

Exact transfer-matrix spectra of $SU(N)$ valence-bond-solid states: the largest eigenvalue by complete positivity, the wedge spectrum as a discrete orthogonal-polynomial system, and a basis criterion for outer multiplicity

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Abstract

The valence-bond-solid states of $SU(N)$ spin chains put a self-conjugate virtual irrep on every bond and project the two virtual halves meeting at a site onto a physical irrep. Their transfer matrices determine the norm of the state, the degeneracy structure of its entanglement spectrum and every correlation length, and four statements about those spectra were published as conjectures. All four are settled here. The leading eigenvalue is the dimension of the physical irrep divided by the square of the dimension of the virtual one: the value follows from a single trace identity, and its maximality follows because the transfer matrix is completely positive with real Kraus operators and sends the identity to a positive multiple of itself. The conjectured degeneracy pattern is false as stated, by three independent exact counterexamples, and the two ways in which it can fail are separated. For the family whose virtual irrep is the middle exterior power, the entire spectrum, including the interior rows that no closed form had reached, is a discrete orthogonal-polynomial system fixed by two known inputs; the correlation lengths follow from it, and one published formula holds only under a different selection rule than the one stated with it. What had blocked the case of an outer-multiplicity physical irrep turns out to be a property of the working basis, and a criterion removes it. Closed forms for the symmetric adjoint copy, for the interior rows beyond the third, and for the location of this polynomial system in the Askey scheme remain open.

1 Introduction

1.1 The state and the operator whose spectrum is at issue

A valence-bond-solid state is built in two steps. One places an auxiliary, or *virtual*, degree of freedom on each half of every bond of the chain and projects each bond onto its singlet, so that the virtual halves of neighbouring sites are tied together; one then projects the two virtual halves meeting at a site onto a chosen *physical* irrep, which becomes the on-site Hilbert space of the spin chain. For $SU(2)$ and the smallest non-trivial choices this is the spin-one chain of Affleck, Kennedy, Lieb and Tasaki [1, 2], whose exact ground state, exact gap and exact correlation length made it the standard example of a gapped, symmetry-protected one-dimensional phase. The construction generalises to $SU(N)$: given a self-conjugate virtual irrep V — self-conjugacy is what makes a singlet in $V \otimes V$ exist at all — and any physical irrep P occurring in $V \otimes V$, one obtains a translation-invariant matrix product state with a

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finite correlation length and edge modes in the irrep V [3]. General recipes for the parent Hamiltonians of such states, and the entanglement properties of the $SU(n)$ families, are in Tu et al. [4] and Katsura et al. [5]; the $SU(3)$ case is a symmetry-protected topological phase whose classification reads off the same data [6].

Everything quantitative about the state passes through one operator. Writing A^σ for the matrix product tensor, with σ running over a basis of the physical irrep P , the *transfer matrix* is

$$T = \sum_{\sigma} A^\sigma \otimes \bar{A}^\sigma, \quad (1)$$

acting on the doubled virtual space. It is the object whose spectrum the general theory of matrix product states makes central: its largest eigenvalue fixes the norm of the state on a periodic chain, the ratio of its two leading eigenvalues fixes the correlation length [7, 8], and the multiplicities of its eigenvalues fix the degeneracy pattern of the entanglement spectrum, which is what distinguishes a symmetry-protected phase from a trivial one [6, 9]. It is a finite matrix whose entries are Clebsch–Gordan coefficients, so its spectrum is exactly computable; what has been missing is a general account of it.

1.2 The four published statements

For $SU(N)$ valence-bond-solid states with an arbitrary self-conjugate virtual irrep V of dimension d and an arbitrary physical irrep $P \subseteq V \otimes V$, Gozel et al. [3] computed the spectrum of (1) in a range of examples and recorded four regularities as conjectures, supported by every example examined up to $N = 12$ but without a general proof. In the notation fixed in Section 2 — W ranges over the irreducible summands of $V \otimes V$, $\lambda_k(j)$ is the eigenvalue of the transfer matrix T_j built from the physical irrep P_j on the summand W_k , and the largest eigenvalue is $\lambda_0(j)$ because the singlet summand is labelled $k = 0$ — the four statements are:

- (a) the eigenvalues of T are degenerate with multiplicities exactly equal to $\dim W$ for the irreps W occurring in $V \otimes V$, and the largest eigenvalue is the unique non-degenerate one, associated with the singlet;
- (b) the largest eigenvalue is $\lambda_{\max} = \dim P/d^2$;
- (c) for $V = \Lambda^{N/2}$ and P the two-column irrep, the correlation length satisfies $\xi^{-1} = \ln(N+1)$;
- (d) for P the adjoint, $\xi^{-1} = \ln((N+1)/(N-3))$.

The two $SU(2)$ specialisations published alongside them are a required consistency check on any general answer.

1.3 Two ambiguities that change the answers

Two features of that statement have to be settled before it can be answered, and both are settled here as part of the problem definition rather than left implicit.

The first concerns the correlation length. The published definition is $\xi^{-1} = \ln |\lambda_1/\lambda_2|$ with λ_1, λ_2 the two eigenvalues of *largest real part*, while the criterion by which statements (c) and (d) are to be validated, and the standard matrix-product-state formula, are statements about the two eigenvalues of *largest modulus*. The transfer matrix is neither positive nor symmetric: its eigenvalues are rational numbers of both signs, so the two selection rules genuinely pick different pairs. Every result below is therefore labelled by the reading under which it holds, and Section 6 reports both. Statement (b) is the exception that needs no label, for a reason that turns out to be the content of its proof: the value $\dim P/d^2$ is positive and bounds the modulus of every eigenvalue, so it is simultaneously the largest real eigenvalue, the largest real part and the largest modulus.

The second concerns statement (a), which can fail in two ways that have different causes and different meanings. A *collision* is an accident of the numbers: two inequivalent summands $W_k \not\cong W_l$ happen to carry the same eigenvalue, so an observed multiplicity is a sum $\dim W_k + \dim W_l$. Schur's lemma is not violated and the transfer matrix is still scalar on each summand. A *split* is a fact about the representation theory: if W occurs in $\text{End}(V)$ with outer multiplicity $\mu > 1$, Schur's lemma forces only a $\mu \times \mu$ block on the isotypic component, and the transfer matrix need not be scalar there, so an observed multiplicity is *smaller* than $\dim W$. Both are tested and reported separately throughout.

1.4 What this paper establishes

Statement (b) is a theorem, in both its halves. The value half needs nothing beyond self-conjugacy: a single trace identity against the singlet projector gives $\lambda_0(j) = \dim P_j/d^2$ for every N , every self-conjugate V , every P , every outer multiplicity and every invariant metric (Section 3). The maximality half, which is the part that had remained open, is a consequence of complete positivity. The transfer matrix has the Kraus form $T_P(Y) = d^{-1} \sum_s Z_s Y Z_s^T$ for any orthonormal basis $\{Z_s\}$ of the physical subspace, with *real* operators Z_s , and it satisfies $T_P(\mathbf{1}) = (\dim P/d^2)\mathbf{1}$. Rescaled, it is therefore a unital positive map on a matrix algebra, hence a contraction of the operator norm, hence of spectral radius at most one. Where the two hypotheses of that argument hold — for the wedge family $V = \Lambda^{N/2}$ at every even N , for P the singlet in every family, and, by the basis change of Section 7, for both adjoint copies at every $N \geq 3$ — statement (b) is proved; elsewhere it is verified exactly, and the reason the proof does not reach those cases is identified.

Statement (a) is false as literally stated, and it is false for three independent reasons, one of them completely general. If P is the singlet then the physical subspace is the line through the identity, the transfer matrix is $T_0 = \mathbf{1}_{\text{End}(V)}/d^2$, and the whole d^2 -dimensional doubled space is a single eigenspace: every sector collides and the largest eigenvalue is d^2 -fold degenerate. That is a theorem, not a sporadic failure, and it holds for every N and every self-conjugate V . The wedge family supplies a second mechanism, collisions in the interior rows $2 \leq j \leq m-1$, exact at $N=6$ and $N=8$ and classified by a criterion at every even N . The adjoint family supplies the third and, in one respect, the strongest: because its antisymmetric transfer matrix turns out to be a function of the quadratic Casimir alone, every conjugate pair $W, \bar{W}$ with $W \not\cong \bar{W}$ collides at every rank, and the same family exhibits the only genuine split found in this work. Section 4 separates the mechanisms and records what survives: the rows $j=1$ and $j=m$ are collision-free at every even N , and there the leading eigenvalue is unique, positive and singlet-associated, which is the physically useful content of statement (a).

The main structural result concerns the wedge family $V = \Lambda^m(\mathbb{C}^N)$, $N = 2m$. Its doubled space is multiplicity-free, with summands W_k labelled by the partitions $(2^k, 1^{N-2k})$, $k = 0, \dots, m$. Set

$$t_k = 1 - \frac{4k(N+1-k)}{N(N+1)}, \quad w_k = \frac{\dim W_k}{d^2}, \quad (2)$$

and let $P_0, \dots, P_m$ be the Gram-Schmidt orthogonal polynomials of the discrete measure $\{(t_k, w_k)\}$, normalised by $P_j(1) = 1$. Then the sector ratios are the values of those polynomials at the nodes,

$$r_k(j) = \frac{\lambda_k(j)}{\lambda_0(j)} = P_j(t_k), \quad \text{so that} \quad \lambda_k(j) = \frac{\dim P_j}{d^2} P_j(t_k), \quad (3)$$

for every j, k and every even N . The nodes (2) are the physical-adjoint row, which is a theorem, and the weights are Weyl dimensions, so both inputs are already known; the derivation (Section 5) is an induction resting on a full orthogonality relation and a uniqueness lemma, and it needs one hypothesis, a degree bound on the rows. That hypothesis is verified exactly at every

even $N \leq 60$ in the strong form $\deg P_j = j$, is unnecessary for $j = 0, 1$, and is a theorem in the prior work cited at the point of use. What (3) delivers is the whole table $\lambda_k(j)$, including the interior rows that no closed form had reached, with no eigensolver, no Racah or walled-Brauer recoupling and no evaluation of a tetrahedral coefficient. Three consequences are worth naming now: the second row has the closed form $P_2(t) = ((N^2 - 1)t^2 - 2(N - 1)t - 1)/(N(N - 2))$; two sectors collide in that row exactly when their Casimir eigenvalues sum to $N^2/2$, which turns a predicted infinite family of collisions into a theorem; and the two-column row $P_m(t_k) = (-1)^k / \binom{N+1}{k}$ is produced by the recursion rather than fitted to it.

For the correlation lengths the outcome is mixed, and the mixture is the answer. Statement (d) holds for every even $N \geq 4$ under both readings, because monotonicity of $k(N + 1 - k)$ makes the same pair of sectors leading in real part and in modulus. Statement (c) does not: under the modulus reading the second eigenvalue is $-1/(N + 1)$ and $\xi^{-1} = \ln(N + 1)$ exactly as published, but under the literal largest-real-part reading the second eigenvalue is $2/(N(N + 1))$, so $\xi^{-1} = \ln(N(N + 1)/2)$, which at $N = 4$ is $\ln 10$ against the published $\ln 5$. The discrepancy grows with N , and its mechanism is the sign alternation of the two-column row, which a real-part ordering cannot see. This is a defect in the definition that accompanied the conjecture, not in the physics, and the exact eigenvalues are the same under both readings.

Finally, the case that had resisted treatment — a physical irrep occurring with outer multiplicity in $V \otimes V$, where V is the adjoint — is resolved by a change of basis. The bond map $\Psi(X) = XS$, whose image is the physical subspace, is a Hilbert–Schmidt homothety exactly when the invariant bilinear form satisfies $SS^T = c\mathbf{1}$ in the working basis of V . By Schur’s lemma the singlet S is unique up to scale, so the only freedom is the basis, and the rational Cartan–Weyl basis in which the construction is usually written is oblique. In a Killing-orthonormal basis the same invariant form is a scalar matrix, both adjoint copies become certified, and the antisymmetric one acquires the closed form $\lambda_f(W) = (2N - C_2(W))/(2N(N^2 - 1))$.

1.5 Relation to earlier work on the same problem

Four earlier treatments of this problem are cited here, and the division of labour matters for reading the results above correctly. Prior work [10] derived, for the same even- N wedge family and before any of the work reported here, both named rows — the physical-adjoint row $\lambda_k(1)$ with multiplicities $\dim W_k = \binom{N}{k}^2 - \binom{N}{k-1}^2$ and the two-column row $\lambda_k(m)$ — together with both correlation-length values, from an $\text{End}(V)$ Casimir argument. Statements (c) and (d), as correlation-length statements, and those two closed forms are therefore rederivations here, not firsts; the independent content of Sections 6 and 5 is the route and its by-products, in particular the proof that statement (d) is independent of the choice of reading, the trace-form derivation of the two frame-operator constants, the orthogonal-polynomial determination of the interior rows, and the collision criterion. The same record lists as out of scope exactly the ground this paper covers: arbitrary self-conjugate V , outer multiplicities, and arbitrary-spin $\text{SU}(2)$.

Two further records established the two row closed forms as theorems by different routes — a frame-operator argument for the physical-adjoint row and a Johnson-scheme argument for the two-column row — together with the third-row closed form, the moments of the discrete measure, its three-term recurrence, and the degree hypothesis as a theorem [11–13]. Everything this paper uses from them is cited at the point of use, and the independent verifications reported here — 23 exact checks reproducing the third row, the moments and the recurrence from this paper’s own nodes and weights by a disjoint route — are described as verifications, not as results. A fourth record supplied the correction to the realignment identities on which two of the proofs below depend, proved the multiplicity-freeness of the wedge family at every even N , published the exact $N = 12$ spectrum that the recursion of Section 5 is checked against, and observed the degree law empirically [14–16]. The basis criterion of Section 7 confirms, from

scratch and with no shared code, a diagnosis posted there [17], and reproduces every exact rational it tabulates.

What remains this paper’s own is the following: the proof of maximality by complete positivity, and the identification of its exact hypothesis; the orthogonal-polynomial determination of the whole wedge table including the interior rows; the basis criterion, the closed form for the antisymmetric adjoint copy and the resulting theorem for statement (b) in the outer-multiplicity family; the three refutations of statement (a) with the collision and split mechanisms separated; and the two-reading analysis of the correlation length, which is what makes statement (c) a claim about a definition. The complete derivations, and the exact arithmetic behind every table, are archived in the source record [18].

1.6 Organisation

Section 2 fixes the construction, the conventions and the realignment identity on which everything rests. Section 3 proves statement (b) and records the four exact identities that constrain any candidate closed form. Section 4 refutes statement (a) and separates its two failure modes. Section 5 contains the orthogonal-polynomial determination of the wedge spectrum and its consequences. Section 6 settles statements (c) and (d) and the $SU(2)$ consistency check. Section 7 resolves the outer-multiplicity family. Section 8 collects the boundaries and the open questions. Appendix A gives the index-level properties of the realignment and the witnesses that show their hypotheses are sharp; Appendix B derives the two Casimir operators, the sector Casimir values and the frame-operator constants; Appendix C holds the exact spectra, the collision census and the status table; and Appendix D states the exact-arithmetic protocol and the coverage of every check.

2 The construction, its conventions and the realignment identity

2.1 The virtual space and its two invariant forms

Let V be a self-conjugate irreducible representation of $SU(N)$ of dimension d , with a fixed rational basis $\{e_a\}$, $a = 0, \dots, d - 1$, and write $\rho(X)$ for the matrix of $X \in \mathfrak{gl}_N$ on V in that basis. Two invariant forms on V enter the construction and must not be confused. The first is the invariant positive hermitian form G , normalised arbitrarily, characterised by

$$\rho(E_{ij})^\top G + G \rho(E_{ji}) = 0, \quad (4)$$

where E_{ij} are the matrix units of $\mathfrak{gl}_N$. The second is the invariant bilinear form S , characterised by $\rho(X)^\top S + S \rho(X) = 0$ for every traceless X ; it satisfies $S^\top = \varepsilon S$ with $\varepsilon = \pm 1$, and it spans the singlet in $V \otimes V$, which is why self-conjugacy of V is the hypothesis that makes the construction possible. Two scalars complete the data: the centre weight w , defined by $\rho(\sum_i E_{ii}) = w\mathbf{1}$, and the quadratic Casimir eigenvalue c_V of V in the normalisation $C_2(\text{fundamental of } SU(N)) = (N^2 - 1)/(2N)$.

Three families are used, and they differ in one respect that turns out to decide where a proof applies: whether G is the identity.

Table 1 lists them. For the wedge family the invariant bilinear form is the Hodge pairing, an integer orthogonal matrix; for $\text{Sym}^n(\mathbb{C}^2)$ it is the $SU(2)$ -invariant pairing; for the adjoint it is $S_{ab} = \text{Tr}(M_a M_b)/(2N)$ in the rational Cartan–Weyl basis. Each of the quantities in Table 1 is checked exactly against an independent source — the invariance residuals (4) entrywise over the rationals, d against the Weyl dimension formula for the partition [19], c_V against the Young-diagram evaluation [20], and self-conjugacy before any form is built; a non-self-conjugate input is rejected rather than repaired.

Table 1: The three families of self-conjugate virtual irreps used in this paper, with the invariant forms in the working basis in which each is constructed. Only the wedge family has $G = \mathbf{1}$, and that single fact is what makes the doubled Casimir symmetric and the sector projectors orthogonal there (Section 3). The sign ε and the overall scale of S never enter a reported number: the transfer matrix depends only on the physical subspace, and the sector eigenvalues only on the transfer matrix.

family	V	basis	G	ε	w
wedge	$\Lambda^m(\mathbb{C}^N)$, $N = 2m$	ordered wedge monomials	$\mathbf{1}$	$(-1)^m$	m
symmetric power, SU(2)	$\text{Sym}^n(\mathbb{C}^2)$, spin $n/2$	monomials $x^a y^b$	$\text{diag}(a! b!)$	$(-1)^n$	n
adjoint	$\text{ad}(\mathfrak{sl}_N)$	E_{ij} for $i \neq j$, and $H_i = E_{ii} - E_{i+1, i+1}$	$\text{Tr}(M_a M_b^\top)/(2N)$	$+1$	0

2.2 The doubled space, the conjugation action and the realignment

The transfer matrix (1) acts on the doubled bond space, which is $\text{End}(V) = V \otimes V^*$, not on $V \otimes V$. A matrix $Z \in \text{End}(V)$ is vectorised row-major, $\text{vec}(Z)_{(ab)} = Z_{ab}$, so an operator on $\text{End}(V)$ is a $d^2 \times d^2$ matrix with indices $((a, b), (c, d))$. The action that matters is the *conjugation* action,

$$\rho_{\text{dbl}}(X) Z = [\rho(X), Z], \quad \text{i.e.} \quad \rho_{\text{dbl}}(X) = \rho(X) \otimes \mathbf{1} - \mathbf{1} \otimes \rho(X)^\top, \quad (5)$$

whose invariant subspace is $\text{End}_{\text{SU}(N)}(V) = \mathbb{C}\mathbf{1}$, one-dimensional because V is irreducible. In a G -orthonormal basis with $G = \mathbf{1}$ the conjugation action and the action on $V \otimes V$ are equivalent through $\mathbf{1} \otimes S$ and the two descriptions give the same spectrum; for $G \neq \mathbf{1}$ they do not, and only the conjugation action is correct here. That is why covariance is always checked against (5).

