the regularity flag on all 94 censused lattices, 55 of them regular and 39 not, and on all 412 specifications with $N \leq 18$, with no exception. Regularity is independent of the degree parity that selects the branch of (20): the 5-regular exceptions are regular with every degree odd, while an open grid is irregular and carries degrees of both parities.

A.2 Theorem 6: bars on an even-sided torus

Let $T = C_L \square C_M$ with $L, M \geq 4$ even. Sites are pairs (x, y) with $x \in \mathbb{Z}_L$, $y \in \mathbb{Z}_M$; the bonds are the LM horizontal pairs $\{(x, y), (x+1, y)\}$ and the LM vertical pairs $\{(x, y), (x, y+1)\}$, so $|E| = 2LM$ and every degree is 4. Both cycles are even, so T is bipartite with colour classes $\{x+y \equiv 0 \pmod{2}\}$ and $\{x+y \equiv 1 \pmod{2}\}$; every degree is even, so T is Eulerian; and $\lambda(T) = 4 = \delta_{\min}(T)$ by Proposition 9.

Selected rows. Take the $M/2$ rows $y = 0, 2, \dots, M-2$. Two of them differ by an even residue modulo M , and since M is even no even residue is ± 1 , so no bond joins two selected rows.

A single bar. For $1 \leq k \leq L-1$ let $B_k = \{(x, 0) : 0 \leq x \leq k-1\}$. Its cut consists of the horizontal bonds $\{(k-1, 0), (k, 0)\}$ and $\{(L-1, 0), (0, 0)\}$, which exist and are distinct because $1 \leq k \leq L-1$, together with the $2k$ vertical bonds incident with the k sites of the bar, which are

distinct from one another because $M \geq 4$ makes the rows $y = 1$ and $y = M - 1$ different from $y = 0$. Hence $|\delta(B_k)| = 2k + 2$, which is the measured value on every torus where the construction was run.

Unions of bars. Let $j \geq 1$ and choose lengths $k_1, \dots, k_j$ and the distinct selected rows $y = 0, 2, \dots, 2(j-1)$; this needs $j \leq M/2$. Write $S = \bigcup_i B_{k_i}^{(i)}$ for the union of the bars placed in those rows. No bond joins two of the bars, because no bond joins two selected rows; therefore an edge has exactly one endpoint in S if and only if it has exactly one endpoint in one of the bars, and the cuts add:

$$|\delta(S)| = \sum_{i=1}^j (2k_i + 2) = 2 \left(\sum_{i=1}^j k_i + j \right). \quad (32)$$

As each k_i ranges independently over $\{1, \dots, L-1\}$, the sum $k_1 + \dots + k_j$ takes every integer value from j to $j(L-1)$: this is immediate for $j = 1$ and follows for $j > 1$ because the intervals $[j-1, (j-1)(L-1)] + [1, L-1]$ cover $[j, j(L-1)]$ without gaps when $L \geq 3$. The attained cut sizes are therefore the even numbers $2t$ with $t \in [2j, jL]$ for some $j \in \{1, \dots, M/2\}$. Consecutive intervals meet or overlap because $2(j+1) \leq jL+1$ is equivalent to $j(L-2) \geq 1$, true for every $j \geq 1$ when $L \geq 3$, so the union over j is $[2, ML/2]$ and the attained values are exactly the even u with $4 \leq u \leq ML$. Since $ML = 2LM/2 = |E|/2$, this is every even cut size from $\lambda = 4$ to $|E|/2$. Every witness is interior: a bar is nonempty because $k \geq 1$ and proper because $k \leq L-1$ and $M \geq 2$.

Completing the spectrum. T is Eulerian, so by Proposition 3(a) no odd value is a level, and by Proposition 3(b) no value in $\{1, 2, 3\}$ is. T is bipartite, so by Theorem 4 $|E|$ is a level and the mirror $u \mapsto |E| - u$ maps the attained even values in $[4, |E|/2]$ onto the even values in $[|E|/2, |E| - 4]$. Together with $u = 0$ this gives $\mathcal{C}(T) = \{0\} \cup \{u \text{ even} : 4 \leq u \leq 2LM - 4\} \cup \{2LM\}$ and

$$|\mathcal{C}(T)| = 1 + \left(\frac{2LM - 4 - 4}{2} + 1 \right) + 1 = LM - 1 = \frac{|E|}{2} - \lambda + 3 = N - 1, \quad (33)$$

the bipartite Eulerian branch of (20). Hypothesis (H) is therefore proved for T , with $t(T) = |E| - \lambda = 2LM - 4$.

Odd second side. If M is odd the same construction uses $\lfloor M/2 \rfloor$ rows and reaches only $u \leq 2\lfloor M/2 \rfloor L = (M-1)L < ML = |E|/2$, and the mirror is unavailable because an odd cycle factor makes T non-bipartite. This is exactly the pattern of the four incomplete cases reported in Section 4.2.