The second ingredient is the *realignment*

$$\mathcal{R}(X)_{(ac),(bd)} = X_{(ab),(cd)}, \quad (6)$$

which regroups the four indices of an operator on $\text{End}(V)$ from the channel pairing $(ab)(cd)$ to the pairing $(ac)(bd)$. It is an involution, $\mathcal{R}^2 = \mathbf{1}$. Three further identities involving $\mathcal{R}$ are used repeatedly, and all three are easy to mis-transcribe, so they are stated with their hypotheses and proved entry by entry in Appendix A:

$$\mathcal{R}(X)^\top = \mathcal{R}(X^{T^4}), \quad \text{Tr}(\mathcal{R}(X) \mathcal{R}(Y)^\top) = \text{Tr}(XY^\top), \quad (7)$$

$$\text{Tr}(\mathcal{R}(X) \mathcal{R}(Y)) = \text{Tr}(XY) \quad \text{whenever } Y^\top = Y \text{ and } Y^{T^4} = Y, \quad (8)$$

$$\text{Tr}(\mathcal{R}(X) Y) = \text{Tr}(\mathcal{R}(Y) X) \quad \text{whenever } X \text{ and } Y \text{ are symmetric,} \quad (9)$$

$$\mathcal{R}(|\text{vec}(\mathbf{1})\rangle\langle\text{vec}(\mathbf{1})|) = \mathbf{1}_{\text{End}(V)}, \quad \text{hence } \mathcal{R}(\Pi_0) = \frac{1}{d} \mathbf{1}_{\text{End}(V)}, \quad (10)$$

where the four-index transpose is $X_{(ab),(cd)}^{T^4} = X_{(ba),(dc)}$ and $\Pi_0 = |\text{vec}(\mathbf{1})\rangle\langle\text{vec}(\mathbf{1})|/d$ is the singlet projector. The unqualified forms of (7) and (8) — with the ordinary matrix transpose in place of the four-index one, and with no hypothesis on Y — are false, and Appendix A prints exact witnesses. The hypothesis of (8) is not a formality: for a projector $\Pi = \sum_l |B_l\rangle\langle B_l|$ onto a subspace $W \subseteq \text{End}(V)$ one has $\Pi^{T^4} = \sum_l |B_l^\top\rangle\langle B_l^\top|$, the projector onto the transposed subspace $W^\top = \{B^\top : B \in W\}$, so $\Pi^{T^4} = \Pi$ says exactly that W is closed under transposition of matrices. In the multiplicity-free wedge family that is automatic, because transposition is an

equivariant involution and therefore permutes isotypic components, and it is verified exactly on every wedge projector at $N = 2, 4, 6$ together with the consequence $\text{Tr}(\mathcal{R}(\Pi_i)\mathcal{R}(\Pi_j)) = \text{Tr}(\Pi_i\Pi_j)$ that the two proofs using (8) need. Where $V \otimes V$ has outer multiplicity and the projectors are built in an oblique basis, neither property holds in that basis; the exact residuals are recorded in Appendix A, and Section 7 identifies the obliqueness as a property of the basis and removes it.

2.3 The matrix product tensor and the transfer-matrix identity

Let P be a physical irrep occurring in $V \otimes V$, hence in $\text{End}(V)$, and let $W_P \subseteq \text{End}(V)$ be a chosen copy of P under the conjugation action, with Π_P the Hilbert–Schmidt orthogonal projector onto it. When P occurs with outer multiplicity $\mu > 1$, Π_P is the projector onto *one* of the μ copies. Normalising the bond singlet to unit length, the matrix product tensor is $A^\sigma = Z_\sigma/\sqrt{d}$, where $\{Z_\sigma\}$, $\sigma = 1, \dots, \dim P$, is any Hilbert–Schmidt-orthonormal basis of W_P . In the rational bases of Table 1 the Z_σ are real, so the complex conjugation in (1) is inert and the transfer matrix is $T = \sum_\sigma Z_\sigma \otimes Z_\sigma/d$.

Theorem 1 (transfer-matrix identity). *With the conventions above,*

$$T_P := \sum_\sigma A^\sigma \otimes A^\sigma = \frac{\mathcal{R}(\Pi_P)}{d}. \quad (11)$$

Proof. $T_P = d^{-1} \sum_\sigma Z_\sigma \otimes Z_\sigma$. Taking $X = \Pi_P = \sum_\sigma \text{vec}(Z_\sigma) \text{vec}(Z_\sigma)^\top$ in (6), so that $X_{(ab),(cd)} = \sum_\sigma (Z_\sigma)_{ab} (Z_\sigma)_{cd}$, gives

$$\mathcal{R}(\Pi_P)_{(ac),(bd)} = \sum_\sigma (Z_\sigma)_{ab} (Z_\sigma)_{cd} = \sum_\sigma (Z_\sigma \otimes Z_\sigma)_{(ac),(bd)},$$

because $(A \otimes B)_{(ac),(bd)} = A_{ab} B_{cd}$. Multiplying by $1/d$ gives (11). $\square$

Two features of Theorem 1 are used constantly below. The first is that no square root and no orthonormalisation is needed to evaluate the right-hand side: $\sum_\sigma Z_\sigma \otimes Z_\sigma$ equals $\sum_\sigma Y_\sigma \otimes Y^\sigma$ for *any* basis $\{Y_\sigma\}$ of W_P together with its Hilbert–Schmidt dual $\{Y^\sigma\}$, defined by $\text{Tr}((Y^\sigma)^\top Y_\tau) = \delta_\tau^\sigma$, because both sides are the image of the identity of W_P under the canonical map $W_P \otimes W_P \rightarrow \text{End}(\text{End}(V))$ and are therefore determined by the subspace alone. This removes every phase and ordering convention from the construction. In every case in which the construction is certified, the difference between the two sides is computed entrywise over the rationals and is exactly zero, and the same holds for the difference between $\mathcal{R}(\Pi_P)$ and $\sum_\sigma Y_\sigma \otimes Y^\sigma$ evaluated in a deliberately skewed basis; that residual is the certificate that the identity is being used rather than assumed.

The second feature is that (11) holds for every invariant metric G and in every rational basis, because Π_P is built as a Lagrange interpolation polynomial in the doubled Casimir, which is G -self-adjoint, so Π_P is the G -Hilbert–Schmidt orthogonal projector whether or not $G = \mathbf{1}$. This is worth dwelling on, because it is where an earlier version of this construction went wrong. Two apparently more general formulas, each inserting G and the singlet S at plausible index positions and each reducing to $\mathcal{R}(\Pi_P)/d$ at $G = \mathbf{1}$, agree with each other exactly in every case including cases with $G \neq \mathbf{1}$; they are nevertheless wrong there. They were found by requiring covariance (12) below to hold exactly, the singlet eigenvalue to be $\dim P/d^2$, and, at $\text{SU}(2)$, agreement with the Wigner- $6j$ closed form of Section 6; those three requirements select $\mathcal{R}(\Pi_P)/d$ uniquely out of the whole family of variants. For $\text{SU}(2)$ with $V = \text{Sym}^2$ and physical spin $S = 1$, where $G = \text{diag}(2, 1, 2)$, the correct operator gives $\{1/3, 1/6, -1/6\}$ on the sectors $j = 0, 1, 2$, which is the $6j$ prediction, while the two general-metric formulas give $\{7/18, 7/36, -7/36\}$ with a nonzero covariance residual and a singlet eigenvalue $26/99 \neq 1/3$.

The failure was silent precisely because the two wrong routes were not independent of each other, both being built from the same mis-transported G insertions; what caught it were checks external to the construction. This is the reason every exact table in Appendix C reports the covariance residual and the dual-basis residual for every case rather than only for a single anchor.

2.4 Covariance, sectors and the two Casimir operators

Because W_P is invariant under the conjugation action and Π_P is the orthogonal projector onto it, $[\rho_{\text{dbl}}(X), \Pi_P] = 0$, and the transfer matrix satisfies the stronger, directly checkable statement

$$[\rho_{\text{dbl}}(X), T_P] = 0 \quad \text{for every } X \in \mathfrak{sl}_N. \quad (12)$$

This is verified exactly on the $2(N-1)$ simple-root generators, which generate $\mathfrak{sl}_N$, and on all N^2 generators wherever the dimension allows; the residual is zero in every certified case. Consequently T_P preserves every isotypic component of $\text{End}(V)$, and by Schur's lemma it acts as $\mathbf{1}_W \otimes B$ on a component $W \otimes \mathbb{C}^\mu$ with B a $\mu \times \mu$ matrix. When the outer multiplicity is $\mu = 1$, B is a number and the transfer matrix is scalar on the sector, with eigenvalue

$$\lambda_k(j) = \frac{\text{Tr}(T_j \Pi_k)}{\text{Tr} \Pi_k} = \frac{\text{Tr}(\mathcal{R}(\Pi_j) \Pi_k)}{d \dim W_k}, \quad (13)$$

where T_j is the transfer matrix of the physical irrep P_j and Π_k the projector onto the doubled sector W_k . Nothing in what follows assumes (13) is available: the scalar by which T_j acts is computed as $\text{Tr}(T_j \Pi_k) / \text{Tr} \Pi_k$ and the entrywise residual $T_j \Pi_k - \lambda_k(j) \Pi_k$ is then evaluated, and a sector is reported as scalar only when that residual is exactly zero. Where a sector splits, the residual is nonzero and both eigenvalues of the block are reported with their multiplicities.

The sector decomposition itself needs no eigensolver. Write $D(X) = \rho(X) \otimes \mathbf{1} + \mathbf{1} \otimes \rho(X)$ for the diagonal action on $V \otimes V$ and ρ_{dbl} for the conjugation action (5), and define the two quadratic Casimir operators by the same expression in each action, $C = \frac{1}{2} [\sum_{ij} D(E_{ij}) D(E_{ji}) - \frac{1}{N} D(\sum_i E_{ii})^2]$ and C_{dbl} with D replaced by ρ_{dbl} . Appendix B derives the two expansions

$$C = \sum_{ij} \rho(E_{ij}) \otimes \rho(E_{ji}) + \left(2c_V - \frac{w^2}{N}\right) \mathbf{1}, \quad C_{\text{dbl}} = \left(2c_V + \frac{w^2}{N}\right) \mathbf{1} - \tilde{\Omega}, \quad (14)$$

where

$$\tilde{\Omega} := \sum_{ij} \rho(E_{ij}) \otimes \rho(E_{ji})^\top. \quad (15)$$

The sign difference in the centre term is not a typo: the conjugation action kills the centre, $\rho_{\text{dbl}}(\sum_i E_{ii}) = w \mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes w \mathbf{1} = 0$, so the $-(1/N)$ correction vanishes identically there. Three consequences are used everywhere. First, $C_{\text{dbl}} \text{vec}(\mathbf{1}) = 0$: the identity of $\text{End}(V)$ spans the singlet. Second, $\tilde{\Omega}$ has integer entries in the wedge and adjoint bases, is symmetric whenever $G = \mathbf{1}$, and commutes with the whole conjugation action, so it is central and every spectral projector of $\tilde{\Omega}$ is an $\text{SU}(N)$ -submodule projector. Third, because C_{dbl} is affine in $\tilde{\Omega}$, the sector decomposition of $\text{End}(V)$ is read off the spectrum of the integer operator $\tilde{\Omega}$: on the summand W it acts as $\varphi(W) = 2c_V + w^2/N - C_2(W)$, so C_{dbl} itself acts on W as $C_2(W)$, and Π_W is the Lagrange interpolation polynomial of $\tilde{\Omega}$ at the values $\varphi(W)$. The completeness of those roots is certified by a minimal-polynomial residual rather than assumed.

2.5 The wedge family

For the rest of the paper $N = 2m$ is even and $V = \Lambda^m(\mathbb{C}^N)$, so $d = \binom{N}{m}$, $w = m$ and $c_V = N(N+1)/8$. The doubled space is multiplicity-free,

$$\begin{aligned} \text{End}(V) &= \bigoplus_{k=0}^m W_k, & W_k &= \lambda^{(k)} = (2^k, 1^{N-2k}), \\ d_k &:= \dim W_k = \binom{N+1}{k} \left[\binom{N}{k} - \binom{N}{k-1} \right], \end{aligned} \tag{16}$$

with $\sum_k d_k = d^2$; multiplicity-freeness for every even N is proved in prior work [14], and both the Weyl-dimension formula and the sum rule are verified exactly here at every even $N \leq 200$. The physical irreps run over the same set, $P_j = \lambda^{(j)}$, $j = 0, \dots, m$, with projectors Π_j and transfer matrices $T_j = \mathcal{R}(\Pi_j)/d$. The Casimir eigenvalues are

$$C_2(V) = \frac{N(N+1)}{8}, \quad C_2(W_k) = c_k = k(N+1-k), \quad \varphi_k = \frac{N(N+2)}{4} - c_k, \tag{17}$$

the first of which is the standard evaluation on the m -th fundamental representation, $m(N-m)(N+1)/(2N)$ at $m = N/2$, and the second of which is the Young-diagram evaluation on $\lambda^{(k)}$. The value $C_2(V)$ is confirmed assumption-free at $N = 2, 4, 6$ by building the Casimir from the Lie-algebra action itself and reading off the scalar ($3/4, 5/2, 21/4$); the sector values $C_2(W_k)$ are confirmed at the operator level at $N = 2, 4, 6$ by dense exact linear algebra and at $N = 8$ by an independent modular route, and at the formula level for every even $N \leq 200$ with $\dim \lambda^{(k)}$ matched against the recorded d_k . Appendix B records the verification table.

Two pieces of notation are used from here on without further comment. The sector ratio is

$$r_k(j) = \frac{\lambda_k(j)}{\lambda_0(j)}, \tag{18}$$

which is the natural object because $\lambda_0(j)$ has a closed form for every j (Theorem 2 below), and the inverse correlation length of the row j is

$$\xi^{-1} = \ln \left| \frac{\lambda_1}{\lambda_2} \right|, \tag{19}$$

where the pair (λ_1, λ_2) is selected either by largest real part or by largest modulus, and the reading is always named.

3 The largest eigenvalue

Statement (b) has two halves, and they need completely different arguments. The value of the singlet eigenvalue is a one-line trace computation with no hypothesis beyond self-conjugacy. That it is the *largest* eigenvalue is a statement about every other eigenvalue at once, and it is where the substance lies.

3.1 The value

Theorem 2 (value of the singlet eigenvalue). *Let V be a self-conjugate irrep of $\text{SU}(N)$ of dimension d and let P_j be any irrep occurring in $\text{End}(V)$, with transfer matrix $T_j = \mathcal{R}(\Pi_j)/d$. Then the eigenvalue of T_j on the singlet sector $W_0 = \mathbb{C}\mathbf{1}$ of $\text{End}(V)$ is*

$$\lambda_0(j) = \frac{\dim P_j}{d^2}. \tag{20}$$

No assumption on N , on V beyond self-conjugacy, on the outer multiplicity of P_j , or on the invariant metric is used.

Proof. The singlet sector is one-dimensional, spanned by $\text{vec}(\mathbf{1})$, and $\Pi_0 = |\text{vec}(\mathbf{1})\rangle\langle\text{vec}(\mathbf{1})|/d$, so $\text{Tr } \Pi_0 = 1$ and (13) gives $\lambda_0(j) = \text{Tr}(\mathcal{R}(\Pi_j)\Pi_0)/d$. Using (10),

$$\begin{aligned}\lambda_0(j) &= \frac{1}{d} \langle \text{vec}(\mathbf{1}) | \mathcal{R}(\Pi_j) | \text{vec}(\mathbf{1}) \rangle = \frac{1}{d^2} \sum_{a,b} \mathcal{R}(\Pi_j)_{(aa),(bb)} = \frac{1}{d^2} \sum_{a,b} (\Pi_j)_{(ab),(ab)} \\ &= \frac{\text{Tr } \Pi_j}{d^2} = \frac{\dim P_j}{d^2}.\end{aligned}\quad \square$$

Since $\dim P_j$ and d are fixed by the representation theory, the value cannot depend on the family: the same formula holds in the wedge family, in the $\text{SU}(2)$ symmetric powers, and for both adjoint copies, and every exact table in Appendix C records it as a separate check. At $\text{SU}(3)$ with V and P both adjoint it reads $1/8$; at $\text{SU}(4)$, $1/15$.

3.2 Complete positivity of the transfer matrix

The maximality half needs the transfer matrix to be thought of as a map on matrices rather than as a $d^2 \times d^2$ array. Lemma 3 below is the same identity as Theorem 1 with the indices unravelled, and it is what makes the operator a completely positive map with real Kraus operators.

Lemma 3 (Kraus form). *Let $\{Z_s\}$ be any Hilbert–Schmidt-orthonormal basis of $\text{im } \Pi_P$, so that $\Pi_P = \sum_s \text{vec}(Z_s) \text{vec}(Z_s)^\top$ with real Z_s . Then T_P acts on $Y \in \text{End}(V)$ as*

$$T_P(Y) = \frac{1}{d} \sum_s Z_s Y Z_s^\top, \quad (21)$$

and the Choi matrix of dT_P is Π_P .

Proof. Unravel (6) at the index pair (a, c) of the output and (b, e) of the input:

$$(\mathcal{R}(\Pi_P)Y)[a, c] = \sum_{b,e} (\Pi_P)_{(ab),(ce)} Y[b, e] = \sum_s Z_s[a, b] Y[b, e] Z_s[c, e] = \left(\sum_s Z_s Y Z_s^\top \right)[a, c].$$

Dividing by d gives (21). The Choi matrix of dT_P is $\sum_s \text{vec}(Z_s) \text{vec}(Z_s)^\top = \Pi_P$ by the same computation read in the other direction. $\square$

Because each Z_s is real, $Z_s^\top = Z_s^\dagger$ and (21) is a Kraus decomposition, so T_P is completely positive on the matrix algebra $M_d(\mathbb{C})$ [21]; equivalently Π_P is a real symmetric idempotent and therefore positive semidefinite. Orthonormalising a rational basis of $\text{im } \Pi_P$ may leave the rationals, but the map (21) does not depend on which orthonormal basis is used — that is the basis-independence recorded after Theorem 1 — so (21) is well defined. Positivity is never decided numerically here: it is checked through exact characteristic polynomials, together with two positivity tests that deliberately avoid the Choi argument, namely $T_P(vv^\top) \succeq 0$ for explicit rational vectors v and the Kadison–Schwarz inequality $\Phi(Y^\top Y) - \Phi(Y)^\top \Phi(Y) \succeq 0$ for explicit Y , with $\Phi = T_P/(\dim P/d^2)$.

3.3 The identity is an eigenvector

Equivariance puts the image of the identity inside the invariant subspace of the conjugation action, which is the line through the identity.

Lemma 4 (unitality up to scale). *Suppose Π_P is an orthogonal, equivariant projector, $\Pi_P^2 = \Pi_P = \Pi_P^\top$ and $[\rho_{\text{dbl}}(E), \Pi_P] = 0$ for every generator E , and set $p = \dim P$. Then $T_P(\mathbf{1}) = (p/d^2)\mathbf{1}$.*

Proof. Equivariance of Π_P gives $T_P(\rho Y \rho^{-1}) = \rho T_P(Y) \rho^{-1}$, so $T_P(\mathbf{1})$ is invariant under the conjugation action and therefore a scalar multiple of the identity, $T_P(\mathbf{1}) = c\mathbf{1}$. Taking the trace of (21) at $Y = \mathbf{1}$ and using orthonormality,

$$cd = \text{Tr } T_P(\mathbf{1}) = \frac{1}{d} \sum_s \text{Tr}(Z_s Z_s^\top) = \frac{1}{d} \sum_s \|Z_s\|_F^2 = \frac{p}{d},$$

hence $c = p/d^2$. $\square$

Lemma 4 is a second, independent derivation of the value (20): Theorem 2 obtains it as a trace against the singlet projector and needs no hypothesis at all, while Lemma 4 obtains it from equivariance and orthonormality and yields it entrywise rather than on the trace. The two agree in every case computed. The step $T_P(\mathbf{1}) = (\dim P/d^2)\mathbf{1}$ also has an independent source that does not use the realignment at all: the partial trace identity $\text{Tr}_2(\Pi_W) = (\dim W/\dim V)\mathbf{1}$, proved for every self-conjugate V and every orthogonal invariant projector in prior work [11], transports through Theorem 1 to exactly Lemma 4, and its hypothesis is precisely the orthogonality that fails in the oblique cases discussed below.

3.4 Maximality

Theorem 5 (statement (b), complete). *Let V , d , P , p and Π_P be as in Lemma 4, and $T_P = \mathcal{R}(\Pi_P)/d$. Then every eigenvalue λ of T_P satisfies $|\lambda| \leq p/d^2$, and p/d^2 is itself an eigenvalue, attained on the singlet sector. Hence*

$$\lambda_{\max} = \frac{\dim P}{d^2} \quad (22)$$

under every reading of “largest”: largest real eigenvalue, largest real part, or largest modulus.

Proof. Set $\mu = p/d^2 > 0$ and $\Phi = T_P/\mu$. By Lemma 3, Φ is completely positive, and by Lemma 4, $\Phi(\mathbf{1}) = \mathbf{1}$: it is unital. A unital positive map on a C^* -algebra is a contraction of the operator norm, because $-\|Y\|\mathbf{1} \leq Y \leq \|Y\|\mathbf{1}$ implies $-\|Y\|\Phi(\mathbf{1}) \leq \Phi(Y) \leq \|Y\|\Phi(\mathbf{1})$, and every power Φ^n is again unital and positive. For an eigenpair $T_P Y = \lambda Y$ with $Y \neq 0$,

$$|\lambda|^n \|Y\| = \|\Phi^n(Y)\| \leq \|Y\| \quad \text{for every } n,$$

so $|\lambda| \leq 1$ and therefore $|\lambda| \leq \mu$. Since μ is attained on $\mathbf{1} \neq 0$, the spectral radius of T_P is μ , which gives (22) in all three senses at once. $\square$

The conclusion is stronger than a bound on the modulus, and the difference is what makes statement (b) immune to the ambiguity of Section 1. Because $\mu > 0$ is an eigenvalue and bounds every eigenvalue in modulus, no ordering convention can select anything else as leading. That is not true of the subleading eigenvalue, and it is the reason statements (c) and (d) have to be reported twice while statement (b) does not.

Two corollaries are used later. First, in the wedge family the sector ratios obey $|r_k(j)| \leq 1$ for every j, k and every even N , by Theorems 2 and 5; this was previously only a verified fact over a finite range, and it is what makes the largest-modulus reading of (19) well defined at every row, with no caveat about ties at the top. Second, the eigenspace of μ contains the strictly positive element $\mathbf{1}$, so the Perron–Frobenius theory of positive maps governs the peripheral spectrum of Φ ; whether μ is *simple* is a different question, and Section 4 shows it has a negative answer in general.

Table 2: Scope of Theorem 5. The two hypotheses are that Π_P be orthogonal and that T_P be equivariant; where they are theorems, statement (b) is proved, and where they fail in the working basis the conclusion is certified exactly instead. “Certified” means the exact spectrum was computed over the rationals and the bound $|\lambda| \leq \dim P/d^2$ was checked on it with $\dim P/d^2$ attained on the singlet sector. The last row is the one case in which no spectrum is claimed at all.

family	$G = \mathbf{1}$	$\Pi^\top = \Pi$	$T_P(\mathbf{1}) = (p/d^2)\mathbf{1}$	statement (b)
wedge $\Lambda^{N/2}$, any even N , any P	yes	theorem	theorem	proved
any family, P the singlet	irrelevant	theorem	theorem	proved
adjoint V , P adjoint, both copies, $N \geq 3$	no, but the basis is changeable	theorem after the basis change	theorem	proved
SU(2), $V = \text{Sym}^n$, $n = 2, 3$, P non-singlet	no	fails, oblique	certified exactly	verified
SU(3), SU(4) adjoint, other P , covariance certified	no	fails, oblique	certified exactly	verified
SU(3) adjoint, P of dimension 10 in the oblique basis	no	fails	fails, covariance residual 1/8	no spectrum claimed

3.5 Where the two hypotheses hold

Theorem 5 carries two hypotheses: Π_P orthogonal, and T_P equivariant. Given them, complete positivity and the contraction bound are automatic, so the whole family dependence of statement (b) sits in those two lines. For the wedge family they are theorems at every even N , not checks at a few ranks. The chain is short and worth stating because it is the only place where the assumption $G = \mathbf{1}$ from Table 1 is used: $G = \mathbf{1}$ means the representation is of real orthogonal type,

$$\rho(E_{ij})^\top = \rho(E_{ji}) \quad \text{for all } i, j, \quad (23)$$

which is the infinitesimal form of $\rho(g)^\top = \rho(g)^{-1}$ for $g \in \text{SO}(N, \mathbb{R})$ and holds because $\Lambda^m(\mathbb{R}^N)$ is a real representation carrying the form induced by the standard one on $\mathbb{R}^N$. It follows that $\text{Tr}(AB^\top)$ is invariant under the conjugation action, so $\{\rho Z_s \rho^{-1}\} = \{\rho Z_s \rho^\top\}$ is again an orthonormal basis of $\text{im } \Pi_P$, and the basis-independence of (21) gives $T_P(\rho Y \rho^{-1}) = \rho T_P(Y) \rho^{-1}$: equivariance. The same involution makes the doubled action orthogonal, $\rho_{\text{dbl}}(E_{ij})^\top = \rho_{\text{dbl}}(E_{ji})$, hence the doubled Casimir symmetric by (14), hence every Lagrange projector $\Pi_P = f(C_{\text{dbl}})$ symmetric: orthogonality. For P the singlet the two hypotheses are theorems in *every* family, since Π_0 is the orthogonal projector onto $\mathbb{C}\mathbf{1}$ and equivariance is trivial. For V and P both adjoint they are theorems at every $N \geq 3$ in a Killing-orthonormal basis, which is the content of Section 7.

Where $G \neq \mathbf{1}$ — $V = \text{Sym}^n$ at SU(2) for $n \geq 2$, and V the adjoint, with P non-singlet — the Casimir–Lagrange projector is oblique, $\Pi^\top \neq \Pi$, with exact nonzero residuals, and the positivity route does not apply to the constructed operator. Table 2 records the position case by case.

The last row deserves a word, because it is the only place where the construction itself is not certified rather than merely inconvenient. There the covariance residual is 1/8 and the physical subspace is not invariant under the constructed operator, with residual 17/240, so no rational spectral resolution exists and no eigenvalue is claimed. Lemma 21 below predicts that the defect is entirely an effect of the oblique basis and disappears in a basis in which the

invariant form is scalar, exactly as it provably does for the adjoint copies; that prediction is not verified for these four cases, and they remain open.

3.6 Four exact identities that constrain any closed form

Four identities hold on every exact table computed here and are used below as inputs rather than as checks. They are consequences of Theorem 1 and of the four properties (7)–(10) of the realignment, and each has a proof of a few lines. Three of them are prior: the sum rule, the second-moment identity and the duality were established, in that order, in prior work [12], and everything said about them here is an independent rederivation by a different route. The fourth, the trace identity, was derived independently and contemporaneously in both records; it is one line from $\mathcal{R}(\Pi_0) = \mathbf{1}/d$ and $\sum_j \Pi_j = \mathbf{1}$, and no priority is claimed for it. What is this paper's own throughout is the route — the trace identity (9) in place of an assertion that $\mathcal{R}$ is self-adjoint — and the explicit statement of the symmetry hypothesis on Π_j that (8) and hence two of the four proofs need.

Proposition 6 (the four identities). *Let the physical irreps P_j run over the same set as the doubled sectors W_k , so that $\sum_j \Pi_j = \mathbf{1}_{\text{End}(V)}$. Then for every j and k ,*

$$(S1) \quad \sum_j \lambda_k(j) = \delta_{k0}, \quad (24)$$

$$(S2) \quad \dim W_k \lambda_k(j) = \dim W_j \lambda_j(k), \quad \text{equivalently} \quad r_k(j) = r_j(k), \quad (25)$$

$$(S3) \quad \sum_k \dim W_k \lambda_k(j)^2 = \frac{\dim P_j}{d^2}, \quad (26)$$

$$(S4) \quad \text{Tr } T_j = \sum_k \dim W_k \lambda_k(j) = \delta_{j0}. \quad (27)$$

Proof. For (24), $\sum_j T_j = \mathcal{R}(\sum_j \Pi_j)/d = \mathcal{R}(\mathbf{1})/d$, and (10) gives $\mathcal{R}(\mathbf{1}) = d \Pi_0$, so $\sum_j T_j = \Pi_0$, whose eigenvalue on W_k is δ_{k0} .

For (25), $\dim W_k \lambda_k(j) = \text{Tr}(\mathcal{R}(\Pi_j)\Pi_k)/d$ by (13), so the identity is the symmetry $\text{Tr}(\mathcal{R}(\Pi_j)\Pi_k) = \text{Tr}(\mathcal{R}(\Pi_k)\Pi_j)$, which is (9). Symmetry of the projectors is automatic when $G = \mathbf{1}$, because then C_{dbl} is a symmetric integer matrix and every Π is a polynomial in it; for $G \neq \mathbf{1}$ the projectors are only G -self-adjoint, and (25) is verified rather than proved. Combining (25) with (20) gives the ratio form, since $\dim W_k \dim P_j = \dim W_j \dim P_k$ when W and P range over the same set.

For (26), the left side is $\text{Tr}(T_j^2)$ whenever T_j is scalar on each sector; where a sector splits, it must be read as $\text{Tr}(T_j^2)$ over the full spectrum, which is what the exact computation sums. Then $\text{Tr}(T_j^2) = \text{Tr}(\mathcal{R}(\Pi_j)\mathcal{R}(\Pi_j))/d^2 = \text{Tr}(\Pi_j^2)/d^2$ by (8), whose hypothesis $\Pi_j^\top = \Pi_j = \Pi_j^{T^4}$ holds in the multiplicity-free wedge family and is verified exactly at $N = 2, 4, 6$, and Π_j is an orthogonal projector of rank $\dim P_j$.

For (27), $\text{Tr } T_j = \text{Tr}(\mathcal{R}(\Pi_j))/d = d^{-1} \sum_{a,c} (\Pi_j)_{(aa),(cc)} = \langle \text{vec}(\mathbf{1}) | \Pi_j | \text{vec}(\mathbf{1}) \rangle / d$, which is $d/d = 1$ when P_j is the singlet and 0 otherwise, because $\text{vec}(\mathbf{1})$ spans the singlet and is orthogonal to every other isotypic component. $\square$

Together with Theorem 2, the four identities determine the first row and the first column of the table $\lambda_k(j)$, the whole block $j, k \in \{0, 1\}$ once $\lambda_1(1)$ is known, and the second moment of every row. What they leave undetermined is the interior $2 \leq j \leq k \leq m-1$ of the wedge table, and it is exactly those rationals that the orthogonal-polynomial system of Section 5 supplies. All four are reported, under unambiguous names, on every exact spectrum in this paper: at $N = 2, 4, 6$ from the dense computation, at $N = 8$ from the independent modular computation, on the recursion table at every even $N \leq 200$, and in every $\text{SU}(2)$ case. An earlier display

of (27) carried a spurious factor $\dim P_j/d$ and conflated it with (20); the two statements are different, the correction changes no table, closed form or count, and it is recorded here because it garbled a displayed identity.

4 Statement (a): degeneracy, collisions and splits

4.1 Two ways in which the conjecture can fail

Statement (a) asserts that the eigenvalues of the transfer matrix are degenerate with multiplicities exactly equal to $\dim W$ for the irreps W occurring in $V \otimes V$, and that the largest eigenvalue is the unique non-degenerate one, associated with the singlet. There are two distinct mechanisms by which an observed multiplicity can differ from $\dim W$, and they have different causes and different meanings, so they are reported separately throughout.

Definition 7 (collision and split). A *collision* is a pair of inequivalent doubled sectors $W_k \not\cong W_l$ with $\lambda_k(j) = \lambda_l(j)$ for some physical irrep P_j . Schur's lemma is not violated and the transfer matrix is still scalar on each sector, but the eigenvalue has multiplicity $\dim W_k + \dim W_l$. A collision is a property of the numbers rather than of the representation theory, and it is therefore sporadic in N and in j . A *split* is a doubled sector occurring with outer multiplicity $\mu > 1$ on which $T_j = \mathbf{1}_W \otimes B$ with B not a scalar, so the eigenvalue multiplicities are *smaller* than $\dim W$. A split is the case a complete proof of statement (a) would have to exclude, and it is what the remark in Gozel et al. [3] about the adjoint occurring twice in $V \otimes V$ is really about.

For the wedge family the doubled space (16) is multiplicity-free, so a split cannot occur there and every failure of statement (a) in that family is a collision. The adjoint family is the other way round: it exhibits both.

4.2 The first refutation: the singlet physical irrep

The second half of statement (a) fails in complete generality, and the refutation is one line.

Proposition 8 (singlet physical irrep). *Let P_j be the trivial representation, for any N and any self-conjugate V . Then*

$$T_0 = \frac{1}{d^2} \mathbf{1}_{\text{End}(V)}, \quad (28)$$

so $\lambda_k(0) = 1/d^2 = \dim P_0/d^2$ for every doubled sector W_k : the whole d^2 -dimensional doubled space is a single eigenspace, every sector collides, and the largest eigenvalue is d^2 -fold degenerate.

Proof. The physical subspace is the singlet line, so $\Pi_j = \Pi_0$, and $T_0 = \mathcal{R}(\Pi_0)/d = \mathbf{1}_{\text{End}(V)}/d^2$ by (10). $\square$

Proposition 8 is not a sporadic counterexample but a theorem about a whole family, and it also explains why the sum rule (24) is saturated by a single row in this case. The exact values are $1/4$, $1/36$, $1/400$ and $1/4900$ in the wedge family at $N = 2, 4, 6, 8$, each with multiplicities summing to d^2 , and, outside that family, $1/64$ for V the adjoint of $\text{SU}(3)$ and $1/225$ for V the adjoint of $\text{SU}(4)$, with sector multiplicities $1, 16, 10, 10, 27$ and $1, 30, 20, 45, 45, 84$. Those two are the cases in which the construction is certified despite outer multiplicity in $V \otimes V$, and the multiplicities $10, 10$ and $45, 45$ are the two conjugate pairs that Section 7 shows are forced to collide by the structure of the decomposition.

Table 3: Exact transfer-matrix spectra $\lambda_k(j)$ of the wedge family $V = \Lambda^{N/2}(\mathbb{C}^N)$, computed over the rationals: rows are the physical irrep $P_j = (2^j, 1^{N-2j})$, columns the doubled sector $W_k = (2^k, 1^{N-2k})$, and the entry is the scalar by which T_j acts on W_k , of multiplicity d_k . At $N = 4$ and $N = 6$ the spectra come from dense exact diagonalisation of a $d^2 \times d^2$ rational matrix; at $N = 8$, where $d^2 = 4900$, from an independent modular route with reconstruction by the Chinese remainder theorem over eight primes. Every sector scalar was certified entrywise, with residual exactly zero. The two collisions discussed in the text are $\lambda_1(2) = \lambda_3(2) = 9/400$ at $N = 6$, of combined multiplicity $35 + 175 = 210$, and $\lambda_2(2) = \lambda_3(2) = -3/980$ at $N = 8$, of combined multiplicity $720 + 2352 = 3072$. Row $j = 0$ is constant by Proposition 8.

N	j	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
4	d_k	1	15	20		
	0	1/36	1/36	1/36		
	1	5/12	1/12	-1/12		
	2	5/9	-1/9	1/18		
6	d_k	1	35	189	175	
	0	1/400	1/400	1/400	1/400	
	1	7/80	3/80	1/240	-1/80	
	2	189/400	9/400	-11/400	9/400	
	3	7/16	-1/16	1/48	-1/80	
8	d_k	1	63	720	2352	1764
	0	1/4900	1/4900	1/4900	1/4900	1/4900
	1	9/700	1/140	1/350	0	-1/700
	2	36/245	8/245	-3/980	-3/980	1/245
	3	12/25	0	-1/100	1/140	-1/175
	4	9/25	-1/25	1/100	-3/700	1/350

4.3 The second refutation: interior rows of the wedge family

Collisions in the wedge family come from the interior physical irreps $2 \leq j \leq m-1$, never from the two named rows. For $j = 1$ the closed form $r_k(1) = 1 - 4k(N+1-k)/(N(N+1))$ makes a collision $r_k = r_l$ with $k \neq l$ require $k+l = N+1$, which is impossible for $k, l \leq N/2$; for $j = m$ the closed form $r_k(m) = (-1)^k / \binom{N+1}{k}$ makes it require $\binom{N+1}{k} = \binom{N+1}{l}$ together with $k \equiv l \pmod{2}$, again impossible. Both statements are theorems, and both are confirmed by a census of every even $N \leq 200$ in Appendix C, in which no collision occurs in row $j = 1$ or row $j = m$ anywhere. In the interior rows collisions are generic. Table 3 gives the exact spectra at $N = 4, 6, 8$; two collisions are visible in it.

The $N = 6$ collision is $\lambda_1(2) = \lambda_3(2) = 9/400$, with combined multiplicity $35 + 175 = 210$; the $N = 8$ collision is $\lambda_2(2) = \lambda_3(2) = -3/980$, with combined multiplicity $720 + 2352 = 3072$. Both are collisions in the sense of Definition 7: the sectors are inequivalent, the transfer matrix is scalar on each, and the residual $T_j \Pi_k - \lambda_k(j) \Pi_k$ is exactly zero on both members of the pair. Section 5 turns this from a list of instances into a classification: in row $j = 2$ two sectors collide exactly when their Casimir eigenvalues sum to $N^2/2$.

4.4 The third refutation: the adjoint family

The family $V = P = \text{adjoint}$ supplies a third and structurally different refutation, and it is the strongest of the three in one respect. The closed form established in Section 7 makes the antisymmetric transfer matrix a function of the quadratic Casimir alone, $\lambda_f(W) = (2N - C_2(W))/(2N(N^2 - 1))$, so every conjugate pair $W, \bar{W}$ with $W \not\cong \bar{W}$ collides at every

rank: not by accident but because the two irreps have equal Casimir. At $SU(3)$ the eigenvalue 0 has multiplicity $20 = \dim \mathbf{10} + \dim \overline{\mathbf{10}}$, and at $SU(4)$ the eigenvalue 0 has multiplicity $90 = \dim \mathbf{45} + \dim \overline{\mathbf{45}}$. The same family exhibits the only genuine split found in this work: the antisymmetric transfer matrix acts as $1/16$ on the whole isotypic component of the adjoint at $SU(3)$, of multiplicity $16 = 2 \dim \mathbf{8}$, and the symmetric copy resolves that component into $1/16$ and $-3/80$, each of multiplicity 8. Both mechanisms are certified rather than suspected, and Section 7 gives the details.

4.5 What survives

Statement (a) is false as stated, but not uniformly, and the part that survives is the part the physics uses. In every case computed here, for the two named rows $j = 1$ and $j = m$ of the wedge family and for both adjoint copies, the leading eigenvalue is unique, positive, non-degenerate and associated with the singlet sector. In the wedge family that is a consequence of Theorem 5 together with the collision-freeness of those two rows proved above. In the adjoint family it is exact computation: $\lambda_{\max} = 1/8$ with multiplicity one at $SU(3)$ and $1/15$ with multiplicity one at $SU(4)$, for both copies. The first half of statement (a), by contrast, fails twice over in that family — once by a structural collision and once by a genuine split — and fails for every N and every self-conjugate V when the physical irrep is the singlet.

There is also a positive statement available about multiplicities, and it is worth separating it from statement (a). Where the doubled space is multiplicity-free and no collision occurs, the multiplicity of $\lambda_k(j)$ is exactly $\dim W_k$, and that is certified entrywise rather than inferred: the exact computation reports both the scalar and the residual against it, and in every sector of every $SU(2)$ case and every wedge case at $N = 2, 4, 6, 8$ the residual is zero and the multiplicity equals $\dim W_k$, which is $2j + 1$ in the $SU(2)$ family and d_k in the wedge family.

5 The wedge spectrum as a discrete orthogonal-polynomial system

The four identities of Proposition 6 and the closed form (20) pin down the first row, the first column and the second moment of every row of the table $\lambda_k(j)$, and nothing else. The interior $2 \leq j \leq k \leq m - 1$ is where the difficulty of this problem has always sat: at $N = 10$ six rationals were published with the explicit remark that no closed form fits them, and at $N = 12$ a complete exact spectrum was published with the same remark attached to its interior. This section shows that the interior is not a collection of unexplained numbers but a classical object: the whole table is a system of discrete orthogonal polynomials on an explicit finite measure, both of whose inputs are already known in closed form.

5.1 The measure

Set

$$c_k = C_2(W_k) = k(N + 1 - k), \quad t_k = 1 - \frac{4c_k}{N(N + 1)} = 1 - \frac{c_k}{2c_V}, \quad w_k = \frac{d_k}{d^2}, \quad (29)$$

where $c_V = N(N + 1)/8$ and d_k is as in (16). The t_k are the *nodes* and the w_k the *weights*. Three properties are immediate and are checked exactly for every even $N \leq 200$: $w_k > 0$, $\sum_k w_k = 1$, and the t_k are pairwise distinct, because c_k is strictly increasing on $0 \leq k \leq m$. So $\{(t_k, w_k)\}$ is a discrete probability measure on $m + 1$ points, and the inner product

$$\langle f, g \rangle = \sum_{k=0}^m w_k f(t_k) g(t_k) \quad (30)$$

is positive definite on polynomials of degree at most m . Let $P_0, \dots, P_m \in \mathbb{Q}[t]$ be the Gram–Schmidt orthogonal polynomials of (30), normalised by $P_j(1) = P_j(t_0) = 1$.