A.3 Proposition 9: $\lambda = \delta_{\min}$ for products of paths and cycles

Let $G = G_1 \square \dots \square G_d$ with $G_a = P_{n_a}$, $n_a \geq 2$, or $G_a = C_{n_a}$, $n_a \geq 3$, and write $c_a = \delta_{\min}(G_a)$, so $c_a = 1$ for a path and $c_a = 2$ for a cycle. Degrees add in a Cartesian product, so $\delta_{\min}(G) = \sum_a c_a$; it remains to prove $\lambda(G) \geq \sum_a c_a$. The proof is by induction on d .

For $d = 1$, a path has a bridge and $\lambda = 1 = c_1$, while a cycle needs two edge removals to disconnect and $\lambda = 2 = c_1$.

For $d \geq 2$ let $\emptyset \neq S \subsetneq V(G)$. Every edge of G lies along exactly one axis, so $|\delta(S)| = \sum_a \delta_a(S)$, where $\delta_a(S)$ counts the edges of $\delta(S)$ along axis a . An a -fibre is a set of sites with all coordinates but the a -th fixed; it induces a copy of G_a , and the edges of $\delta(S)$ along axis a are precisely the cuts of the restrictions $S \cap F$ inside the a -fibres F . Call a fibre *mixed* if $0 < |S \cap F| < |F|$ and let m_a be the number of mixed a -fibres. Since a nonempty proper subset of G_a has cut at least $\lambda(G_a) = c_a$,

$$|\delta(S)| \geq \sum_{a=1}^d c_a m_a. \quad (34)$$

If $m_a \geq 1$ for every a , then (34) gives $|\delta(S)| \geq \sum_a c_a$ and we are done. Otherwise fix an axis a with $m_a = 0$. Then every a -fibre is either contained in S or disjoint from it, so $S = \pi_a^{-1}(T)$ for a subset T of the vertex set of $H = \square_{b \neq a} G_b$, with $\emptyset \neq T \subsetneq V(H)$ because S is a nonempty proper subset. Every edge of $\delta(S)$ joins two a -fibres, and each crossing edge of H gives exactly n_a such edges, so

$$|\delta(S)| = n_a |\delta_H(T)| \geq n_a \lambda(H) = n_a \sum_{b \neq a} c_b \quad (35)$$

by the induction hypothesis, which applies because H is a product of $d - 1$ paths and cycles. It remains to check $n_a \sum_{b \neq a} c_b \geq \sum_b c_b$, i.e. $(n_a - 1) \sum_{b \neq a} c_b \geq c_a$. Here $d \geq 2$, so the sum on the left has at least one term and $\sum_{b \neq a} c_b \geq 1$. If $n_a \geq 3$ then $(n_a - 1) \sum_{b \neq a} c_b \geq 2 \geq c_a$; if $n_a = 2$ then $G_a = P_2$, $c_a = 1$ and $(n_a - 1) \sum_{b \neq a} c_b \geq 1 = c_a$. This completes the induction.

A.4 Proposition 10: the doubled-bond violations

Take the smallest case, the 2×2 specification with boundary conditions *op* under the doubled convention: axis one open, axis two periodic of length two and doubled. It has $N = 4$ sites, $|E| = 6$ bonds — two open bonds and two pairs of parallel bonds — and every site has degree $1 + 2 = 3$, so $\delta_{\min} = 3$. Let S be the two sites sharing the first coordinate. The only bonds with exactly one endpoint in S are the two open bonds, so $|\delta(S)| = 2$ and $\lambda \leq 2$; no bond is a bridge, since a parallel pair survives the loss of either of its bonds and each open bond lies on the four-cycle that runs along the periodic axis and back, so $\lambda \geq 2$. Hence $\lambda = 2 < 3 = \delta_{\min}$.

Vertex-transitivity: the four product translations $i \mapsto i + t$, $t \in \mathbb{Z}_2 \times \mathbb{Z}_2$, each map the bond multiset onto itself — translation along the doubled periodic axis permutes the two parallel bonds, and translation along the open axis of length two interchanges the endpoints of each open bond, leaving the bond set fixed — and the orbit of any one site under them is all of V . Both facts were checked bond by bond rather than assumed, and the same check was run for the other four certified cases, where the translation group has order N and again has a single orbit.

A.5 Proposition 7: the certificates for the missing levels

Both exception lattices are d -regular with $d = 5$. For a subset S of size n with $e = |E(S)|$ internal edges, Proposition 3(a) specialised to a regular graph gives $u = |\delta(S)| = 5n - 2e$, so u has the parity of n and, for each n , determines $e = (5n - u)/2$ uniquely. Exhaustive enumeration of all 2^{24} and 2^{18} subsets gives, for every n , the exact set of attainable values of e ; its minimum and maximum are tabulated in Table 6 for $n \leq 12$; the remaining sizes follow from $u(S) = u(V \setminus S)$ and $e(V \setminus S) = |E| - e(S) - u(S)$, the first of which was verified on every size.