The nodes are not new numbers. Since $t_k = (d^2/d_1)\lambda_k(1)$, the node set is exactly the physical-adjoint row rescaled, so the measure is built from the physical-adjoint row and the Weyl dimensions and from nothing else. At $N = 8$, $d = 70$, and

$$\begin{aligned} c_k &= [0, 8, 14, 18, 20], & d_k &= [1, 63, 720, 2352, 1764], \\ t_k &= \left[1, \frac{5}{9}, \frac{2}{9}, 0, -\frac{1}{9}\right], & w_k &= \left[\frac{1}{4900}, \frac{9}{700}, \frac{36}{245}, \frac{12}{25}, \frac{9}{25}\right], \end{aligned}$$

with $\sum_k d_k = 4900 = d^2$ and $\sum_k w_k = 1$. The first moments are $M_p = \sum_k w_k t_k^p$:

$$M_0 = 1, \quad M_1 = 0, \quad M_2 = \frac{1}{N^2 - 1}, \quad M_3 = \frac{2(N - 1)}{(N^2 - 1)^2}, \quad (31)$$

and, from the cited recurrence theory,

$$M_4 = \frac{9N - 25}{(N - 3)(N - 1)(N + 1)^3}, \quad M_5 = \frac{4(11N^2 - 35N + 16)}{N(N - 3)(N - 1)(N + 1)^4}; \quad (32)$$

at $N = 8$ these read 1, 0, $1/63$, $2/567$, $47/25515$ and $44/45927$. Every one of (31) and (32) is recomputed here from the measure, exactly, for every even $N \leq 60$; M_0 – M_3 are also verified for every even $N \leq 200$. Of these, $M_0 = 1$ and $M_1 = 0$ are theorems, $M_2 = 1/(N^2 - 1) = 1/d_1$ is a theorem (Lemma 10 below), M_3 is a cited theorem verified here, and M_4 , M_5 are cited and verified.

5.2 The single hypothesis

Hypothesis 9 (degree law). For every $j = 0, \dots, m$ there is a polynomial $Q_j \in \mathbb{Q}[t]$ of degree at most j such that $r_k(j) = Q_j(t_k)$ for all $k = 0, \dots, m$.

Hypothesis 9 has an operator form that makes its content clear. Since $t_k = (d^2/d_1)\lambda_k(1)$, it says exactly that every T_j is a polynomial of degree at most j in T_1 , namely

$$T_j = \frac{d_j}{d^2} Q_j\left(\frac{d^2}{d_1} T_1\right). \quad (33)$$

In a multiplicity-free doubled space *some* polynomial in T_1 of degree at most m reproduces T_j , by interpolation on the $m + 1$ distinct eigenvalues; Hypothesis 9 says the degree can be taken to be j . At $N = 8$, where $d_1 = 63$ and $d^2 = 4900$, the polynomials (33) are

$$\begin{aligned} T_2 &= -\frac{3}{980} - \frac{10}{3}T_1 + \frac{3500}{3}T_1^2, \\ T_3 &= \frac{1}{140} + \frac{16}{9}T_1 - \frac{38500}{9}T_1^2 + \frac{4900000}{9}T_1^3, \\ T_4 &= -\frac{3}{700} + \frac{10}{3}T_1 + \frac{33250}{9}T_1^2 - \frac{12250000}{9}T_1^3 + \frac{857500000}{9}T_1^4, \end{aligned}$$

each of the announced degree and each reproducing the exact spectrum of Table 3.

Three remarks fix the status of Hypothesis 9, and they matter for reading everything that follows. First, $j = 0$ and $j = 1$ need no hypothesis: $Q_0 = 1$ because $T_0 = \mathbf{1}_{\text{End}(V)}/d^2$ (Proposition 8), and $Q_1(t) = t$ is a theorem, because the operator $U = (4/(N(N + 1)))\tilde{\Omega} - (1/(N + 1))\mathbf{1}$ has spectrum $\{t_k\}$ by (17) and $T_1 = (4(N - 1)/(Nd^2))(\tilde{\Omega} - (N/4)\mathbf{1})$, which is proved twice, once by the frame-operator route of Appendix B and once as a consequence of Lemma 10 together with $M_1 = 0$. Second, Hypothesis 9 is verified exactly at every even $N \leq 60$ in the strong form $\deg P_j = j$, with the polynomials built from the measure by Gram–Schmidt; it is a theorem in prior work [11, 22], where a degree bound on a triangular system is composed

with a closed-form inverse, and it is cited at the point of use in Section 5.6. Third, it is the only unproved input to the whole determination below, and the degree law is exactly the kind of statement for which a representation-theoretic reason ought to exist: the nodes are affine in the Casimir, and the duality $P_j(t_k) = P_k(t_j)$ of Corollary 14 is Askey–Wilson duality in its self-dual form. The polynomial data that the Leonard-pair correspondence attaches to such a structure are two monic sequences satisfying three-term recurrences, two distinct eigenvalue sequences, the vanishing of the final polynomials at the eigenvalues, non-vanishing at the base point, and the duality $p_i(\theta_j)/p_i(\theta_0) = p_j^*(\theta_i^*)/p_j^*(\theta_0^*)$ [23]. Since $t_0 = 1$ and $P_j(1) = 1$, the identity of Corollary 14 is exactly that duality with the two sequences identified and the base-point normalisation already applied, and the recurrence (41), the distinctness of the nodes and $\deg P_j = j$ supply the rest. What is not supplied is the part of a parameter array by which a family is recognised, namely the split sequences, and no identification within the Askey scheme [24] is claimed here; Section 8 records what is missing and why the search that was run could not have found it.

5.3 Two unconditional lemmas

Lemma 10 (full orthogonality). *For all i, j ,*

$$\sum_k d_k \lambda_k(i) \lambda_k(j) = \delta_{ij} \frac{d_i}{d^2}, \quad \text{equivalently} \quad \langle Q_i, Q_j \rangle = \frac{\delta_{ij}}{d_j}. \quad (34)$$

Proof. In the multiplicity-free doubled space T_i and T_j are simultaneously diagonal with eigenvalue λ_k on W_k , so the left-hand side is $\text{Tr}(T_i T_j)$. Then $\text{Tr}(T_i T_j) = \text{Tr}(\mathcal{R}(\Pi_i) \mathcal{R}(\Pi_j))/d^2 = \text{Tr}(\Pi_i \Pi_j)/d^2$ by (8), whose hypothesis $\Pi^T = \Pi = \Pi^{T^4}$ holds for every isotypic projector of the multiplicity-free wedge family, as explained after (8) and verified exactly at $N = 2, 4, 6$ together with the consequence $\text{Tr}(\mathcal{R}(\Pi_i) \mathcal{R}(\Pi_j)) = \text{Tr}(\Pi_i \Pi_j)$. The Π are orthogonal projectors onto distinct isotypic components, so $\text{Tr}(\Pi_i \Pi_j) = \delta_{ij} d_i$. Dividing by $\lambda_0(i) \lambda_0(j) = d_i d_j / d^4$ gives the second form. $\square$

Lemma 10 is strictly stronger than the diagonal case (26), and the off-diagonal part is what makes the Q_j an orthogonal *system* rather than a family of individually normalised vectors. Two specialisations, both now theorems rather than empirical facts, are

$$(M1) \quad \sum_k w_k t_k = \langle Q_1, 1 \rangle = 0, \quad (M2) \quad \sum_k w_k t_k^2 = \langle Q_1, Q_1 \rangle = \frac{1}{d_1} = \frac{1}{N^2 - 1}. \quad (35)$$

Both have an independent source in the trace form of Λ^m , which Appendix B uses to compute $\text{Tr} \tilde{\Omega} = Nd^2/4$ and $\text{Tr}(\tilde{\Omega}^2) = N^2(\alpha^2 + \beta^2) + 2\alpha\beta N$ with $\alpha = \binom{N-2}{m-1}$, $\beta = \binom{N-2}{m-2}$; since $\tilde{\Omega} = \sum_k \varphi_k \Pi_k$ and $\text{Tr} \Pi_k = d_k$, those two traces give $\sum_k d_k \varphi_k$ and $\sum_k d_k \varphi_k^2$ in closed form, hence M_1 and M_2 again. The two routes are checked against each other exactly at $N = 2, 4, 6, 8, 10, 12, 16$.

Lemma 11 (uniqueness). *Let $Q \in \mathbb{Q}[t]$ have degree at most j , satisfy $\langle Q, t^i \rangle = 0$ for $i = 0, \dots, j-1$, and $Q(t_0) = 1$. Then $Q = P_j$. In particular the monic orthogonal polynomial π_j satisfies $\pi_j(t_0) \neq 0$.*

Proof. $\deg \pi_i = i$ gives $\text{span}\{1, t, \dots, t^{j-1}\} = \text{span}\{\pi_0, \dots, \pi_{j-1}\}$. Writing $Q = \sum_{i \leq j} a_i \pi_i$, the orthogonality forces $a_i = 0$ for $i < j$, so $Q = a_j \pi_j$, and $1 = Q(t_0) = a_j \pi_j(t_0)$ gives both $\pi_j(t_0) \neq 0$ and $Q = \pi_j / \pi_j(t_0) = P_j$. $\square$

5.4 The determination

Theorem 12 (orthogonal-polynomial determination of the wedge spectrum). *Assume Hypothesis 9. Then for every even $N \geq 2$ and all $0 \leq j, k \leq m$,*

$$r_k(j) = P_j(t_k), \quad \lambda_k(j) = \frac{d_j}{d^2} P_j(t_k) = \frac{\dim P_j}{d^2} P_j(t_k), \quad (36)$$

where P_j is the Gram–Schmidt polynomial of the discrete measure (29) normalised by $P_j(1) = 1$.

Proof. Induction on j . The cases $j = 0, 1$ are theorems by the first remark of Section 5.2. Assume $Q_i = P_i$ for $i < j$. Lemma 10 gives $\langle Q_j, Q_i \rangle = 0$ for $i < j$, hence $\langle Q_j, P_i \rangle = 0$, hence $\langle Q_j, t^i \rangle = 0$ for $i \leq j - 1$. Hypothesis 9 gives $\deg Q_j \leq j$, and $Q_j(t_0) = r_0(j) = 1$. Lemma 11 gives $Q_j = P_j$. $\square$

Remark 13 (what the theorem does not use). No evaluation of a tetrahedral or Racah coefficient, no walled-Brauer recoupling, and no eigensolver enters the proof or the computation. That is worth stating because the negative structural result recorded in prior work [17] concluded that a Racah or walled-Brauer evaluation of the coefficient $F_{jk} = \langle \Pi_j, \mathcal{R}(\Pi_k) \rangle$ was the only available route even to the two-column row, and a fortiori to the interior rows. Theorem 12 replaces that route by two closed-form inputs and a recursion.

Corollary 14 (duality). $P_j(t_k) = P_k(t_j)$ for all j, k , and therefore $r_k(j) = r_j(k)$.

Proof. Identity (25) gives $d_k \lambda_k(j) = d_j \lambda_j(k)$, and (20) gives $\lambda_0(j) = d_j/d^2$, so

$$r_k(j) = \frac{\lambda_k(j)}{\lambda_0(j)} = \frac{d^2 \lambda_k(j)}{d_j} = \frac{d^2 \lambda_j(k)}{d_k} = \frac{\lambda_j(k)}{\lambda_0(k)} = r_j(k),$$

and under Hypothesis 9 the left side is $P_j(t_k)$ while the right side is $P_k(t_j)$. $\square$

Corollary 14 is used below as a computational device: it propagates a collision in one row into two further rows, and it is the reason a single numerical accident at $N = 14$ shows up three times in the table.

5.5 The minimal bootstrap

Theorem 12 is proved by orthogonality, but the polynomials do not have to be computed that way. Because $\langle Q_j, t^i \rangle = 0$ for $i < j$ is equivalent to the two identities (27) and (25), P_j is the unique solution of the $(j+1) \times (j+1)$ exact rational linear system

$$\sum_k d_k P_j(t_k) = 0, \quad P_j(t_k) = P_k(t_j) \quad (k = 1, \dots, j-1), \quad P_j(1) = 1, \quad \deg P_j \leq j, \quad (37)$$

in which the values $P_k(t_j)$ for $k < j$ are already known from the previous rows. System (37) uses the duality, the trace identity and the degree bound, and no inner product at all; it is nonsingular because its first j rows are evaluations at the distinct nodes $t_0, \dots, t_{j-1}$ and its last row is a positive functional on those same nodes. Two independent constructions of the same object — Gram–Schmidt on the measure, and (37) — are compared coefficient by coefficient here for every even $N \leq 60$ and agree exactly, which is a real check rather than a tautology because the two routes share no step. The bootstrap form is also what makes the interior rows computable at large N in seconds, with no spectrum computation whatsoever.

5.6 The rows

Row $j = 0$ is $P_0 = 1$ and row $j = 1$ is $P_1(t) = t$, both theorems. The first interior row has a closed form.

Proposition 15 (the second row). *Assume Hypothesis 9 at $j = 2$ and*

$$(I3) \quad M_3 = \sum_k w_k t_k^3 = \frac{2(N-1)}{(N^2-1)^2}. \quad (38)$$

Then for every even $N \geq 4$,

$$P_2(t) = \frac{(N^2-1)t^2 - 2(N-1)t - 1}{N(N-2)}, \quad r_k(2) = P_2(t_k). \quad (39)$$

Proof. $P_2 = at^2 + bt + c$ is determined by $\langle P_2, 1 \rangle = 0$, which is (27) at $j = 2$ and is proved; by $\langle P_2, t \rangle = 0$, which is Lemma 10 with $Q_1 = t$ and is proved; and by $P_2(1) = 1$. Using $M_0 = 1$, $M_1 = 0$ and $M_2 = 1/(N^2-1)$ from (35),

$$aM_2 + bM_1 + cM_0 = 0 \Rightarrow c = -\frac{a}{N^2-1}, \quad aM_3 + bM_2 + cM_1 = 0 \Rightarrow b = -a(N^2-1)M_3,$$

and $a + b + c = 1$ gives $a(1 - (N^2-1)M_3 - 1/(N^2-1)) = 1$. With (38), $(N^2-1)M_3 = 2(N-1)/(N^2-1)$ and the bracket is $(N^2-1-2N+2-1)/(N^2-1) = N(N-2)/(N^2-1)$, so $a = (N^2-1)/(N(N-2))$, $b = -2(N-1)/(N(N-2))$ and $c = -1/(N(N-2))$. $\square$

The hypothesis (38) is verified exactly for every even $N \leq 200$ and is a cited theorem; it is not proved from the trace form here. It is equivalent to a closed form for $\text{Tr}(\tilde{\Omega}^3) = \sum_{p,q,r,s,t,u} (\text{Tr} \rho(E_{pq}) \rho(E_{rs}) \rho(E_{tu}))^2$, because M_3 is a fixed rational combination of $\sum_k d_k \varphi_k^i$ for $i \leq 3$ and the first two of those are computed in Appendix B. That is the cleanest remaining target in this direction.

Row $j = 3$ has a cited closed form,

$$P_3(t) = [N(N-3)(N+1)^2 t^3 - 2(N-3)(N+1)(3N-2)t^2 + (3N^2 - 19N + 24)t + 4(N-3)] / [N(N-4)(N-2)(N-1)], \quad (40)$$

together with $\dim W_3 = N^2(N-5)(N-1)^2(N+1)/36$, and the three-term recurrence of the system in both normalisations,

$$tP_j = A_j P_{j+1} + B_j P_j + C_j P_{j-1}, \quad tp_j = p_{j+1} + b_j p_j + u_j p_{j-1}, \quad (41)$$

where $p_j = P_j/l_j$ is monic with $l_0 = 1$, $l_{j+1} = l_j/A_j$, and

$$b_j = \frac{2j(N+1-j)}{N(N+1)} = \frac{1-t_j}{2}, \quad u_j = \frac{h_j}{h_{j-1}} = A_{j-1}^2 \frac{\dim W_{j-1}}{\dim W_j}, \quad M_s = (L^s)_{00}, \quad (42)$$

with L the Jacobi matrix $L_{jj} = b_j$, $L_{j,j+1} = 1$, $L_{j+1,j} = u_{j+1}$ and $h_j = \langle p_j, p_j \rangle$. None of (40) or (41)–(42) is claimed as a result here; all of it is verified independently, from this paper's own nodes and weights, by two constructions that share no step with the cited derivation and none with each other. The verified range is: (40) coefficient-wise and node-wise for every even $6 \leq N \leq 200$, with all three constructions agreeing for every even $6 \leq N \leq 60$; the characterising property $\langle P_3, P_i \rangle = 0$ for $i \leq 2$ together with $\langle P_3, P_3 \rangle = 1/\dim W_3$, which identifies (40) as an object of the measure rather than as a matching table, for every even $6 \leq N \leq 200$; $\dim W_3$ against the Weyl binomials for every even $6 \leq N \leq 200$; and (41) as a polynomial identity on all $j+2$ coefficients, not on three of them, for every $j < m$ and

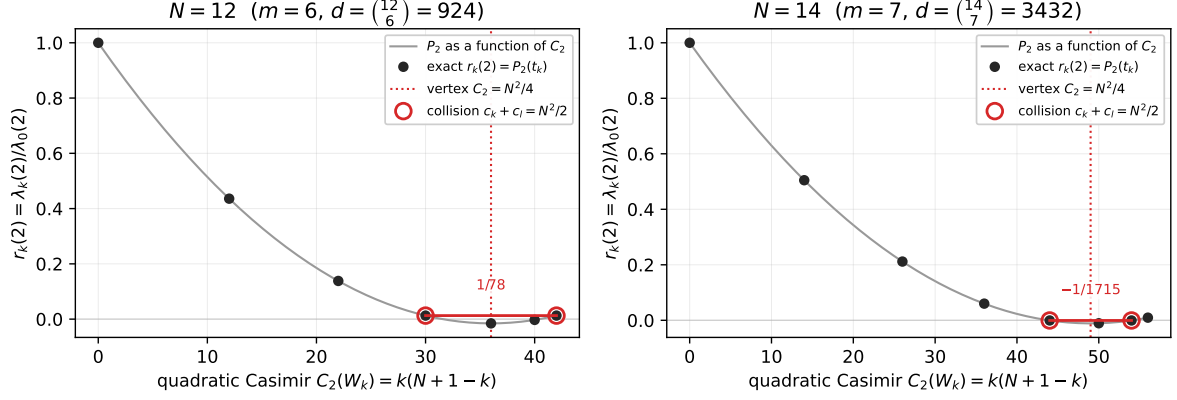


Figure 1: The row $j = 2$ collision criterion of Proposition 16, at $N = 12$ and $N = 14$. The plotted function is P_2 as a function of the quadratic Casimir $C_2(W_k) = k(N + 1 - k)$, with the exact node values $r_k(2) = P_2(t_k)$ marked. The parabola has its vertex at $c = N^2/4$, so two sectors share a value exactly when their Casimir eigenvalues are symmetric about the vertex, that is, when they sum to $N^2/2$. At $N = 12$ that selects the pair $\{3, 6\}$, with $c_3 + c_6 = 30 + 42 = 72 = N^2/2$ and common value $1/78$; at $N = 14$ it selects the pair $\{4, 6\}$, with $c_4 + c_6 = 44 + 54 = 98 = N^2/2$ and common value $-1/1715$, while $c_2 + c_4 = 70$ and $c_2 + c_6 = 80$ do not qualify and $r_2(2) = 363/1715$ is different.

every even $N \leq 60$, which is 465 recurrences against the 440 reported for $N \leq 40$ by the cited source. The endpoint b_m is computed directly from the measure as $\langle tp_m, p_m \rangle / \langle p_m, p_m \rangle$, and $u_1 = 1/(N^2 - 1) = M_2$.

For orientation, the explicit polynomials at $N = 8$ are

$$\begin{aligned} P_0(t) &= 1, & P_1(t) &= t, \\ P_2(t) &= \frac{21}{16}t^2 - \frac{7}{24}t - \frac{1}{48}, & P_3(t) &= \frac{135}{56}t^3 - \frac{165}{112}t^2 + \frac{1}{21}t + \frac{5}{336}, \\ P_4(t) &= \frac{405}{56}t^4 - \frac{225}{28}t^3 + \frac{95}{56}t^2 + \frac{5}{42}t - \frac{1}{84}, \end{aligned}$$

each with $P_j(1) = 1$ and $\langle P_j, P_j \rangle = 1/d_j$; P_2 is (39) at $N = 8$ and P_3 is (40) at $N = 8$. No rational function of N is fitted or claimed for $j \geq 4$, and none is needed: the recursion determines every row exactly at any given N .

5.7 Collisions in the second row are forced

P_2 is a quadratic in t with vertex at $t^* = (N - 1)/(N^2 - 1) = 1/(N + 1)$, so two distinct nodes give the same value exactly when they are symmetric about the vertex.

Proposition 16 (row-2 collision criterion). *Assume Hypothesis 9 at $j = 2$ and (38). Then for distinct $k, l \in \{0, \dots, m\}$,*

$$r_k(2) = r_l(2) \iff t_k + t_l = \frac{2}{N + 1} \iff c_k + c_l = \frac{N^2}{2} = 2m^2. \quad (43)$$

Proof. $P_2(t_k) = P_2(t_l)$ with $t_k \neq t_l$ gives $(t_k - t_l)[(N^2 - 1)(t_k + t_l) - 2(N - 1)] = 0$, hence $t_k + t_l = 2(N - 1)/(N^2 - 1) = 2/(N + 1)$. Since $t_k + t_l = 2 - 4(c_k + c_l)/(N(N + 1))$, this is $4(c_k + c_l)/(N(N + 1)) = 2N/(N + 1)$, i.e. $c_k + c_l = N^2/2$. $\square$

Figure 1 shows the mechanism at $N = 12$ and $N = 14$. The criterion converts a predicted infinite family of collisions into a theorem. An independent computation reported in prior work [11] found that, with $T_s = s(s + 1)/2$ the s -th triangle number, at $N = (s + 1)(s + 2)$

the pair $\{T_s, m\}$ with $m = N/2 = T_{s+1}$ collides, verified numerically for $s = 1, \dots, 6$, that is $N = 6, 12, 20, 30, 42, 56$, and asked whether the family is a theorem and in which rows it occurs. Substituting into (43),

$$\begin{aligned} c_{T_s} &= T_s(N + 1 - T_s) = \frac{s(s+1)(s+2)(s+3)}{4}, \\ c_{T_{s+1}} &= T_{s+1}(N + 1 - T_{s+1}) = \frac{(s+1)(s+2)(s^2 + 3s + 4)}{4}, \\ c_{T_s} + c_{T_{s+1}} &= \frac{(s+1)(s+2)}{4}(2s^2 + 6s + 4) = \frac{((s+1)(s+2))^2}{2} = \frac{N^2}{2}, \end{aligned}$$

identically in s . The family is therefore a theorem in row $j = 2$ for every $s \geq 1$, given Hypothesis 9 at $j = 2$ and (38): not a numerical coincidence that happens to survive to $s = 6$. The census locates the rows as well, and prints

s	N	colliding pair	common value at $j = 2$
1	6	$\{1, 3\}$	1/21
2	12	$\{3, 6\}$	1/78
3	20	$\{6, 10\}$	1/210
4	30	$\{10, 15\}$	1/465
5	42	$\{15, 21\}$	1/903
6	56	$\{21, 28\}$	1/1596
7	72	$\{28, 36\}$	1/2628
8	90	$\{36, 45\}$	1/4095
9	110	$\{45, 55\}$	1/6105

where $s = 7, 8, 9$ go beyond the range of that engine and are predictions of the criterion, not of a fit. The same pair also collides at $j = 4$ from $s = 2$ on, with common values 1/715, 1/5985, 1/31465, 1/123410, 1/395010, 1/1088430, 1/2672670, 1/5989005 for $s = 2, \dots, 9$. Whether the $j = 4$ half is also forced is open: the criterion of Proposition 16 is specific to $j = 2$, and at $j \geq 3$ a collision is a further coincidence of the recursion that this paper cannot explain — at $N = 14$ the pair $\{5, 7\}$ collides in row $j = 5$ with value $-1/3003$, and at $N = 72$ row $j = 2$ carries the unrelated pair $\{30, 31\}$ at $-35/94608$ in addition to the triangle-number pair.

The census of every even $N \leq 200$ lists 102 colliding pairs $(j, \{k, l\})$, at 62 of the 100 values of N ; restricting to $N \leq 56$ gives 33 pairs at 19 of the 28 values. The convention is stated so that the count can be compared rather than argued about: unordered index pairs $\{k, l\}$ with $0 \leq k < l \leq m$, rows $1 \leq j \leq m$, equality of the exact rationals $r_k(j)$. Under that convention the *set* of 19 values of N agrees exactly with the set reported by the independent computation in prior work [11], which is a strong cross-check of two computations that share nothing, while the number of pairs differs, 33 against their 40. That discrepancy is not resolved here; it is flagged where it arises rather than reconciled silently. An independent recomputation of the census by the orthogonal-polynomial route, reported in Appendix D, reproduces the 33 pairs at 19 values for $N \leq 56$ and extends the count to 34 pairs at 20 values for $N \leq 60$.

Two further facts about the census are exact and worth separating. First, at $N = 14$ the criterion selects precisely one row-2 pair, $c_4 + c_6 = 98 = N^2/2$, so $r_4(2) = r_6(2) = -1/1715$ while $r_2(2) = 363/1715$; the value $-1/1715$ nevertheless occurs in three rows, because Corollary 14 propagates that single collision to $r_2(4) = r_6(4)$ and $r_2(6) = r_4(6)$. It is a duality orbit of one collision, not a triple collision. Second, no collision occurs in row $j = 1$ or row $j = m$ anywhere in the census to $N \leq 200$, which is what the injectivity arguments of Section 4 require: $r_k(1) = 1 - 4c_k/(N(N+1))$ is injective in c_k , and $r_k(m) = (-1)^k / \binom{N+1}{k}$ alternates in sign while decreasing in modulus.

5.8 Partial maximality from the second moment alone

The bound $|r_k(j)| \leq 1$, which makes the largest-modulus reading of (19) well defined at every row, is verified for every even $N \leq 200$ on the table produced by (36). Part of that verification can be replaced by a theorem.

Lemma 17 (partial maximality). *Assume Hypothesis 9. Then for every even $N \geq 4$, every $j \geq 1$ and every $k \geq 1$,*

$$\lambda_k(j)^2 \leq \frac{\lambda_0(j)(1 - \lambda_0(j))}{d_k}, \quad (44)$$

and consequently

$$\lambda_0(j)(1 + d_k) \geq 1 \quad \implies \quad |\lambda_k(j)| \leq \lambda_0(j). \quad (45)$$

Proof. Lemma 10 at $i = j$, which is unconditional, gives $\sum_{k'} d_{k'} \lambda_{k'}(j)^2 = d_j/d^2 = \lambda_0(j)$. Under Hypothesis 9 each $\lambda_{k'}(j) = (d_j/d^2)P_j(t_{k'})$ is rational, so every summand is non-negative; keeping only $k' = 0$ and $k' = k$ and using $d_0 = 1$ gives $\lambda_0(j)^2 + d_k \lambda_k(j)^2 \leq \lambda_0(j)$, and $\lambda_0(j) < 1$ turns that into (44). If the hypothesis of (45) holds then $\lambda_k(j)^2 \leq \lambda_0(j)(1 - \lambda_0(j))/d_k \leq \lambda_0(j)^2$. $\square$

For $j = 0$ no criterion is needed, since $\lambda_k(0) = 1/d^2$ for every k and maximality holds with equality. Lemma 17 is genuinely partial, and it is worth being precise about which part it certifies. It is easiest to satisfy when d_k is large, so it certifies the large- k sectors and leaves small k open; and since $\lambda_0(j) = d_j/d^2 \leq 1$, the whole row j is certified as soon as $d_j/d^2 \geq 1/(1 + d_1) = 1/N^2$, that is, as soon as $\dim P_j \geq d^2/N^2$. The census over every even $4 \leq N \leq 200$ is that Lemma 17 certifies 294706 of the 338349 pairs $(j \geq 1, k \geq 1)$, with the certified fraction falling from 1 at $N = 4$ and $N = 6$ through 15/16 at $N = 8$, 33/36 at $N = 12$ and 356/400 at $N = 40$ to 8690/10000 at $N = 200$. At $N = 10$ the row $j = 1$ is the only incomplete one, with $\lambda_0(1) = 11/7056$ and $k = 1$ left open; at $N = 12$ rows $j = 1$ and $j = 2$ are incomplete. The physically important row $j = 1$ is exactly the one Lemma 17 cannot reach at large N , because $\lambda_0(1) = (N^2 - 1)/d^2 < 1/N^2$ for $N \geq 8$; that is why statements (c) and (d) rest on the closed forms of Section 6 and on the verified table rather than on Lemma 17. A proof of $|r_k(j)| \leq 1$ for all j, k would follow from a general bound $|P_j| \leq 1$ on the nodes, and remains open; Theorem 5 supplies it wherever its two hypotheses hold, which in this family is everywhere.

5.9 Independent verification of the determination

Theorem 12 would be worthless if it only reproduced the numbers it was fitted to, and it is not fitted to anything: the measure is built from the Weyl dimensions and the proved physical-adjoint row, and the recursion runs forward. The table it produces is compared, entry by entry in exact rational arithmetic, against every independently computed spectrum available.

At $N = 2, 4, 6$ the comparison is against dense exact diagonalisation of the $d^2 \times d^2$ rational transfer matrix built from the $SU(N)$ generators on Λ^m , and the full $\lambda_k(j)$ table is identical. At $N = 8$, where $d^2 = 4900$, the comparison is against a modular route — exact integer arithmetic reduced modulo eight primes, with a proven bound $D p_{\max}^2 = 4900 \cdot 524389^2 < 2^{53}$ on the entries and reconstruction by the Chinese remainder theorem — and the full table is identical again, including both the collision $\lambda_2(2) = \lambda_3(2) = -3/980$ and the four identities of Proposition 6. At $N = 10$ the six interior rationals that were published as fitting no closed form, 79/1375, -29/1375, -1/125, -7/4125, 23/4125 and -31/8250, are reproduced exactly. At $N = 12$ the complete published table — dimensions [1, 143, 4212, 44044, 196625, 382239, 226512], both named rows, the two published collisions $r_3(2) = r_6(2) = 1/78$ and $r_3(4) = r_6(4) = 1/715$, and the two published collision eigenvalues 3/47432 and 25/77616 — is reproduced exactly, including every interior rational. Rows $j = 0$ and $j = 1$ and the identities (24)–(27) hold on the produced

table for every even $N \leq 200$, as does the two-column closed form $r_k(m) = (-1)^k / \binom{N+1}{k}$, which is thereby not fitted but produced: the evidence for it is now the evidence for Hypothesis 9 and nothing else. The two rows $N = 14$ and $N = 16$ in Appendix C are predictions of the recursion, with no independent spectrum computed at those ranks.

6 Correlation lengths: statements (c) and (d)

6.1 The two readings

The correlation length is defined by (19), $\xi^{-1} = \ln |\lambda_1/\lambda_2|$, and the whole content of this section is which pair (λ_1, λ_2) that is. Two selection rules are in play.

Definition 18 (the two readings of ξ^{-1}). Let $\{\lambda_k\}$ be the spectrum of a transfer matrix, listed with multiplicity. The *literal reading* orders the eigenvalues by real part and takes λ_1, λ_2 to be the two largest; the *modulus reading* orders them by absolute value and takes the two largest. The two agree whenever the two leading eigenvalues are positive, and they can differ whenever the spectrum contains a negative eigenvalue of large modulus, because a real-part ordering cannot see the sign.

The literal reading is the one that accompanies the published statements, and the modulus reading is the one that accompanies the validation criterion published with statement (c) and the standard matrix-product-state formula, in which the correlation length is governed by the ratio of the two largest *moduli* of the transfer matrix [8]. Both are reported for every case in this paper. For the leading eigenvalue the question does not arise: Theorem 5 makes $\lambda_0(j) = \dim P_j/d^2$ simultaneously the largest real eigenvalue, the largest real part and the largest modulus, and attained on the one-dimensional singlet sector whenever that sector is non-degenerate. The ambiguity is entirely in the subleading eigenvalue.

6.2 Statement (d) is a theorem under both readings

Theorem 19 (statement (d)). *Let $N \geq 4$ be even, $V = \Lambda^{N/2}(\mathbb{C}^N)$, and P the adjoint, that is, the physical irrep $P_1 = (2, 1^{N-2})$. Then*

$$\xi^{-1} = \ln \frac{N+1}{N-3}, \quad (46)$$

and the value is the same whether the two leading eigenvalues are selected by largest real part or by largest modulus.

Proof. Write $f(k) = k(N+1-k)$, so that $r_k(1) = 1 - 4f(k)/(N(N+1))$ by the closed form of Section 5.6. The function f is strictly increasing on $0 \leq k \leq (N+1)/2$, and $N/2 < (N+1)/2$, so $r_k(1)$ is strictly decreasing in k across the whole range $k = 0, \dots, m$. Three consequences follow. First, $r_0(1) = 1$ is the unique largest ratio, attained on the one-dimensional singlet sector W_0 , and it is positive, so the leading eigenvalue is unique, positive and singlet-associated. Second, the second largest real part is $r_1(1) = 1 - 4N/(N(N+1)) = (N-3)/(N+1)$. Third, the most negative value is

$$r_m(1) = 1 - \frac{4 \cdot \frac{N}{2} \left(\frac{N}{2} + 1 \right)}{N(N+1)} = 1 - \frac{N+2}{N+1} = -\frac{1}{N+1},$$

and monotonicity puts every r_k with $k \geq 1$ between $r_1(1)$ and $r_m(1)$, so $\max_{k \geq 1} |r_k(1)| = \max(r_1(1), 1/(N+1))$. For $N \geq 4$, $(N-3)/(N+1) \geq 1/(N+1)$, with equality only at $N = 4$, where the leading modulus is attained twice, $|r_1(1)| = |r_2(1)| = 1/5$, which does not change the ratio $|r_0/r_1| = 5$. Both readings therefore select the same pair and give (46). $\square$

Table 4: Statement (d) at the four even ranks at which the row is available from an independently computed exact spectrum: $N = 4, 6$ from dense exact diagonalisation, $N = 8$ from the modular route, $N = 10$ from the published exact values. The row $r_k(1)$ is the physical-adjoint row, and the second entry by real part and by modulus coincide at every one of these ranks, which is the content of Theorem 19. At $N = 2$ the formula $\ln((N+1)/(N-3))$ is not defined because $N-3 = -1 < 0$; the computed value is $\xi^{-1} = \ln 3$ under both readings, and the published target $-1/3$ fails to match. That is a statement about the target, not about the spectrum.

N	$r_k(1), k = 0, \dots, m$	2nd, real part	2nd, modulus	ξ^{-1}	$\ln \frac{N+1}{N-3}$
4	1, 1/5, -1/5	1/5	1/5, tie with -1/5	$\ln 5$	$\ln 5$
6	1, 3/7, 1/21, -1/7	3/7	3/7	$\ln(7/3)$	$\ln(7/3)$
8	1, 5/9, 2/9, 0, -1/9	5/9	5/9	$\ln(9/5)$	$\ln(9/5)$
10	1, 7/11, 19/55, 7/55, -1/55, -1/11	7/11	7/11	$\ln(11/7)$	$\ln(11/7)$
2	1, -1/3	-1/3	-1/3	$\ln 3$	undefined

Table 4 records the exact numbers. The value (46) and the row $r_k(1)$ from which it comes are not first derived here: prior work [10] obtained both, for the same even- N wedge family, from an $\text{End}(V)$ Casimir argument, including the $N = 4$ tie and the $N = 2$ value $-1/3$. The proof above is independent — it is monotonicity of f composed with the closed form — and its content is that statement (d) is *reading-independent*, which is not part of the earlier derivation and is what makes it the one statement of the four whose verdict does not depend on Definition 18.

6.3 Statement (c) is reading-dependent, and the reading decides it

For P the two-column irrep $P_m = (2^m)$ the row is

$$r_k(m) = \frac{(-1)^k}{\binom{N+1}{k}}, \quad k = 0, \dots, m, \quad (47)$$

which is produced by the recursion of Section 5 rather than fitted to it, and is verified exactly for every even $N \leq 200$. Its values alternate in sign and decrease in modulus strictly, because $\binom{N+1}{k}$ increases on $0 \leq k \leq m$. That single feature decides everything.

Proposition 20 (statement (c)). *Let $N \geq 4$ be even, $V = \Lambda^{N/2}(\mathbb{C}^N)$ and P the two-column irrep. Under the modulus reading of Definition 18,*

$$\xi^{-1} = \ln(N+1), \quad (48)$$

as published. Under the literal largest-real-part reading,

$$\xi^{-1} = \ln \frac{N(N+1)}{2} \neq \ln(N+1), \quad (49)$$

so statement (c) is false for every even $N \geq 4$, and $N = 4$ is the smallest explicit counterexample. At $N = 2$ there is no $k = 2$ sector, the two readings coincide, and the published value $\xi^{-1} = \ln 3$ is correct.

Proof. The leading eigenvalue is $r_0(m) = 1$ on the singlet sector in both readings. Under the modulus reading the second eigenvalue is $|r_1(m)| = 1/(N+1)$, since (47) decreases in modulus,

Table 5: Statement (c) at every even rank for which the two-column row is available. The modulus reading reproduces the published $\ln(N+1)$ exactly; the literal largest-real-part reading gives $\ln(N(N+1)/2)$, and the gap between the two grows like $\ln(N/2)$. The exact eigenvalues are identical in the two columns — only the selection rule differs. Values at $N = 8, 10, 12, 14$ are those produced by the orthogonal-polynomial recursion; the decimal values are printed for comparison only.

N	2nd, real part	2nd, modulus	ξ^{-1} , real part	ξ^{-1} , modulus	published	reading that matches
2	$-1/3$	$-1/3$	$\ln 3 \approx 1.0986$	$\ln 3 \approx 1.0986$	$\ln 3$	both
4	$1/10$	$-1/5$	$\ln 10 \approx 2.3026$	$\ln 5 \approx 1.6094$	$\ln 5$	modulus only
6	$1/21$	$-1/7$	$\ln 21 \approx 3.0445$	$\ln 7 \approx 1.9459$	$\ln 7$	modulus only
8	$1/36$	$-1/9$	$\ln 36 \approx 3.5835$	$\ln 9 \approx 2.1972$	$\ln 9$	modulus only
10	$1/55$	$-1/11$	$\ln 55 \approx 4.0073$	$\ln 11 \approx 2.3979$	$\ln 11$	modulus only
12	$1/78$	$-1/13$	$\ln 78 \approx 4.3567$	$\ln 13 \approx 2.5649$	$\ln 13$	modulus only
14	$1/105$	$-1/15$	$\ln 105 \approx 4.6540$	$\ln 15 \approx 2.7081$	$\ln 15$	modulus only

giving (48). Under the real-part reading the second largest real part is $r_2(m) = 2/(N(N+1))$, because

$$\frac{2}{N(N+1)} > 0 > -\frac{1}{N+1} = r_1(m)$$

for every even $N \geq 4$ and every further term of (47) is smaller in modulus than $1/(N+1)$, giving $\xi^{-1} = \ln(1/r_2(m)) = \ln(N(N+1)/2)$. At $N = 2$ the row is $[1, -1/3]$ and both readings give $\ln 3$. $\square$

Table 5 gives the numbers. The mechanism is worth naming, because it is the same mechanism that makes Proposition 20 a statement about a definition rather than about the physics: the row (47) alternates in sign, so the second eigenvalue in modulus is negative, and a real-part ordering promotes a much smaller positive eigenvalue in its place. The two readings select *different sectors*, $k = 1$ and $k = 2$, and the difference between the answers is not a matter of convention in the logarithm but of which physical mode decays slowest. For a correlation length the modulus reading is the physically meaningful one, since it is the slowest-decaying mode that sets the length scale, and under that reading statement (c) is exactly right. Under the reading published with it, it is wrong at every even $N \geq 4$.

This is the reason the four statements do not have comparable verdicts. Statement (d) is a theorem about the spectrum. Statement (c) is true about the spectrum and false about the definition that was attached to it, and the distinction cannot be removed by choosing a convention, because the two conventions give different numbers that grow apart with N .

6.4 The $SU(2)$ consistency check

The two $SU(2)$ formulas published alongside the four statements are a required check on any general answer, and both are specialisations of Theorem 1. Let the virtual irrep be spin s , so $V = \text{Sym}^{2s}(\mathbb{C}^2)$ and $d = 2s + 1$, and the physical irrep spin S with $S \in \{0, 1, \dots, 2s\}$ and $2s - S$ an integer, so that S occurs in $s \otimes s$. The doubled space is multiplicity-free,

$\text{End}(V) = \bigoplus_{j=0}^{2s} (\text{spin } j)$ with $\dim W_j = 2j + 1$, the transfer matrix is scalar on each W_j , and

$$\lambda_j(s, S) = (-1)^{2s+S+j} \frac{2S+1}{2s+1} \left\{ \begin{matrix} s & s & S \\ s & s & j \end{matrix} \right\}, \quad (50)$$

where $\{\cdot\}$ is the Wigner $6j$ symbol [25]. For the tetrahedral argument pattern $\{s s S; s s j\}$ the four triangle factors of the Racah formula occur in two equal pairs, so the radicand is a perfect fourth power and (50) is an exact rational number; that is asserted rather than assumed, and no floating-point value enters any number reported here.