For the $2 \times 3 \times 4$ box with boundary conditions *opp* ($|E| = 60$): $u = 6$ would need $e = (5n - 6)/2$, that is $e = 2, 7, 12, 17, 22, 27$ at $n = 2, 4, 6, 8, 10, 12$, against $e_{\max}(n) = 1, 4, 9, 12, 17, 24$; every requirement exceeds the maximum, and at $n = 24$ the requirement $e = 57$ falls below $e_{\min}(24) = 60$. $u = 7$ would need $e = 4, 9, 14, 19, 24, 29$ at $n = 3, 5, 7, 9, 11, 13$ against $e_{\max} = 3, 6, 10, 15, 20, 25$, again all exceeded. $u = 51$ would need $e = 2$ at $n = 11$ and $e = 7$ at $n = 13$, against $e_{\min} = 3$ and 8 , both too small. $u = 50$ would need $e = 0$ at $n = 10$, where $e_{\min}(10) = 2$, or $e = 5$ at $n = 12$, which lies inside $[e_{\min}, e_{\max}] = [4, 24]$ but is not attained: the attainable internal-edge counts at size 12 are $4, 6, 7, \dots, 21, 24$, and 5 is absent. The same test applied to all 45 attained levels of this lattice excluded none, which is the control that the certificate is not vacuous.

For the $2 \times 3 \times 3$ box ($|E| = 45$) the identical computation gives $u = 6$ needing $e = 2, 7, 12, 17, 22, 27$ at $n = 2, 4, \dots, 12$ against $e_{\max} = 1, 4, 9, 14, 19, 24$, and $u = 7$ needing $e = 4, 9, 14, 19, 24, 29$ at

$n = 3, 5, \dots, 13$ against $e_{\max} = 3, 6, 11, 18, 21, 26$; both are excluded at every size, and $u_{\max} = 33$ leaves no near-maximum gap inside the predicted interval.

A.6 The mirror refinements of Section 3

The global flip. The map $s \mapsto -s$ sends S to $V \setminus S$. Since $\delta(V \setminus S) = \delta(S)$, it preserves u ; since $M(-s) = -M(s)$, it negates the magnetisation. It is an involution on configurations, so $g(b, M) = g(b, -M)$ for every bond graph, which is (15).

The colour-class flip. Let G be bipartite with $V = A \sqcup A'$ and define $\varphi(s)_i = -s_i$ for $i \in A$ and $\varphi(s)_i = s_i$ for $i \in A'$. Every bond joins A to A' , so φ reverses the sign of $s_i s_j$ on every bond: $u(\varphi(s)) = |E| - u(s)$. Writing $M_A = \sum_{i \in A} s_i$ and $M_{A'} = \sum_{i \in A'} s_i$, so that $M = M_A + M_{A'}$ and $M_s = M_A - M_{A'}$, the flip gives $M(\varphi(s)) = -M_A + M_{A'} = -M_s(s)$ and $M_s(\varphi(s)) = -M_A - M_{A'} = -M(s)$, which is (16). As φ is an involution, the triple histogram satisfies $\mathcal{G}(u, M, M_s) = \mathcal{G}(|E| - u, -M_s, -M)$.

The marginal. Summing over M_s and then applying the global flip to the staggered magnetisation, which leaves u unchanged,

$$\tilde{g}(u, M) = \sum_{M_s} \mathcal{G}(u, M, M_s) = \sum_{M_s} \mathcal{G}(|E| - u, -M_s, -M) = \tilde{h}(|E| - u, -M) = \tilde{h}(|E| - u, M), \quad (36)$$

which is (17).

Why the naive mirror fails. On the 4×4 torus the level $u = 4$ carries 16 configurations at $M = +14$ and 16 at $M = -14$, and none at $M = \pm 2$; its mirror image $u = |E| - 4 = 28$ carries 16 at $M = +2$ and 16 at $M = -2$, and none at $M = \pm 14$. In terms of u the naive identity reads $\tilde{g}(u, M) = \tilde{g}(|E| - u, -M)$, and here $\tilde{g}(4, 14) = 16$ while $\tilde{g}(28, -14) = 0$: it is false already at the first excited level. Of the $33 \times 33 = 1089$ entries of the (u, M) table of this lattice, 104 differ from their naive mirror image. The same failure was measured on the $2 \times 2 \times 2$ box, the open 3×4 grid and every other bipartite lattice tested, while the colour-class flip (16) and its marginal (17) held entry by entry on all of them. Physically, the two states are a single flipped spin, which leaves the magnetisation nearly saturated, and its image under the colour-class flip, which is a single flipped spin of the opposite sublattice and leaves the staggered magnetisation nearly saturated.