The origin of (50) is the recoupling interpretation of the realignment. The projector Π_S onto the spin- S summand of $s \otimes s$ is, as a four-index tensor, $\sum_{\sigma} |S, \sigma\rangle \langle S, \sigma|$, and Theorem 1 regroups its indices from the channel pairing $(ab)(cd)$ to the pairing $(ac)(bd)$. In $\text{SU}(2)$ representation theory that regrouping *is* the Racah recoupling transformation between the two ways of coupling four spins s, s, s, s , namely $|(ss)_S(ss)_S; 0\rangle$ and $|(ss)_j(ss)_j; 0\rangle$, and the overlap of the two bases is the tetrahedral spin network with four external edges s and internal edges S and j , evaluated as $(-1)^{2s+S+j} \sqrt{(2S+1)(2j+1)} \{s s S; s s j\}$. Contracting against the projector, which fixes the second channel to S , and dividing by $d = 2s + 1$ collapses one $\sqrt{2j+1}$ and leaves (50). The argument fixes the shape of the answer; the normalisation and the overall sign are what the exact computation checks, and they are checked by two routes that share nothing — one builds T from explicit representation matrices and sector projectors with no $6j$ symbol anywhere, the other builds the $6j$ symbol from factorials with no matrices anywhere — and they agree on all 21 cases, for virtual spins $s = 1/2, 1, 3/2, 2, 5/2, 3$ and every allowed physical spin. The sign convention is pinned by the anchor $s = 1/2, S = 1$, whose spectrum $\{3/4, -1/4\}$ is the textbook value for the spin-one AKLT chain. On those 21 cases the following all hold exactly: (50) equals the matrix construction as a rational number for every j ; the multiplicity of λ_j is $2j + 1 = \dim W_j$; the residual $T\Pi_j - \lambda_j\Pi_j$ is zero entrywise; and $\lambda_0 = (2S+1)/(2s+1)^2 = \dim P_S/d^2$, which is Theorem 2 in this family. The full table is in Appendix C.

Two published formulas have to be read against that table. The first, the spin-one physical irrep, is correct: for $S = 1$ both readings of Definition 18 select the sector $j = 1$ at every virtual spin, so the published value is a theorem and not merely a quotation, and at $s = 1/2$ it gives $\xi^{-1} = \ln 3$, hence $\xi = 1/\ln 3$.

The second is the Casimir-ratio formula

$$\xi = -\frac{1}{\ln |1 - C_2(S)/(2C_2(s))|}, \quad C_2(x) = x(x+1), \quad (51)$$

which is published without a statement of which pair of eigenvalues the ratio $1 - C_2(S)/(2C_2(s))$ describes. There are two natural candidates, $\lambda_{j=S}/\lambda_0$ and $\lambda_{j=1}/\lambda_0$, and the computation settles the question in favour of the second: for all 21 cases,

$$\frac{\lambda_{j=1}(s, S)}{\lambda_0(s, S)} = 1 - \frac{C_2(S)}{2C_2(s)} \quad (52)$$

exactly, while $\lambda_{j=S}/\lambda_0$ equals it only when $S = 1$. At $S = 1$ the two candidates coincide, which is why the two published formulas agree with each other and with $\xi^{-1} = \ln 3$ at $s = 1/2$. Identity (52) is the $\text{SU}(2)$ instance of the adjoint-row closed form $r_k(1) = 1 - C_2(W_k)/(2c_V)$, proved for the wedge family in Section 5.6 and reproduced here by an independent dense rational computation at $N = 2, 4, 6$.

Whether (51) is a correct formula for the correlation length is then a separate question, because $\lambda_{j=1}$ is not always the subleading eigenvalue. Table 6 gives the six cases in which the two readings can be compared. Read off it: for $S = 1$ both readings always select $j = 1$, so (51) is correct for physical spin one at every virtual spin; for $S \geq 2$ the two readings disagree, and

Table 6: The $SU(2)$ cases in which the two readings of Definition 18 differ or could differ. All eigenvalues are exact rationals, obtained identically by the matrix construction and by the $6j$ formula (50), with multiplicity $2j + 1$ and entrywise residual zero. The last column is the value predicted by the Casimir-ratio formula (51), which is exactly $\lambda_{j=1}/\lambda_0$ in every case; it coincides with the correlation length only when the $j = 1$ sector is subleading in the chosen sense.

s	S	$\lambda_j, j = 0, \dots, 2s$	ξ^{-1} , real part	ξ^{-1} , modulus	(51)
1/2	1	3/4, -1/4	$\ln 3$	$\ln 3$	$\ln 3$
1	1	1/3, 1/6, -1/6	$\ln 2$	$\ln 2$	$\ln 2$
1	2	5/9, -5/18, 1/18	$\ln 10$	$\ln 2$	$\ln 2$
3/2	1	3/16, 11/80, 3/80, -9/80	$\ln(15/11)$	$\ln(15/11)$	$\ln(15/11)$
3/2	2	5/16, 1/16, -3/16, 1/16	$\ln 5$	$\ln(5/3)$	$\ln 5$
3/2	3	7/16, -21/80, 7/80, -1/80	$\ln 5$	$\ln(5/3)$	$\ln(5/3)$

(51) matches one or the other depending on the case — it is the modulus answer at $(s, S) = (1, 2)$ and $(3/2, 3)$ and the real-part answer at $(3/2, 2)$. Under the literal largest-real-part reading, therefore, (51) is false at $(s, S) = (1, 2)$ and at $(3/2, 3)$, where it predicts $\ln 2$ and $\ln(5/3)$ while the definition gives $\ln 10$ and $\ln 5$; both are in scope, since $s \geq S/2$ in both. Under the modulus reading it is false at $(3/2, 2)$.

The correct general statement, which Table 6 supports and which follows structurally from the construction, is this: the Casimir ratio $1 - C_2(S)/(2C_2(s))$ is exactly $\lambda_{j=1}/\lambda_0$, and it equals $|\lambda_2/\lambda_1|$ in the notation of (19) — hence gives ξ^{-1} — if and only if the $j = 1$ sector is subleading in the chosen sense. For physical spin $S = 1$ it always is. For $S \geq 2$ it need not be, and whether it is depends on the reading.

The reason the ratio law holds on the $j = 1$ sector and on no other is also structural, and it explains why (51) could never have been a general formula. The identity $\lambda_W/\lambda_0 = 1 - C_2(W)/(2C_2(V))$ for *all* sectors W is equivalent to T being proportional to $\sum_a T_a \otimes T_a$ over a basis $\{T_a\}$ of $\mathfrak{sl}_N$, that is, to $\mathcal{R}(\Pi_P)$ being proportional to the doubled Casimir $\tilde{\Omega}$. That happens precisely when Π_P is proportional to the frame operator $K = \sum_{p,q} |\rho(E_{pq})\rangle\langle\rho(E_{pq})|$, whose image is $\rho(\mathfrak{gl}_N) = W_0 \oplus W_1$, and hence precisely when P is the adjoint. For P the singlet, T is a multiple of the identity and the ratio is trivially 1 on every sector. For every other P the ratio law fails, and what Table 6 shows in its $j = 1$ column is the residue of that law: the ratio equals the adjoint Casimir ratio because the *sector* $j = 1$ is the adjoint sector, while no other sector obeys it. The same mechanism gives, for $V = \Lambda^{N/2}$ of $SU(N)$ and P the adjoint, the row

$$\frac{\lambda_k(1)}{\lambda_0(1)} = 1 - \frac{C_2(\lambda^{(k)})}{2c_V} = 1 - \frac{4k(N+1-k)}{N(N+1)}, \quad \lambda_0(1) = \frac{N^2 - 1}{d^2}, \quad (53)$$

which is the closed form used in Theorem 19 and which Section 5 obtains again, without a frame operator, from $M_1 = 0$.

7 Outer multiplicity: the basis criterion and the adjoint family

Every family treated so far has a multiplicity-free $V \otimes V$, so each physical irrep occurs once, its projector is unique, and the sector formula (13) applies. The smallest family in which that fails is the one in which the virtual and physical irreps are both the adjoint of $\mathfrak{sl}_N$, where the adjoint occurs in $V \otimes V$ with outer multiplicity two, as the symmetric and the antisymmetric copy. This is the family that the published construction leaves unresolved, and the reason is worth isolating precisely, because it is not a reason about the physics.

7.1 What fails, and where

For $V = \text{ad}(\mathfrak{sl}_N)$ in the rational Cartan–Weyl basis $\{E_{ij}\}_{i \neq j} \cup \{H_i = E_{ii} - E_{i+1, i+1}\}$, with the invariant bilinear form $S_{ab} = \text{Tr}(M_a M_b)/(2N)$, the two adjoint copies are spanned by the structure-constant tensors $(F_c)_{ab} = f_{cab}$ and by the symmetric tensors $(D_c)_{ab} = d_{cab}$. Assembling copy projectors from them as $\Pi_c = \mu_c^{-1} \sum_a |\text{vec}(M_a)\rangle \langle \text{vec}(M_a)|$ gives operators that are idempotent, of trace $\dim P$, and mutually orthogonal in the sense that their cross Gram matrix vanishes — and yet the identity $\sum_\sigma Y_\sigma \otimes Y^\sigma = \mathcal{R}(\Pi_c)$, which is what certifies $T_c = \mathcal{R}(\Pi_c)/d$ through Theorem 1, fails, with exact nonzero residuals $-8/15$, $-5/12$, $13/30$, $7/22$, $-68/285$, $5/12$, $-76/291$, $-1/2$ and $-3/8$ across the two ranks and the copies and gauges examined. In the same basis the isotypic projector is oblique, $\Pi^\top \neq \Pi$, so the complete-positivity argument of Section 3 does not apply either.

Localising the failure is the first step, and it is exact. The covariance residual $[\rho_{\text{dbl}}(E), T_P]$ is zero on every generator for the two adjoint copies and for the copies of dimension 27 at $\text{SU}(3)$ and 84 at $\text{SU}(4)$, but it is $1/8$ for the two copies of dimension 10 at $\text{SU}(3)$ and $31/480$ for the two copies of dimension 45 at $\text{SU}(4)$. For those four the physical subspace is not invariant under the constructed operator — the exact invariance residual is $17/240$ at $\text{SU}(3)$ and $608353/6260760$ at $\text{SU}(4)$ — so there is no rational spectral resolution to report, and no eigenvalue is claimed for them anywhere in this paper. They are recorded as unresolved cases, not as results. For the six copies with zero covariance residual, exact spectra exist and are reported in Section 7.6.

7.2 The criterion

The bond contraction of the valence-bond-solid construction is the map

$$\Psi: \text{End}(V) \rightarrow \text{End}(V), \quad \Psi(X) = XS, \quad (54)$$

whose image is the physical subspace, and whose failure to be a homothety of the Hilbert–Schmidt inner product is exactly what makes the dual-basis identity fail.

Lemma 21 (basis criterion). *Let V be a self-conjugate irrep of $\text{SU}(N)$ with a fixed working basis, and let S be the matrix of the invariant bilinear form in that basis, so that $\rho(X)^\top S + S \rho(X) = 0$ for every $X \in \mathfrak{sl}_N$. Then Ψ is a Hilbert–Schmidt homothety, that is, there is a scalar c with $\langle XS, YS \rangle_{\text{HS}} = c \langle X, Y \rangle_{\text{HS}}$ for all X, Y , if and only if $SS^\top = c\mathbf{1}$ in the working basis.*

Proof. $\langle XB, YB \rangle_{\text{HS}} = \text{Tr}(XBB^\top Y^\top)$. If $BB^\top = c\mathbf{1}$ this is $c \text{Tr}(XY^\top) = c \langle X, Y \rangle_{\text{HS}}$. Conversely, taking $X = E_{ij}$ and $Y = E_{kl}$ gives $(BB^\top)_{jl} \delta_{ik} = c \delta_{jl} \delta_{ik}$ for every choice of indices, hence $BB^\top = c\mathbf{1}$. $\square$

By Schur’s lemma the singlet S is unique up to scale, so there is no freedom to repair a failure of Lemma 21 by changing the form on the physical subspace: the freedom is entirely in the basis of V . That is the content of the criterion, and it converts an obstruction into a choice.

Corollary 22 (what the criterion buys). *If the working basis of V satisfies $SS^\top = c\mathbf{1}$ — in particular if it is S -orthonormal, so that $S = \sigma\mathbf{1}$ — then: the Casimir–Lagrange isotypic projector of any physical isotypic component is symmetric, because the doubled Casimir is symmetric and the projector is a polynomial in it; the copy projectors assembled from Clebsch–Gordan tensors are symmetric by construction, idempotent because their Gram matrix is $\mu_c\mathbf{1}$, mutually orthogonal because the cross Gram matrix vanishes, and of trace $\dim P$; the dual-basis identity holds for every basis of the physical subspace together with its Hilbert–Schmidt dual; and therefore Theorem 1 certifies $T_c = \mathcal{R}(\Pi_c)/d$ for every copy c , and the complete-positivity argument of Theorem 5 applies.*

One further fact makes the criterion sufficient rather than merely convenient. Under a change of basis C of V the tensor transforms as $A^\sigma \mapsto CA^\sigma C^{-1}$, hence $T \mapsto (C \otimes C)T(C \otimes C)^{-1}$, so the spectrum of T is basis-independent up to similarity. Anything proved about the spectrum in a convenient basis therefore holds in all of them, and the oblique-basis exceptions are exceptions of the basis rather than of the claim.

7.3 The Killing-orthonormal basis

The defect in the rational Cartan–Weyl basis is located exactly. The E -block of S is a signed permutation matrix, because E_{ij} pairs with E_{ji} , and it satisfies Lemma 21; the Cartan block is the Cartan matrix of A_{N-1} divided by $2N$, which is not a scalar matrix. Hence $SS^\top$ is not scalar, and the whole defect sits on the Cartan block: the exact residual of $SS^\top$ against its scalar part is $1/9$ at $SU(3)$ in the Cartan–Weyl basis and 0 in the basis below.

Take the generalised Gell-Mann matrices λ_a normalised by $\text{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$, and set $t_a = \text{im } \lambda_a/2$. Then

$$\begin{aligned} \text{Tr}(t_a t_b) &= -\frac{1}{2}\delta_{ab}, & [t_a, t_b] &= \sum_c f_{abc} t_c, \\ \{\lambda_a, \lambda_b\} &= \frac{4}{N}\delta_{ab}\mathbf{1} + 2 \sum_c d_{abc} \lambda_c, & [\lambda_a, \lambda_b] &= 2 \text{im} \sum_c f_{abc} \lambda_c, \end{aligned} \tag{55}$$

and every one of these is verified entrywise for every ordered pair at $N = 3$, which is 64 pairs, and at $N = 4$, which is 225 pairs, together with the total antisymmetry of f , the total symmetry of d , the Jacobi identity over all d^4 quadruples, which is 4096 and 50625 of them, the antisymmetry $\rho_a^\top = -\rho_a$ of $\rho_a = \text{ad}(t_a)$, the representation identity $[\rho_a, \rho_b] = \sum_c f_{abc} \rho_c$, and $\sum_a \rho_a^2 = -N\mathbf{1}$, that is, $C_2(\text{adjoint}) = N$. At $N = 3$ the extracted symbols reproduce the standard tables, including $f_{123} = 1$, $f_{147} = f_{246} = f_{257} = f_{345} = 1/2$, $f_{156} = f_{367} = -1/2$, $f_{458} = f_{678} = \sqrt{3}/2$, $d_{118} = d_{228} = d_{338} = \sqrt{3}/3$, $d_{448} = \dots = d_{778} = -\sqrt{3}/6$, $d_{888} = -\sqrt{3}/3$ and $d_{146} = d_{157} = d_{256} = d_{344} = d_{355} = 1/2$, $d_{247} = d_{366} = d_{377} = -1/2$.

In this basis $S = -\frac{1}{2}\mathbf{1}$, so $SS^\top = \frac{1}{4}\mathbf{1}$ and Lemma 21 holds with $c = 1/4$: Ψ is a homothety of factor $1/2$. Transporting the Cartan–Weyl form by the exact change-of-basis matrix C gives $S_{\text{new}} = C^\top S_{\text{old}} C = -\mathbf{1}/(4N)$ with residual zero, which is the point worth making: the Cartan–Weyl form was always the correct invariant bilinear form, and every residual quoted in Section 7.1 measured the obliqueness of that basis and nothing else. No new form on the physical subspace was ever needed.

There is a limit to how far this can be checked exactly. The diagonal Gell-Mann normalisation requires $\sqrt{2/(l(l+1))} \in \mathbb{Q}(\sqrt{2}, \sqrt{3})$, which holds for $l \leq 3$, that is, for $N \leq 4$. That is the exact-arithmetic scope of the verification in this section. Nothing in the derivations of Sections 7.5 and 7.6 uses $N \leq 4$; they are stated for general N and verified exactly where the field allows.

7.4 The two adjoint copies

For $V = P = \text{adjoint}$ the physical irrep occurs with outer multiplicity two, and the two copies are spanned by $(F_c)_{ab} = f_{cab}$ and $(D_c)_{ab} = d_{cab}$. The exact identities are

$$\begin{aligned} F_c^\top &= -F_c, & D_c^\top &= D_c, & F_c &= \rho_c^\top = -\rho_c, & \text{Tr}(F_a D_b^\top) &= 0, \\ \text{Tr}(F_a F_b^\top) &= \mu_f \delta_{ab}, & \text{Tr}(D_a D_b^\top) &= \mu_d \delta_{ab}, \end{aligned} \tag{56}$$

with the copy scalars

$$\mu_f = N, \quad \mu_d = \frac{N^2 - 4}{N}, \tag{57}$$

and the two frame sums

$$\sum_a F_a^2 = -N\mathbf{1}, \quad \sum_a D_a^2 = \mu_d\mathbf{1}. \quad (58)$$

The last line is the whole asymmetry between the two copies: $\sum_a F_a^2$ is a Casimir, while $\sum_a D_a^2$ is only a scalar. It is why the antisymmetric copy has a closed form on every sector and the symmetric copy does not.

With $\Pi_c = \mu_c^{-1} \sum_a |\text{vec}(M_a)\rangle\langle\text{vec}(M_a)|$ for $M = F$ or $M = D$, the following are exact:

$$\begin{aligned} \Pi_c^\top &= \Pi_c, \quad \Pi_c^2 = \Pi_c, \quad \text{Tr } \Pi_c = \dim P = d, \quad \Pi_f \Pi_d = 0, \\ \Pi_f + \Pi_d &= \Pi_{\text{iso}}, \quad C_{\text{dbl}} \text{vec}(F_a) = C_{\text{dbl}} \text{vec}(D_a) = N \text{vec}, \end{aligned} \quad (59)$$

so Π_{iso} is exactly the $C_2 = N$ isotypic projector of $\text{End}(V)$, of trace $2 \dim P$: the outer multiplicity two is now certified rather than suspected. Idempotency is verified by brute force at $\text{SU}(3)$, on a 64×64 matrix, and follows at $\text{SU}(4)$ from $\text{Gram} = \mu_c \mathbf{1}$ by $\Pi^2 = \mu^{-2} \sum_{a,b} v_a (v_a v_b) v_b^\top$. The dual-basis identity $\sum_\sigma Y_\sigma \otimes Y^\sigma = \mathcal{R}(\Pi_c)$ is verified by three independent routes for each copy — the orthonormal basis $M/\sqrt{\mu}$, the raw basis with its exact Hilbert–Schmidt dual, and a deliberately skewed basis such as $Y_0 + Y_1$, $Y_1 - Y_2$, and so on — and all residuals are zero. This is the identity that failed with residuals 13/30 and 7/22 in the oblique basis.

7.5 The antisymmetric copy is a Casimir

Theorem 23 (closed form for the antisymmetric copy). *Let $V = \text{ad}(\mathfrak{sl}_N)$, so $d = N^2 - 1$, and let P be the adjoint, with T_f the transfer matrix of the antisymmetric copy in the normalisation of Theorem 1. Then*

$$T_f = \frac{\mathcal{R}(\Pi_f)}{d} = \frac{2N\mathbf{1} - C_{\text{dbl}}}{2dN}, \quad \lambda_f(W) = \frac{2N - C_2(W)}{2N(N^2 - 1)} \quad (60)$$

for every irrep $W \subseteq V \otimes V$.

Proof. With $\rho_{\text{dbl}}(t_a) = \rho_a \otimes \mathbf{1} - \mathbf{1} \otimes \rho_a^\top$ and $C_{\text{dbl}} = -\sum_a \rho_{\text{dbl}}(t_a)^2$, the identity $\rho_a^\top = -\rho_a$ from (55) makes $\rho_{\text{dbl}}(t_a) = \rho_a \otimes \mathbf{1} + \mathbf{1} \otimes \rho_a$, so squaring and summing, and using $\sum_a \rho_a^2 = -N\mathbf{1}$, gives

$$C_{\text{dbl}} = 2N\mathbf{1} - 2 \sum_a \rho_a \otimes \rho_a. \quad (61)$$

On the other hand Theorem 1 and (56) give

$$\mathcal{R}(\Pi_f) = \frac{1}{\mu_f} \sum_a F_a \otimes F_a = \frac{1}{N} \sum_a \rho_a \otimes \rho_a,$$

because $\mathcal{R}(|\text{vec}(u)\rangle\langle\text{vec}(v)|) = u \otimes v$ for row-major vectorisation and $F_a = -\rho_a$. Substituting into (61) gives $\mathcal{R}(\Pi_f) = (2N\mathbf{1} - C_{\text{dbl}})/(2N)$, and dividing by d gives the first formula in (60). By (14) the doubled Casimir acts on the summand W as $C_2(W)$, so (60) gives the eigenvalue $(2N - C_2(W))/(2dN) = (2N - C_2(W))/(2N(N^2 - 1))$. $\square$

Theorem 23 is verified entrywise on the whole doubled space at $N = 3$, on a 64×64 matrix, and at $N = 4$, on a 225×225 matrix, with residual zero in both cases. It is worth pausing on what it says, because it is the only closed form for an interior sector anywhere in this paper that is not confined to the wedge family: the antisymmetric adjoint transfer matrix is an affine function of the doubled Casimir, so its entire spectrum follows from the Casimir eigenvalues of $V \otimes V$ and from nothing else.

The normalisation has to be fixed before any comparison with published numbers is meaningful. In the raw Clebsch–Gordan gauge, $A^\sigma = M^\sigma S$ with un-orthonormalised structure constants, the transfer matrix is larger by exactly μ_c :

$$T_c(\text{raw}) = \mu_c \frac{\mathcal{R}(\Pi_c)}{d}, \quad \lambda_f(\text{raw})(W) = \frac{2N - C_2(W)}{2(N^2 - 1)}. \quad (62)$$

The second form is the one reported in prior work [17]. The normalisation of Theorem 1 is used throughout this paper because it is the one in which statement (b) reads $\lambda_{\max} = \dim P/d^2$ with no extra factor: it gives 1/8 at SU(3) and 1/15 at SU(4) on the singlet, where the raw gauge gives 3/8 and 4/15. Both are the same operator and only the scale of A^σ differs.

7.6 Statement (b) in this family, and the exact spectra

The transfer matrix of either copy has the Kraus form $T_c(Y) = (d\mu_c)^{-1} \sum_a M_a Y M_a^\top$ with real M_a , so it is completely positive, and

$$T_c(\mathbf{1}) = \frac{1}{d\mu_c} \sum_a M_a M_a^\top = \frac{1}{d\mu_c} \mu_c \mathbf{1} = \frac{1}{d} \mathbf{1} = \frac{\dim P}{d^2} \mathbf{1}, \quad (63)$$

using $\sum_a F_a F_a^\top = -\sum_a F_a^2 = N\mathbf{1} = \mu_f \mathbf{1}$ and $\sum_a D_a D_a^\top = \sum_a D_a^2 = \mu_d \mathbf{1}$ from (58). Both are general identities, being $C_2(\text{adjoint}) = N$ and $\sum_a d_{acd} d_{bcd} = ((N^2 - 4)/N) \delta_{ab}$, and both are verified exactly at $N = 3, 4$. So $(d^2/\dim P)T_c$ is unital and completely positive, hence a contraction, hence of spectral radius at most one, and

$$|\lambda| \leq \frac{\dim P}{d^2} \quad \text{for every eigenvalue of } T_c, \quad (64)$$

with equality attained on the singlet sector. Statement (b) — value and maximality — is therefore a theorem for $V = \text{ad}(\mathfrak{sl}_N)$, $P = \text{ad}(\mathfrak{sl}_N)$, for both copies and every $N \geq 3$. This is the third family for which it is proved here, after the wedge family and P the singlet in any family. More generally, the hypothesis needed by Theorem 5 is exactly the hypothesis of Lemma 21: a working basis of V in which the invariant form is a scalar matrix. Such a basis exists whenever the invariant hermitian form G is real symmetric positive definite in a real basis of V , which holds for the wedge family, where $G = \mathbf{1}$ already, for Sym^n of SU(2), where $G = \text{diag}(\prod k!)$, and for the adjoint, in the Killing-orthonormal basis. Since the spectrum is basis-independent up to similarity, statement (b) is a theorem for every pair (V, P) in those families once the construction has been certified in such a basis, which is what Corollary 22 supplies; for the four copies whose covariance residual is nonzero that certification has not been carried out, and they are recorded as open in Section 7.8. The oblique-basis exceptions are exceptions of the basis, not of the claim.

Table 7 is the complete exact spectrum at SU(3). The symmetric copy T_d is an intertwiner of the conjugation action, so it acts as $M_{\text{mult}} \otimes \mathbf{1}_V$ on the $C_2 = N$ isotypic component, and exact computation gives

$$M_{\text{mult}} = \text{diag}(1/16, -3/80) \text{ at SU(3),} \quad M_{\text{mult}} = \text{diag}(1/30, 1/90) \text{ at SU(4),} \quad (65)$$

with off-block entries exactly zero and each $d \times d$ block exactly scalar, so that each eigenvalue has multiplicity $\dim P = d$. These are precisely the two numbers reported earlier from the oblique basis and reproduced from a different Clebsch–Gordan gauge in prior work [17]: the split is real, it is the outer multiplicity, and it is now certified by Theorem 1. At SU(4) the complete spectra are $\{1/15, 1/30, 1/60, 0, 0, -1/60\}$ with multiplicities 1, 30, 20, 45, 45, 84 for the antisymmetric copy, which is (60) with $\mu_f = 4$, and $\{1/15, 1/30, 1/90, -1/30, -1/90, -1/90, 1/90\}$ with

Table 7: The complete exact spectrum of both adjoint copies at $SU(3)$, where $d = 8$ and $\dim \text{End}(V) = 64 = 1 + 8 + 8 + 20 + 27$. The antisymmetric column is Theorem 23: $\lambda_f(W) = (6 - C_2(W))/48$. The symmetric column is computed, not fitted, and no closed form for it is claimed. The isotypic component of the adjoint has outer multiplicity two, and the symmetric copy resolves it into two eigenspaces of dimension 8 while the antisymmetric copy acts as a scalar on all 16 dimensions; that difference is the outer multiplicity made visible. Neither copy separates the conjugate pair $\mathbf{10}$ and $\overline{\mathbf{10}}$, which share $C_2 = 6$. The leading eigenvalue is $1/8 = \dim P/d^2$ in both columns, unique and on the singlet. The correlation length of both copies under both readings of Definition 18 is $\xi^{-1} = \ln 2$.

sector	C_2	dim	λ_f	λ_d
singlet	0	1	1/8	1/8
adjoint isotypic, f -copy	3	8	1/16	1/16
adjoint isotypic, d -copy	3	8	1/16	-3/80
$\mathbf{10} \oplus \overline{\mathbf{10}}$	6	20	0	-1/20
27	8	27	-1/24	1/40

the isotypic component resolved by (65) for the symmetric copy, with $\mu_d = 3$. Explicitly, the symmetric copy at $SU(4)$ has $1/15$ with multiplicity 1; $1/30$ and $1/90$ with multiplicity 15 each, resolving the 30-dimensional isotypic component; $-1/30$ with multiplicity 20; $-1/90$ with multiplicity 45 on each of the two conjugate sectors of Casimir 8; and $1/90$ with multiplicity 84, the multiplicities summing to $225 = 1 + 30 + 20 + 45 + 45 + 84$.

Remark 24 (statement (a) in this family). The second half of statement (a) holds here: $\lambda_{\max} = 1/d$ with multiplicity one, for both copies and both ranks. The first half fails in two distinct ways. It fails by *collision*, because Theorem 23 makes λ_f a function of $C_2(W)$ alone, so any two inequivalent irreps of equal Casimir share an eigenvalue: $\lambda_f = 0$ has multiplicity $20 = \dim \mathbf{10} + \dim \overline{\mathbf{10}}$ at $SU(3)$ and multiplicity $90 = \dim \mathbf{45} + \dim \overline{\mathbf{45}}$ at $SU(4)$, and in general every conjugate pair $W, \bar{W}$ with $W \not\cong \bar{W}$ collides at every rank. The same collision occurs for T_d , at $-1/20$ for $SU(3)$. And it fails by *outer multiplicity*: $\lambda_f = 1/16$ at $SU(3)$ has multiplicity $16 = 2 \dim \mathbf{8}$, and the symmetric copy splits the isotypic component into two eigenspaces of dimension 8, so the multiplicity is smaller than $\dim W$ in the sense of Definition 7. This is a third and independent refutation of statement (a), and it is the strongest of the three in one respect: the collision is structural rather than sporadic.

Statements (c) and (d) concern $V = \Lambda^{N/2}$ only, so this family is outside them. For completeness, the adjoint-virtual family gives $\xi^{-1} = \ln 2$ at $SU(3)$ for both copies and under both readings, from the pair $1/8$ and $1/16$.

7.7 The oblique gauge, reproduced

The diagnosis of Section 7.1 makes a prediction that can be tested against numbers obtained by a different route. Conjugating by the rational matrix $g = \mathbf{1} + E_{01} + E_{12}$ and applying the rule $M \mapsto gMg^\top$ to the index space of V reproduces exactly the three quantities reported in prior work [17] from that gauge: the maximum off-diagonal entry of $B'B'^\top$ is $11/4$, up to the sign convention for that entry; $\text{Tr } F'_f = 71/2$ with idempotency residual $888/5041$; and $\text{Tr } F'_d = 43/2$ with residual $1636/5547$. The alternative reading $M \mapsto gMg^{-1}$ gives $5360/27889$ and $8672/24843$ instead, so the gauge used there is pinned by this comparison rather than assumed. The two denominators are $5041 = 71^2$ and $5547 = 3 \cdot 43^2$, the squares of the two traces, which is exactly what $\Pi' = F' / (\text{Tr } F' / d)$ predicts. Symmetry of F'_c survives any gauge, because $vv^\top$ is always symmetric; idempotency is lost exactly when $BB^\top$ stops being scalar, which is Lemma 21.

7.8 What is not claimed here

Three things are open in this family and are not concealed by the theorem above. The symmetric copy T_d has no closed form on the interior sectors: $\sum_a D_a^2 = \mu_d \mathbf{1}$ fixes only the singlet eigenvalue, and T_d is not a polynomial in C_{dbl} . The exact values in Table 7 and (65) are computed, not fitted, but a general formula for $\lambda_d(W)$ is open. So is a Racah or Casimir formula for the copy scalars μ_c at an arbitrary pair (V, P) with outer multiplicity: $\mu_f = N$ and $\mu_d = (N^2 - 4)/N$ for $V = P = \text{adjoint}$ and $\mu_c = 1$ for $V = \Lambda^{N/2}$, where Pieri’s rule gives multiplicity one, and two families are not a theorem. Finally, the exact-arithmetic scope of this section is $N \leq 4$, and the four copies with nonzero covariance residual — the two of dimension 10 at $\text{SU}(3)$ and the two of dimension 45 at $\text{SU}(4)$ — are not resolved. Lemma 21 predicts that their defect is likewise an effect of the oblique basis and disappears in a basis in which the invariant form is scalar, exactly as it provably does for the adjoint copies, but that prediction is not verified for them, and they remain open.

8 Discussion

The four statements have four different verdicts, and the pattern across them is more informative than any single one. Statement (b) is true and is now a theorem in every family where its two hypotheses can be certified, which is the wedge family at every even rank, the singlet physical irrep in every family, and both adjoint copies at every rank from three up. Statement (d) is true and, uniquely among the four, independent of the ambiguity in the definition of the correlation length. Statement (c) is true about the spectrum and false about the selection rule that was published with it. Statement (a) is false as stated, for three unrelated reasons, and the part of it that survives — a unique, positive, singlet-associated leading eigenvalue — is precisely the part that the physics of these chains needs.

Physically the results are of three kinds. The norm of the state on a periodic chain is governed by the leading eigenvalue, and Theorem 5 gives it as $\dim P/d^2$ with a proof that also bounds every other eigenvalue, so the thermodynamic limit of the norm is controlled without any assumption about the rest of the spectrum. The correlation length is governed by the ratio of the two leading eigenvalues, and here the substantive finding is not a number but a caution: in a chain whose transfer matrix has rational eigenvalues of both signs, “the two leading eigenvalues” is ambiguous, and the two readings select different physical modes. Proposition 20 exhibits the consequence — the published $\ln(N + 1)$ for the two-column family is the modulus answer, while the real-part answer is $\ln(N(N + 1)/2)$, and the two diverge logarithmically in N . Any comparison of correlation lengths across families of these states has to fix the reading first, in the same way that any comparison of adjoint-family eigenvalues has to fix the normalisation of A^σ first. The degeneracy structure of the entanglement spectrum is governed by the multiplicities, and here the honest answer is that the multiplicities are not $\dim W$ in general, for two reasons that no refinement of the conjecture can merge: collisions between inequivalent sectors, which are sporadic in the wedge family and structural in the adjoint family, and splits on isotypic components of outer multiplicity greater than one, which are representation-theoretic and are what distinguishes a symmetry-protected phase with edge multiplicity from one without.

The methodological content is worth separating from the results. Two devices do most of the work. The first is the realignment identity of Theorem 1, which replaces a construction involving Clebsch–Gordan coefficients, a choice of orthonormal basis, a phase convention and an invariant metric with a single regrouping of four indices applied to a projector. Everything that follows — the trace identity for the singlet eigenvalue, complete positivity, the four spectral identities, the orthogonality of the polynomial system — is a consequence of that one identity together with four properties of the regrouping map, and two of those four properties are false as usually

transcribed. Appendix A records the witnesses, because the false transcriptions are exactly the kind that produce plausible-looking spectra, and the earlier version of this construction did produce them. The second device is complete positivity. The bound $|\lambda| \leq \dim P/d^2$ had been the open half of statement (b), and it needs no representation theory at all once the transfer matrix is recognised as a unital completely positive map rescaled by its value on the identity: the contraction property of such maps does the rest in three lines. The price is a hypothesis, and identifying that hypothesis precisely — an orthogonal, equivariant projector, which is a property of the basis of V rather than of the representation — is what makes the theorem usable and what explains every exception found in this family.

The orthogonal-polynomial structure of the wedge spectrum is the result with the most consequence beyond this problem. What Theorem 12 says is that a table of rational numbers which had resisted every closed-form attempt, and whose interior entries had been published as unexplained, is a classical discrete orthogonal-polynomial system whose measure is built from two known inputs. The practical effect is immediate: the interior rows at any even rank are obtained by an exact recursion in seconds, with no spectrum computation, no eigensolver and no Racah coefficient. The structural effect is a question. The duality $P_j(t_k) = P_k(t_j)$ of Corollary 14 is Askey–Wilson duality in the self-dual case, and the polynomial data the Leonard-pair correspondence asks for are all present: monic three-term recurrences, distinct eigenvalues, the final polynomials vanishing at the nodes, non-vanishing at the base point, and the duality itself [23]. One therefore expects these polynomials to be a known family of the Askey scheme [24] in a disguised variable. They have not been identified as one, and the reason the search failed is now specific rather than merely negative. Recognising a family is not done from the recurrences and the trace formulas: it needs the parameter array, that is, the two eigenvalue sequences together with the two split sequences, and the terminating branch of the scheme is then matched against the q -Racah array and its seven parameters [23]. For this system the eigenvalue sequence is known in closed form, $t_k = 1 - 4k(N+1-k)/(N(N+1))$, and no split sequence has ever been computed. What the bounded search tested instead was whether the norm ratios u_j for $j \geq 2$ factor into the standard Racah form over the computed ranges, and whether a rational function of N of low degree reproduces the sixth moment, with 91 degree pairs tried; both answers are negative. Neither condition is known to be necessary for a q -Racah identification, so the negative answers do not exclude a member of the scheme; what excludes one is the absence of the input the standard route consumes. Since the nodes are affine in the quadratic Casimir and the weights are Weyl dimensions, an identification would presumably explain *why* the degree law of Hypothesis 9 holds, which is at present the only unproved input to the whole determination.

Three boundaries should be stated plainly. First, Hypothesis 9 is verified exactly in the strong form $\deg P_j = j$ for every even $N \leq 60$ and is a cited theorem, but it is not proved here from the representation theory, and Theorem 12 is conditional on it. Second, the exact-arithmetic scope of Section 7 is $N \leq 4$, because the Killing-orthonormal basis of the adjoint requires $\sqrt{2/(l(l+1))}$ to lie in $\mathbb{Q}(\sqrt{2}, \sqrt{3})$; the derivations there are stated for general N , and Theorem 23 and (64) are general, but the entrywise verification stops at $\mathrm{SU}(4)$. Third, four copies — the two of dimension 10 at $\mathrm{SU}(3)$ and the two of dimension 45 at $\mathrm{SU}(4)$ — have a nonzero covariance residual and are not T -invariant as constructed, so no spectrum is claimed for them at all. That is a defect of the construction in an oblique basis rather than of any theorem, and Lemma 21 predicts it disappears in a scalar-form basis, but the prediction is untested.

The open questions that follow from this paper are these. A closed form for the symmetric adjoint copy T_d on the interior sectors, which $\sum_a D_a^2 = \mu_d \mathbf{1}$ cannot supply because T_d is not a polynomial in the doubled Casimir. A Racah or Casimir formula for the copy scalars μ_c at an arbitrary pair (V, P) with outer multiplicity, of which $\mu_f = N$, $\mu_d = (N^2 - 4)/N$ and

$\mu_c = 1$ for the wedge family are three data points and not a theorem. A closed form for P_j at $j \geq 4$, and the trace-form route to $\text{Tr}(\tilde{\Omega}^3)$ that would turn the third-moment identity (38) into a theorem and with it Proposition 15 and the collision criterion of Proposition 16. A proof that the second row of the wedge family is the only one in which collisions are forced, or a criterion for the $j = 4$ family that the census keeps producing. A general bound $|P_j| \leq 1$ on the nodes, which would replace the verified range of Lemma 17 by a theorem in the cases where Theorem 5 does not already apply. The identification of the system within the Askey scheme, which needs the split sequences of its parameter array before the standard route can be run at all, and with it a reason for Hypothesis 9. And, beyond the spectrum, the parent Hamiltonians and the interpolation between these families, which are outside the scope of this paper entirely.

A The realignment: index proofs, witnesses and the sharpness of the hypotheses

Every proof in this paper that involves a trace of a realigned operator uses one of the four identities (7)–(10), and two of the four are false in the form in which they are usually written. This appendix gives the index-level proofs, states the identities that do hold without hypotheses, prints exact witnesses for the false transcriptions, shows that the hypothesis of (8) is sharp, and records where the hypotheses hold and where they fail. Throughout, $D = \dim V$, a matrix X on $\text{End}(V)$ has composite indices $X_{(ab),(cd)}$ with $a, b, c, d \in \{0, \dots, D-1\}$, the composite index (ab) means $aD + b$, the four-index transpose is $X_{(ab),(cd)}^{T4} = X_{(ba),(dc)}$, and $\mathcal{R}(X)_{(ac),(bd)} = X_{(ab),(cd)}$.

A.1 Proofs

$\mathcal{R}$ is an involution. Applying the rule twice, $\mathcal{R}(\mathcal{R}(X))_{(ab),(cd)} = \mathcal{R}(X)_{(ac),(bd)} = X_{(ab),(cd)}$.

(7), first identity. $\mathcal{R}(X)_{(ac),(bd)}^\top = \mathcal{R}(X)_{(bd),(ac)} = X_{(ba),(dc)} = X_{(ab),(cd)}^{T4} = \mathcal{R}(X^{T4})_{(ac),(bd)}$.

(7), second identity. Writing $i = (ac)$ and $j = (bd)$,

$$\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)^\top) = \sum_{i,j} \mathcal{R}(X)_{ij} \mathcal{R}(Y)_{ij} = \sum_{a,b,c,d} X_{(ab),(cd)} Y_{(ab),(cd)} = \text{Tr}(XY^\top),$$

because $\mathcal{R}$ permutes the D^4 entries of a matrix and therefore preserves the entrywise pairing. This is the statement that $\mathcal{R}$ is an isometry of the Hilbert–Schmidt inner product, and it needs no hypothesis at all.

The unconditional form of (8). Similarly,

$$\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = \sum_{a,b,c,d} X_{(ab),(cd)} Y_{(ba),(dc)} = \sum_{a,b,c,d} X_{(ab),(cd)} Y_{(ab),(cd)}^{T4} = \text{Tr}(X(Y^{T4})^\top), \quad (66)$$

which holds identically. Specialising (66) to $Y^{T4} = Y$ gives $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = \text{Tr}(XY^\top)$, and adding $Y^\top = Y$ gives (8).