B Derivation of the one-dimensional closed forms

B.1 The ring

A configuration of the ring C_n is a circular word of length n in two symbols; let m be the number of down spins, so $M = n - 2m$, and let j be the number of maximal runs of down spins around the circle. Every run has two ends and every end is an unsatisfied bond, and no other bond is unsatisfied, so

$$u = 2j, \quad b = n - 2j. \quad (37)$$

For $1 \leq m \leq n - 1$ a word with j runs is assembled from the j run lengths, each a positive integer summing to m , the j gap lengths between consecutive runs, each a positive integer summing to $n - m$ — positivity is what makes the runs maximal — and a choice of which site the first run starts at. The number of positive compositions of m into j parts is $\binom{m-1}{j-1}$ and of $n - m$ into j parts is $\binom{n-m-1}{j-1}$. Marking a starting site in n ways counts every word exactly j times, once for each choice of which of its runs is called first, so

$$g(n - 2j, n - 2m) = \frac{n}{j} \binom{m-1}{j-1} \binom{n-m-1}{j-1}, \quad 1 \leq j \leq \min(m, n - m), \quad (38)$$

which is (27). The quotient n/j times the product is an integer because it counts words; the same integrality was asserted and checked arithmetically at every index in every evaluation of the formula. The two uniform configurations are the $j = 0$ case: $g(n, \pm n) = 1$, giving $m(0) = 2$ as required for a connected bond graph.

Summing over m at fixed j and applying Vandermonde's convolution in the form

$$\sum_m \binom{m-1}{j-1} \binom{n-m-1}{j-1} = \binom{n-1}{2j-1}$$

gives the zero-field marginal (28). The level set follows from (37): the admissible run counts are $0 \leq j \leq \lfloor n/2 \rfloor$, since $j \leq \min(m, n-m)$ forces $2j \leq n$, and every such j is attained, for instance by $m = j$ down spins placed one per run, which is admissible because $n-j \geq j$. Hence $\mathcal{C}(C_n) = \{0, 2, \dots, 2\lfloor n/2 \rfloor\}$ and $|\mathcal{C}(C_n)| = \lfloor n/2 \rfloor + 1$.

At zero field the partition function is therefore a sum of exactly $\lfloor n/2 \rfloor + 1$ distinct exponent values,

$$Z(\beta, J, 0) = e^{\beta J n} \sum_{j=0}^{\lfloor n/2 \rfloor} n(n-2j) e^{-4\beta J j}, \quad (39)$$

with the multiplicities (28), and at nonzero field a sum over the attainable pairs (j, m) weighted by $e^{-4\beta J j} e^{\beta h(n-2m)}$.

B.2 The open chain

A configuration of P_n is a linear word; let t be its number of domain walls, that is, of positions i with $s_i \neq s_{i+1}$. Then

$$u = t, \quad b = (n-1) - t, \quad 0 \leq t \leq n-1. \quad (40)$$

Fix m down spins and $n-m$ up spins. If $t = 2k$ is even the word begins and ends with the same symbol; there are two cases. Beginning and ending with an up spin, the word has $k+1$ up-runs and k down-runs, counted by $\binom{n-m-1}{k} \binom{m-1}{k-1}$; beginning and ending with a down spin it has $k+1$ down-runs and k up-runs, counted by $\binom{m-1}{k} \binom{n-m-1}{k-1}$. If $t = 2k-1$ is odd the word begins and ends with different symbols, so it has k runs of each kind, counted by $\binom{m-1}{k-1} \binom{n-m-1}{k-1}$, and there are two choices of which symbol comes first. This is (29).

The convention $\binom{-1}{-1} = 1$, with $\binom{a}{b} = 0$ outside $0 \leq b \leq a$, is what makes the two cases cover the uniform words without special treatment: for $m = 0$ and $t = 0$ the first branch gives $\binom{n-1}{0} \binom{-1}{-1} + \binom{-1}{0} \binom{n-1}{-1} = 1$, the single all-up word, and similarly for $m = n$.

Every t between 0 and $n-1$ is attained — an alternating prefix of length $t+1$ followed by a constant tail has exactly t walls — so $\mathcal{C}(P_n) = \{0, 1, \dots, n-1\}$ and $|\mathcal{C}(P_n)| = n$, with no parity constraint because the degrees $1, 2, \dots, 2, 1$ are not all even.

B.3 The partition function as a check

Assembling (2) from either closed form and evaluating the classical 2×2 transfer matrix

$$T = \begin{pmatrix} e^{\beta J + \beta h} & e^{-\beta J} \\ e^{-\beta J} & e^{\beta J - \beta h} \end{pmatrix} \quad (41)$$

in exact rational arithmetic gives two routes to the same integer polynomial in $x = e^{2\beta J}$ and $y = e^{\beta h}$. At x, y rational the two agree exactly; the comparison was run at eight exact points, the pairs $(2, 3)$,

$(3, 2)$, $(2, 5)$, $(5, 7)$, $(4, 1)$, $(7, 11)$, $(1, 1)$ and $(2, 1)$, and at 43 sizes — $n = 3$ to 40 together with 50, 62, 100, 200 and 400 — for both families, $8 \times 43 \times 2 = 688$ comparisons in all, with zero mismatches. An independent implementation repeated the comparison over $n = 3, \dots, 40$, $8 \times 38 \times 2 = 608$ further agreements, also with zero mismatches. Because the transfer matrix is the classical route [4] and shares no recursion or data structure with the run-counting formulas, the agreement is a genuine check of (27) and (29) rather than a tautology.

C Data tables

The tables in this appendix list every lattice of the census with the invariants that enter Proposition 5: the number of sites N , the number of bonds $|E|$, the edge connectivity λ , the maximum cut $u_{\max}$, bipartiteness, whether every degree is even, the exact level count $|\mathcal{C}(G)|$ and the count predicted by (20). All values are exact integers. The column “agree” records whether $|\mathcal{C}(G)|$ equals the prediction, that is, whether hypothesis (H) holds for that lattice; the two exceptions are discussed in Section 4.3 and certified in Appendix A.5. Minimum degree is not tabulated because $\lambda = \delta_{\min}$ on every lattice listed (Propositions 9 and 10).

Table 3: One-dimensional lattices of the census: rings C_n and open chains P_n . The simple convention is the only one that matters here except for the doubled ring at $n = 2$, whose bond graph is a digon. Level counts agree with Corollary 16 throughout, and with (20) in the branch indicated by bipartiteness and degree parity.

shape	boundaries	form	bonds	N	$ E $	$\delta_{\min}$	λ	$u_{\max}$	bip.	Eul.	levels	Prop. 5	agree
3	p	ring	simple	3	3	2	2	2	no	yes	2	2	yes
4	p	ring	simple	4	4	2	2	4	yes	yes	3	3	yes
5	p	ring	simple	5	5	2	2	4	no	yes	3	3	yes
6	p	ring	simple	6	6	2	2	6	yes	yes	4	4	yes
7	p	ring	simple	7	7	2	2	6	no	yes	4	4	yes
8	p	ring	simple	8	8	2	2	8	yes	yes	5	5	yes
10	p	ring	simple	10	10	2	2	10	yes	yes	6	6	yes
12	p	ring	simple	12	12	2	2	12	yes	yes	7	7	yes
16	p	ring	simple	16	16	2	2	16	yes	yes	9	9	yes
20	p	ring	simple	20	20	2	2	20	yes	yes	11	11	yes
24	p	ring	simple	24	24	2	2	24	yes	yes	13	13	yes
30	p	ring	simple	30	30	2	2	30	yes	yes	16	16	yes
40	p	ring	simple	40	40	2	2	40	yes	yes	21	21	yes
62	p	ring	simple	62	62	2	2	62	yes	yes	32	32	yes
2	o	chain	simple	2	1	1	1	1	yes	no	2	2	yes
4	o	chain	simple	4	3	1	1	3	yes	no	4	4	yes
6	o	chain	simple	6	5	1	1	5	yes	no	6	6	yes
8	o	chain	simple	8	7	1	1	7	yes	no	8	8	yes
12	o	chain	simple	12	11	1	1	11	yes	no	12	12	yes
20	o	chain	simple	20	19	1	1	19	yes	no	20	20	yes
31	o	chain	simple	31	30	1	1	30	yes	no	31	31	yes
40	o	chain	simple	40	39	1	1	39	yes	no	40	40	yes
62	o	chain	simple	62	61	1	1	61	yes	no	62	62	yes
200	p	ring	simple	200	200	2	2	200	yes	yes	101	101	yes
1000	p	ring	simple	1000	1000	2	2	1000	yes	yes	501	501	yes
200	o	chain	simple	200	199	1	1	199	yes	no	200	200	yes
1000	o	chain	simple	1000	999	1	1	999	yes	no	1000	1000	yes

Table 4: Two-dimensional lattices of the census: tori (pp), grids (oo) and cylinders (op, po), simple and doubled. Rows with $N > 62$ carry an exact level set from the reachability route but no multiplicity table.

shape	boundaries	form	bonds	N	$ E $	$\delta_{\min}$	λ	$u_{\max}$	bip.	Eul.	levels	Prop. 5	agree
3×3	pp	torus	simple	9	18	4	4	12	no	yes	6	6	yes
3×4	pp	torus	simple	12	24	4	4	20	no	yes	10	10	yes
3×5	pp	torus	simple	15	30	4	4	22	no	yes	11	11	yes
3×6	pp	torus	simple	18	36	4	4	30	no	yes	15	15	yes
4×4	pp	torus	simple	16	32	4	4	32	yes	yes	15	15	yes
4×5	pp	torus	simple	20	40	4	4	36	no	yes	18	18	yes
4×6	pp	torus	simple	24	48	4	4	48	yes	yes	23	23	yes
4×8	pp	torus	simple	32	64	4	4	64	yes	yes	31	31	yes
5×5	pp	torus	simple	25	50	4	4	40	no	yes	20	20	yes
5×6	pp	torus	simple	30	60	4	4	54	no	yes	27	27	yes
5×12	pp	torus	simple	60	120	4	4	108	no	yes	54	54	yes
3×20	pp	torus	simple	60	120	4	4	100	no	yes	50	50	yes
4×15	pp	torus	simple	60	120	4	4	116	no	yes	58	58	yes
2×10	pp	torus	simple	20	30	3	3	30	yes	no	27	27	yes
2×30	pp	torus	simple	60	90	3	3	90	yes	no	87	87	yes
6×6	pp	torus	simple	36	72	4	4	72	yes	yes	35	35	yes
6×10	pp	torus	simple	60	120	4	4	120	yes	yes	59	59	yes
3×3	oo	grid	simple	9	12	2	2	12	yes	no	11	11	yes
4×4	oo	grid	simple	16	24	2	2	24	yes	no	23	23	yes
5×5	oo	grid	simple	25	40	2	2	40	yes	no	39	39	yes
6×6	oo	grid	simple	36	60	2	2	60	yes	no	59	59	yes
4×8	oo	grid	simple	32	52	2	2	52	yes	no	51	51	yes
7×8	oo	grid	simple	56	97	2	2	97	yes	no	96	96	yes
12×5	oo	grid	simple	60	103	2	2	103	yes	no	102	102	yes
10×6	oo	grid	simple	60	104	2	2	104	yes	no	103	103	yes
6×10	oo	grid	simple	60	104	2	2	104	yes	no	103	103	yes
31×2	oo	grid	simple	62	91	2	2	91	yes	no	90	90	yes
20×3	oo	grid	simple	60	97	2	2	97	yes	no	96	96	yes
15×4	oo	grid	simple	60	101	2	2	101	yes	no	100	100	yes
8×7	op	cylinder	simple	56	105	3	3	97	no	no	96	96	yes
20×3	op	cylinder	simple	60	117	3	3	97	no	no	96	96	yes
15×4	op	cylinder	simple	60	116	3	3	116	yes	no	113	113	yes
12×5	op	cylinder	simple	60	115	3	3	103	no	no	102	102	yes
2×10	pp	torus	doubled	20	40	4	4	40	yes	yes	19	19	yes
2×30	pp	torus	doubled	60	120	4	4	120	yes	yes	59	59	yes
100×6	oo	grid	simple	600	1094	2	2	1094	yes	no	1093	1093	yes
300×3	oo	grid	simple	900	1497	2	2	1497	yes	no	1496	1496	yes
500×2	oo	grid	simple	1000	1498	2	2	1498	yes	no	1497	1497	yes
40×7	op	cylinder	simple	280	553	3	3	513	no	no	512	512	yes
80×5	op	cylinder	simple	400	795	3	3	715	no	no	714	714	yes
7×7	pp	torus	simple	49	98	4	4	84	no	yes	42	42	yes
8×8	pp	torus	simple	64	128	4	4	128	yes	yes	63	63	yes
8×12	pp	torus	simple	96	192	4	4	192	yes	yes	95	95	yes
8×15	pp	torus	simple	120	240	4	4	232	no	yes	116	116	yes

Table 5: Three-dimensional lattices of the census. The last four rows are the doubled-convention boxes, whose bond graphs carry parallel pairs; the two rows with boundary conditions opp and no doubling are the exceptions of Proposition 7.

shape	boundaries	form	bonds	N	$ E $	$\delta_{\min}$	λ	$u_{\max}$	bip.	Eul.	levels	Prop. 5	agree
$2 \times 2 \times 2$	<i>ooo</i>	open box	simple	8	12	3	3	12	yes	no	9	9	yes
$2 \times 2 \times 3$	<i>ooo</i>	open box	simple	12	20	3	3	20	yes	no	17	17	yes
$2 \times 2 \times 4$	<i>ooo</i>	open box	simple	16	28	3	3	28	yes	no	25	25	yes
$2 \times 2 \times 5$	<i>ooo</i>	open box	simple	20	36	3	3	36	yes	no	33	33	yes
$2 \times 2 \times 6$	<i>ooo</i>	open box	simple	24	44	3	3	44	yes	no	41	41	yes
$2 \times 3 \times 3$	<i>ooo</i>	open box	simple	18	33	3	3	33	yes	no	30	30	yes
$3 \times 3 \times 2$	<i>ooo</i>	open box	simple	18	33	3	3	33	yes	no	30	30	yes
$2 \times 3 \times 4$	<i>ooo</i>	open box	simple	24	46	3	3	46	yes	no	43	43	yes
$3 \times 3 \times 3$	<i>ooo</i>	open box	simple	27	54	3	3	54	yes	no	51	51	yes
$2 \times 2 \times 2$	<i>ppp</i>	periodic box	simple	8	12	3	3	12	yes	no	9	9	yes
$2 \times 2 \times 3$	<i>ppp</i>	periodic box	simple	12	24	4	4	20	no	yes	10	10	yes
$2 \times 6 \times 2$	<i>ppp</i>	periodic box	simple	24	48	4	4	48	yes	yes	23	23	yes
$2 \times 2 \times 6$	<i>ppp</i>	periodic box	simple	24	48	4	4	48	yes	yes	23	23	yes
$2 \times 3 \times 4$	<i>opp</i>	mixed box	simple	24	60	5	5	52	no	no	45	49	no
$2 \times 3 \times 3$	<i>opp</i>	mixed box	simple	18	45	5	5	33	no	no	28	30	no
$2 \times 5 \times 2$	<i>opp</i>	mixed box	simple	20	40	4	4	36	no	yes	18	18	yes
$2 \times 3 \times 2$	<i>opp</i>	mixed box	simple	12	24	4	4	20	no	yes	10	10	yes
$2 \times 6 \times 2$	<i>opp</i>	mixed box	simple	24	48	4	4	48	yes	yes	23	23	yes
$2 \times 4 \times 2$	<i>opp</i>	mixed box	simple	16	32	4	4	32	yes	yes	15	15	yes
$2 \times 2 \times 2$	<i>ppp</i>	periodic box	doubled	8	24	6	6	24	yes	yes	9	9	yes
$2 \times 2 \times 6$	<i>ppp</i>	periodic box	doubled	24	72	6	6	72	yes	yes	33	33	yes
$2 \times 6 \times 2$	<i>ppp</i>	periodic box	doubled	24	72	6	6	72	yes	yes	33	33	yes
$2 \times 6 \times 2$	<i>opo</i>	mixed box	doubled	24	48	4	4	48	yes	yes	23	23	yes

Table 6: The per-size induced-edge ranges behind the certificates of Proposition 7. For a subset of size n in a 5-regular bond graph, $u = 5n - 2e$ with e the number of internal edges, so a level u is attainable only if some n of the right parity admits $e = (5n - u)/2$ within $[e_{\min}(n), e_{\max}(n)]$, and — for the sharper certificate at $u = 50$ — only if that value of e is actually attained. Both endpoints were obtained by enumerating all 2^N subsets; sizes above 12 follow from $u(S) = u(V \setminus S)$ and $e(V \setminus S) = |E| - e(S) - u(S)$.

	n	0	1	2	3	4	5	6	7	8	9	10	11	12
$e_{\min}(n)$	$(2 \times 3 \times 4, \text{opp})$	0	0	0	0	0	0	0	0	0	1	2	3	4
$e_{\max}(n)$	$(2 \times 3 \times 4, \text{opp})$	0	0	1	3	4	6	9	10	12	15	17	20	24
$e_{\min}(n)$	$(2 \times 3 \times 3, \text{opp})$	0	0	0	0	0	0	0	2	4	6	9	12	15
$e_{\max}(n)$	$(2 \times 3 \times 3, \text{opp})$	0	0	1	3	4	6	9	11	14	18	19	21	24

Table 7: All 16 violations of $\lambda = \delta_{\min}$ among the 204 uniform lattice specifications with $N \leq 12$, each found by exhaustive enumeration of all 2^N vertex subsets. $Q = \sum t_{ij}(t_{ij} - 1)$ counts ordered pairs of distinct bonds sharing both endpoints; every violation has $Q > 0$, and no simple specification violates the equality. The last column reports the vertex-transitivity test: *certified* means the axis-flip translations were checked bond by bond to be automorphisms of the bond multiset and the orbit of a single site under them is all N sites; *not certified* means the same test was run and the orbit was smaller, which does not exclude vertex-transitivity under automorphisms outside the tested family.

shape	boundaries	bonds	N	$ E $	Q	$\delta_{\min}$	λ	vertex-transitive
2×2	<i>op</i>	doubled	4	6	4	3	2	certified
2×2	<i>po</i>	doubled	4	6	4	3	2	certified
2×3	<i>po</i>	doubled	6	10	6	3	2	not certified
2×4	<i>po</i>	doubled	8	14	8	3	2	not certified
2×5	<i>po</i>	doubled	10	18	10	3	2	not certified
2×6	<i>po</i>	doubled	12	22	12	3	2	not certified
3×2	<i>op</i>	doubled	6	10	6	3	2	not certified
4×2	<i>op</i>	doubled	8	14	8	3	2	not certified
5×2	<i>op</i>	doubled	10	18	10	3	2	not certified
6×2	<i>op</i>	doubled	12	22	12	3	2	not certified
$2 \times 2 \times 2$	<i>opp</i>	doubled	8	20	16	5	4	certified
$2 \times 2 \times 2$	<i>pop</i>	doubled	8	20	16	5	4	certified
$2 \times 2 \times 2$	<i>ppo</i>	doubled	8	20	16	5	4	certified
$2 \times 2 \times 3$	<i>ppo</i>	doubled	12	32	24	5	4	not certified
$2 \times 3 \times 2$	<i>pop</i>	doubled	12	32	24	5	4	not certified
$3 \times 2 \times 2$	<i>opp</i>	doubled	12	32	24	5	4	not certified

D Enumeration protocol, coverage and provenance

D.1 Construction of a bond graph from a specification

Sites are the tuples of $\prod_a \{0, \dots, n_a - 1\}$ in lexicographic order. For each axis a and each site i , the neighbour $i + e_a$ is formed with the a -th coordinate reduced modulo n_a when that axis is periodic and discarded when it is open and the coordinate is already $n_a - 1$. The bond multiset collects the unordered pairs $\{i, i + e_a\}$ over all sites and axes. When $n_a = 2$ and the axis is periodic, the forward and backward steps produce the same pair twice; the doubled convention keeps both copies, the simple convention keeps one. Writing t_{ij} for the multiplicity of a site pair, $Q = \sum t_{ij}(t_{ij} - 1)$ is the number of ordered pairs of distinct bonds with the same endpoints, as in Section 2; degrees are counted with multiplicity, and every structural statement in this paper is a statement about the resulting multigraph, not about the declared convention. Degrees, $|E|$, bipartiteness (by two-colouring), the parallel-pair count and the triangle count are recomputed from the multiset rather than read from the specification.

D.2 Routes and caps

Table 8 lists the four routes of Section 6, what each of them returns and where each of them stops.

Table 8: The four enumeration routes, what each returns and where each stops. The caps are enforced in code, not inferred from timings: a route refuses a lattice it cannot finish exactly rather than returning an approximate answer.

route	returns	cap
exhaustive enumeration	the exact table $g(b, M)$ and the sublattice refinement $g(b, m_A, m_B)$	$N \leq 24$
dictionary transfer	the exact table $g(b, M)$ and its refinement	cross-sections of at most 2^6 slice configurations
dense integer transfer	the exact table $g(b, M)$ and its refinement	multiplicities below 2^{63} , so $N \leq 62$; cross-sections of 2^6 periodic or 2^{12} open slice configurations
boolean reachability	the attained level set $\mathcal{C}(G)$ only	cross-sections of 2^8 periodic or 2^{10} open slice configurations; no cap on the size of the integers

Coverage of the 94 censused lattices: 50 confirmed by the first three routes together, 2 by the two transfer routes (the entries at $N = 25$, immediately above the exhaustive cap), 29 by the dense integer route alone, and 13 by the reachability route alone. Of those 13, twelve exceed the 64-bit magnitude ceiling — the largest is the ring with $n = 1000$, whose multiplicities have 302 digits — and one, the 7×7 torus at $N = 49$, exceeds only the width cap of the counting routes. The 29 single-route tables are the main coverage limitation of the census; one of them, the 6×6 torus, is confirmed externally (Section 10).

D.3 Exactness

Every reported quantity is an exact integer or an exact truth value. Spin sums, bond counts, cut sizes, subset counts and multiplicities are carried in machine integers where they fit and in arbitrary-precision integers otherwise; the one-dimensional partition-function comparison is carried

in exact rational arithmetic; and the level-set route uses bitwise operations on exact bitmasks. No floating-point value enters any number quoted in this paper, and no tolerance is used in any comparison: two tables agree when they are equal entry by entry. The only non-integer quantities recorded anywhere are wall-clock timings, which are reported only as an order of magnitude and never as evidence.

D.4 Independent reimplementations

The verification of Section 10 used a second implementation built from the model description alone. It reconstructs lattices from the shape, boundary conditions and bond convention by an independent routine; enumerates spin configurations and, separately, vertex subsets, so that the spin formulation of “unsatisfied bond” and the set formulation $|\delta(S)|$ are compared rather than assumed equal; implements the four-branch count (20) from the statement of Proposition 5 rather than from any existing code; builds the coincidence-class certificates from $K_2 \square K_2 = C_4$ by explicit coordinate maps; and re-derives the per-size induced-edge ranges of Appendix A.5 by exhaustive enumeration. It agrees with the census on every overlapping quantity, and it is the source of the numbers reported for the mirror refinements of Section 3 and for the certificates of Proposition 7.

D.5 External recomputations

The first external check was executed here after its code was inspected for network access and for file writes, neither of which it has; it uses only the standard library, enumerates vertex subsets rather than spin configurations, and shares no data structure with any route used here. Its outputs were compared against the tables of this paper entry by entry, with maximum absolute difference zero on all 24 ladder tables and on all 35 marginals and level counts.

The second was not executed. Its published integers were transcribed and re-derived from the routes used here, which makes it a value-level cross-check rather than a code-level one; this distinction is stated wherever its numbers are quoted. Its convention $u = (|E| - b')/2$ with $b' = \sum s_i s_j$ coincides with the convention here because $b' = 2b - |E|$ by (6), so the two histograms are the same integers and were compared without rescaling.

D.6 Availability

The enumeration code, the exact records behind every table and figure, the three figures and the verification scripts are archived with the source repository [8]. Each table is produced by a single deterministic, single-threaded computation from that archive, and each figure is drawn only from those records, with no hand-entered value. The environment of record is CPython 3.12.3 with NumPy 1.26.4 on a Linux x86-64 host; nothing in the results depends on that environment beyond the availability of arbitrary-precision integers, which CPython provides natively.

Bibliographic provenance is reported in Section 10: twelve entries resolve through Crossref or the arXiv API with matching titles, authors, venues, volumes, pages and years, three carry complete records without a DOI, and abstract-level content — no full text — was consulted for three of them during the preparation of this paper. The twelve entries verified only at the level of their bibliographic record are cited for attribution and context and support no technical claim.

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