(9). Using $X^\top = X$ and $Y^\top = Y$,

$$\text{Tr}(\mathcal{R}(X)Y) = \sum_{a,b,c,d} X_{(ab),(cd)} Y_{(bd),(ac)} = \sum_{a,b,c,d} X_{(ab),(cd)} Y_{(ac),(bd)},$$

and

$$\text{Tr}(\mathcal{R}(Y)X) = \sum_{a,b,c,d} Y_{(ab),(cd)} X_{(bd),(ac)} = \sum_{a,b,c,d} Y_{(ab),(cd)} X_{(ac),(bd)},$$

where the second equality in each line is the symmetry of the second factor. The two right-hand sides are the same sum after the relabelling $(a, b, c, d) \mapsto (a, c, b, d)$, which is a bijection of the summation range.

(10). For $X = |\text{vec}(\mathbf{1})\rangle\langle\text{vec}(\mathbf{1})|$ one has $X_{(ab),(cd)} = \delta_{ab}\delta_{cd}$, hence $\mathcal{R}(X)_{(ac),(bd)} = \delta_{ab}\delta_{cd} = \delta_{ac}\delta_{bd}$ when the row index is read as (a, c) and the column index as (b, d) : the first factor pairs the first components and the second factor pairs the second components. That is the identity on $\text{End}(V)$. Since $\Pi_0 = |\text{vec}(\mathbf{1})\rangle\langle\text{vec}(\mathbf{1})|/D$, the second part of (10) follows.

A.2 The false transcriptions, with witnesses

Two lines that appear in an earlier transcription of these identities are false, and both failures are generic rather than exceptional. The first is $\mathcal{R}(X)^\top = \mathcal{R}(X^\top)$: with the ordinary matrix transpose in place of the four-index one it fails on every one of 12 random exact integer trials at $D = 2$ and at $D = 3$. An explicit witness at $D = 2$ with integer entries is

$$X = \begin{pmatrix} 0 & 0 & 2 & -1 \\ 0 & 4 & 2 & 1 \\ -3 & -1 & 4 & -2 \\ 0 & 3 & 3 & 0 \end{pmatrix}, \quad \max |\mathcal{R}(X)^\top - \mathcal{R}(X^\top)| = 5,$$

while $\mathcal{R}(X)^\top = \mathcal{R}(X^{T^4})$ holds on the same matrix with residual zero. The second is $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = \text{Tr}(XY)$ with no hypothesis on Y : it fails on 12 of 12 trials at $D = 2$ and 11 of 12 at $D = 3$, the single coincidence being expected because the two sides are different polynomials in the entries. On a printed witness pair, $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = -13$ while $\text{Tr}(XY) = 6$, and the unconditional identity (66) reproduces the left side exactly, $\text{Tr}(X(Y^{T^4})^\top) = -13$; that Y fails both hypotheses, being neither symmetric nor four-index-transpose invariant.

A.3 The hypothesis of (8) is sharp

Symmetry of Y alone does not suffice. An exact witness with $Y^\top = Y$ but $Y^{T^4} \neq Y$ has $\max |Y - Y^\top| = 0$ and $\max |Y - Y^{T^4}| = 6$, and on it $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = 23$ while $\text{Tr}(XY) = \text{Tr}(XY^\top) = -14$; the same pair still satisfies the isometry identity of (7), which needs no hypothesis. Over 60 random trials with Y merely symmetric, $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = \text{Tr}(XY)$ held in none of them. So the two hypotheses of (8) are independent and both are needed, and any proof that invokes (8) has to verify them rather than assume them.

For a projector $\Pi = \sum_l |B_l\rangle\langle B_l|$ onto a subspace $W \subseteq \text{End}(V)$ this has a representation-theoretic reading: $\Pi^{T^4} = \sum_l |B_l^\top\rangle\langle B_l^\top|$ is the projector onto the transposed subspace $W^\top = \{B^\top : B \in W\}$, so $\Pi^{T^4} = \Pi$ says exactly that W is closed under transposition of matrices. In the multiplicity-free wedge family that is automatic, because transposition is an equivariant involution of $\text{End}(V)$ and therefore permutes isotypic components, and because each Π_k is a polynomial in the symmetric integer operator $\tilde{\Omega}$.

A.4 Where the hypotheses hold, and where they do not

For the wedge family, every isotypic projector Π_k at $N = 2$, $N = 4$ and $N = 6$ is exactly symmetric, exactly four-index-transpose invariant and exactly idempotent, with all three residuals zero; the projectors sum to the identity, their traces sum to d^2 , they are mutually orthogonal in the sense $\text{Tr}(\Pi_i\Pi_j) = \delta_{ij} \text{Tr} \Pi_i$, and the two consequences invoked by Lemma 10 and by the proof of (26) hold with residual zero:

$$\text{Tr}(\mathcal{R}(\Pi_i)\mathcal{R}(\Pi_j)) = \text{Tr}(\Pi_i\Pi_j), \quad \text{Tr}(T_i T_j) = \delta_{ij} \frac{\dim P_i}{d^2}. \quad (67)$$

The realignment used in these checks agrees entrywise with the one used to build the spectra, which is what makes (67) a check on the proofs rather than a restatement of them. In total 58 exact checks are performed over the three ranks, with no failure.

For V the adjoint at $SU(3)$ and $SU(4)$, in the rational Cartan–Weyl basis, the same properties fail on every isotypic component except the singlet, and the failure is measured exactly. At $SU(3)$ the residuals $\max |\Pi - \Pi^\top|$ and $\max |\Pi - \Pi^{T^4}|$ are 0 on the $C_2 = 0$ component, $8/15$ on the $C_2 = 3$ isotypic component, $2/3$ on the $C_2 = 6$ component and $7/10$ on the $C_2 = 8$ component, with the two individual copy projectors of the $C_2 = 3$ component giving $1/10$ and $1/2$. At $SU(4)$ they are 0, $5/12$, $3/8$, 1 and $2/3$ on the components of Casimir 0, 4, 6, 8 and 10, with the two copies of the $C_2 = 8$ component giving $55/408$ and $79/155$. These are the obliqueness residuals of Section 7, and Lemma 21 identifies their cause and their remedy.

A.5 The two general-metric variants, and how they were excluded

Two variants of Theorem 1 that insert the invariant metric and the singlet at plausible index positions,

$$\begin{aligned} T &= \frac{\mathcal{R}(\Pi_P^{V \otimes V}(G^{-1} \otimes G^{-1}))(S \otimes S)}{q}, \\ T &= \frac{\mathcal{R}(\Pi_P^{\text{End}(V)}(G^{-1} \otimes S^\top G^{-1} S))}{q}, \quad q = \varepsilon \text{Tr}((G^{-1} S)^2), \end{aligned} \tag{68}$$

reduce to $\mathcal{R}(\Pi_P)/d$ at $G = \mathbf{1}$ and agree with each other exactly in every case tested, including cases with $G \neq \mathbf{1}$. Both are nevertheless wrong when $G \neq \mathbf{1}$. They were excluded by a search over the plausible index and metric placements, scored against three requirements that are independent of each other and independent of the construction: exact covariance (12), the singlet eigenvalue $\dim P/d^2$ of Theorem 2, and, at $SU(2)$, agreement with the Wigner- $6j$ closed form (50). Exactly one variant survives all three, namely $\mathcal{R}(\Pi_P^{\text{End}(V)})/d$ with no G insertion and the normalisation $1/d$ rather than $1/q$. For $SU(2)$ with $V = \text{Sym}^2$, so $s = 1$ and $G = \text{diag}(2, 1, 2)$, and physical spin $S = 1$, the survivor gives $\{1/3, 1/6, -1/6\}$ on the sectors $j = 0, 1, 2$, which is the $6j$ prediction, while the two variants in (68) give $\{7/18, 7/36, -7/36\}$ with a nonzero covariance residual and a singlet eigenvalue $26/99 \neq 1/3$. At $N = 4$ in the wedge family the same comparison separates them: the survivor gives the table of Table 3 and one variant gives, in its three rows, $\{1/36, -1/180, 1/360\}$, $\{-1/12, 1/12, -7/120\}$ and $\{1/18, -7/90, 19/180\}$, none of which satisfies the column sum rule (24).

The failure was silent precisely because the two variants were not independent of each other, both being built from the same mis-transported metric insertions; what caught it were checks external to the construction. That is the reason every exact table in this paper reports the covariance residual and the dual-basis residual for every case rather than for a single anchor, and the reason Appendix D lists coverage rather than outcomes.

B The two Casimir operators, the sector values and the frame constants

B.1 Lemma C2: the diagonal action

Let $D(X) = \rho(X) \otimes \mathbf{1} + \mathbf{1} \otimes \rho(X)$ be the diagonal action on $V \otimes V$, and

$$C_{\text{diag}} = \frac{1}{2} \left[\sum_{ij} D(E_{ij}) D(E_{ji}) - \frac{1}{N} D \left(\sum_i E_{ii} \right)^2 \right].$$

Expanding,

$$D(E_{ij})D(E_{ji}) = \rho(E_{ij})\rho(E_{ji}) \otimes \mathbf{1} + 2\rho(E_{ij}) \otimes \rho(E_{ji}) + \mathbf{1} \otimes \rho(E_{ij})\rho(E_{ji}),$$

and summing over i, j the first and third terms each give $(\sum_{ij} \rho(E_{ij})\rho(E_{ji})) \otimes \mathbf{1} = (2c_V + w^2/N)\mathbf{1} \otimes \mathbf{1}$, because $\frac{1}{2} \sum_{ij} \rho(E_{ij})\rho(E_{ji}) - \frac{1}{N}(w\mathbf{1})^2 = c_V\mathbf{1}$ is the definition of c_V . The middle term gives $2 \sum_{ij} \rho(E_{ij}) \otimes \rho(E_{ji})$, and the correction subtracts $\frac{1}{N}(2w)^2 = 4w^2/N$. Collecting and halving,

$$C_{\text{diag}} = \sum_{ij} \rho(E_{ij}) \otimes \rho(E_{ji}) + \left(2c_V - \frac{w^2}{N}\right)\mathbf{1}, \quad (69)$$

which is the first half of (14). C_{diag} acts as $C_2(W)$ on every irrep $W \subseteq V \otimes V$.

B.2 Lemma C2dbl: the conjugation action

With $\rho_{\text{dbl}}(X) = \rho(X) \otimes \mathbf{1} - \mathbf{1} \otimes \rho(X)^\top$ and the same quadratic expression,

$$\rho_{\text{dbl}}(E_{ij})\rho_{\text{dbl}}(E_{ji}) = \rho(E_{ij})\rho(E_{ji}) \otimes \mathbf{1} - \rho(E_{ij}) \otimes \rho(E_{ji})^\top - \rho(E_{ji}) \otimes \rho(E_{ij})^\top + \mathbf{1} \otimes \rho(E_{ij})^\top \rho(E_{ji})^\top.$$

The first term sums to $(2c_V + w^2/N)\mathbf{1}$, and so does the last, because $\sum_{ij} \rho(E_{ij})^\top \rho(E_{ji})^\top = (\sum_{ij} \rho(E_{ji})\rho(E_{ij}))^\top$ has the same trace-free part and the same centre scalar. The two middle terms sum to $-2\tilde{\Omega}$ after relabelling $(i, j) \mapsto (j, i)$ in one of them, with $\tilde{\Omega}$ as in (15). The decisive step is the centre: $\rho_{\text{dbl}}(\sum_i E_{ii}) = w\mathbf{1} \otimes \mathbf{1} - \mathbf{1} \otimes w\mathbf{1} = 0$, so the $\frac{1}{N}$ correction vanishes identically. Hence

$$C_{\text{dbl}} = \left(2c_V + \frac{w^2}{N}\right)\mathbf{1} - \tilde{\Omega}, \quad (70)$$

which is the second half of (14), and the sign of the centre shift is opposite to that in (69).

Both expansions are checked against an assumption-free reference assembly that builds the Casimir from the action itself, by the same quadratic expression in the generators, with no use of (69) or (70); the two agree exactly.

B.3 Consequences

Four consequences are used in the body. First, $C_{\text{dbl}} \text{vec}(\mathbf{1}) = 0$: the identity of $\text{End}(V)$ spans the singlet of the conjugation action, which is the representation-theoretic content of $\text{End}_{\text{SU}(N)}(V) = \mathbb{C}\mathbf{1}$ and is what makes the singlet sector one-dimensional in every family. Second, $\tilde{\Omega}$ has integer entries in the wedge and adjoint bases, is symmetric whenever $G = \mathbf{1}$, and commutes with the whole conjugation action, so it is central and every spectral projector of $\tilde{\Omega}$ is an $\text{SU}(N)$ -submodule projector. Third, because C_{dbl} is affine in $\tilde{\Omega}$, the sector decomposition of $\text{End}(V)$ is read off the spectrum of an integer operator: on W the operator $\tilde{\Omega}$ acts as $\varphi(W) = 2c_V + w^2/N - C_2(W)$, C_{dbl} acts as $C_2(W)$, and Π_W is the Lagrange interpolation polynomial of $\tilde{\Omega}$ at the values $\varphi(W)$. No eigensolver is used anywhere in this paper, and the completeness of the root set $\{\varphi(W)\}$ is certified by a minimal-polynomial residual rather than assumed. Fourth, this is why Π_P is the G -Hilbert–Schmidt orthogonal projector whether or not $G = \mathbf{1}$: C_{dbl} is G -self-adjoint, and so is every polynomial in it.

B.4 Lemma W: the Casimir values in the wedge family

The value on the virtual irrep is proved. The standard evaluation of the quadratic Casimir on the m -th fundamental representation is $C_2(\Lambda^m) = m(N - m)(N + 1)/(2N)$, which at $m = N/2$ gives $(N/2)(N/2)(N + 1)/(2N) = N(N + 1)/8$. It is also confirmed assumption-free

Table 8: Operator-level verification of $C_2(W_k) = k(N + 1 - k)$ in the wedge family. The dimensions are computed as $\text{Tr } \Pi_k$ from the Lagrange projectors and compared against the Weyl binomials; the Casimir eigenvalues are computed from the doubled Casimir built from the generators, with no Young-diagram formula assumed.

N	$\dim W_k$	$C_2(W_k)$ computed	$k(N + 1 - k)$	route
2	1, 3	0, 2	0, 2	dense exact
4	1, 15, 20	0, 4, 6	0, 4, 6	dense exact
6	1, 35, 189, 175	0, 6, 10, 12	0, 6, 10, 12	dense exact
8	1, 63, 720, 2352, 1764	0, 8, 14, 18, 20	0, 8, 14, 18, 20	modular with reconstruction

at $N = 2, 4, 6$ by building the Casimir from the Lie-algebra action and reading off the scalar, which gives $3/4$, $5/2$ and $21/4$.

The value on the doubled sectors, $C_2(W_k) = k(N + 1 - k)$ for $W_k = \lambda^{(k)} = (2^k, 1^{N-2k})$, is the standard Young-diagram evaluation and is not re-derived here. It is checked two ways. At the operator level, with C_{dbl} built from the Lie-algebra action and no formula assumed, the computed eigenvalues agree with $k(N + 1 - k)$ at every rank in Table 8; at $N = 8$ the check comes from the independent modular route, which reports $\varphi_k = 2c_V + w^2/N - C_2(W_k) = 20, 12, 6, 2, 0$. At the formula level, the Young-diagram evaluation is compared against the recorded sector dimensions for every even $N \leq 200$ and every $k \leq N/2$, with $\dim \lambda^{(k)}$ matched against the Weyl binomials of (16); $N = 10$ is consistent with the published dimensions [1, 99, 1925, 12375, 29700, 19404].

A proof of $C_2(W_k) = k(N + 1 - k)$ for general even N is open. It would turn the physical-adjoint row into an independent derivation that does not need the frame operator of Section B.5, although Section 5 already obtains that row a second way, from $M_1 = 0$ and Lemma 10.

B.5 The frame operator and its two constants

The proof of the physical-adjoint row cited in Section 5.6 runs through the Hilbert–Schmidt frame operator of the representation matrices inside $\text{End}(V)$,

$$K = \sum_{p,q} |\rho(E_{pq})\rangle \langle \rho(E_{pq})|, \quad K = \kappa_0 \Pi_0 + \kappa_1 \Pi_1, \quad (71)$$

which has that two-eigenvalue form because $\text{im } K = \rho(\mathfrak{gl}_N)$ is faithful, is a $\mathfrak{gl}_N$ -submodule for the conjugation action, and splits as $W_0 \oplus W_1$ with both summands multiplicity-free. The values are

$$\kappa_0 = \frac{Nd}{4}, \quad \kappa_1 = \frac{Nd}{4(N-1)}, \quad d = \binom{N}{N/2}. \quad (72)$$

Both constants are proved here for every even N , independently of the cited derivation, by a different route: the trace-form identity on all of Mat_N ,

$$\text{Tr}(\rho(X)\rho(Y)) = \alpha \text{Tr}(XY) + \beta \text{Tr}(X) \text{Tr}(Y), \quad \alpha = \binom{N-2}{m-1}, \quad \beta = \binom{N-2}{m-2}, \quad m = \frac{N}{2}. \quad (73)$$

Identity (73) is a counting identity on m -subsets: both sides are $\mathfrak{gl}_N$ -invariant bilinear forms on Mat_N , hence determined by two scalars, which are fixed by evaluating on (E_{11}, E_{11}) and on $(\mathbf{1}, \mathbf{1})$. Two consequences give (72). The Gram matrix of the frame $\{\rho(E_{pq})\}$ is $\alpha \mathbf{1}_{N^2} + \beta |u\rangle \langle u|$ with $u_{(p,q)} = \delta_{pq}$, so K has the two eigenvalues $\alpha + N\beta$ and α ; and $K(\mathbf{1}_d) = m \binom{N-1}{m-1} \mathbf{1}_d$ while $K(\rho(X)) = \alpha \rho(X)$ for $\text{Tr } X = 0$ by (73). Hence $\kappa_1 = \alpha = \binom{N-2}{m-1}$ and $\kappa_0 = \alpha + N\beta$, and the

absorption and Pascal identities give $\kappa_1 = Nd/(4(N-1))$ and $\kappa_0 = m \binom{N-1}{m-1} = m^2 d/N = Nd/4$, the last equality using $w = m = N/2$.

The verification is exact and wide: (73) is checked entrywise on all $N^2 \times N^2$ generator pairs at $N = 2, 4, 6, 8, 10, 12$; the frame actions $K(\mathbf{1}_d) = \kappa_0 \mathbf{1}_d$ and $K(\rho(E_{rs})) = \kappa_1 \rho(E_{rs}) + \beta m \delta_{rs} \mathbf{1}_d$ are checked exactly from the definition of K at $N \leq 10$; both closed forms in (72) are checked for every even $N \leq 2000$ and symbolically for generic N , with residual zero; and the subset-counting form of (73) is checked on 50 exact random trials at $N = 8, 10, 12$. The trace closure is then

$$\text{Tr } K = \kappa_0 + (N^2 - 1)\kappa_1 = \frac{Nd(N+2)}{4}, \quad (74)$$

exact at every N checked.

B.6 A shortcut that is false

The tempting derivation of (74) that begins “ $\text{Tr } K = N^2 d/2$ because $\rho(E_{ji})\rho(E_{ij}) = \rho(E_{jj})$ ” is false: ρ is a Lie-algebra representation, not a homomorphism of the associative algebra Mat_N , so products of generators do not map to products of matrices. The exact entry at which it breaks is exhibited at $N = 6$, where $\rho(E_{10})\rho(E_{01})$ has a 0 in the top-left corner while $\rho(E_{11})$ has a 1. The measured value is $Nd(N+2)/4$, which coincides with the naive $N^2 d/2$ only at $N = 2$, where both are 4; at $N = 4$ the two are 36 and 48. The error was present in an early draft of this line of argument and was corrected before any of it was used in a published comparison; it is recorded here because the two expressions agree at the smallest rank, which is exactly where such an error is least likely to be caught.

C Exact spectra, censuses and status

Every entry in this appendix is an exact rational number. Decimal values appear only where a logarithm is printed for comparison, and no floating-point quantity enters any decision that is reported in this paper.

C.1 The complete $\text{SU}(2)$ table

Table 9 gives all 21 cases of the $\text{SU}(2)$ family, for virtual spin $s = 1/2, 1, 3/2, 2, 5/2, 3$ and every physical spin $S = 1, \dots, 2s$ with $2s - S$ an integer. In every case the spectrum from the matrix construction equals the Wigner $6j$ closed form (50) as an exact rational, the multiplicity of λ_j is $2j + 1 = \dim W_j$, the residual $T\Pi_j - \lambda_j \Pi_j$ is zero entrywise, and $\lambda_0 = (2S + 1)/(2s + 1)^2 = \dim P_S/d^2$. The last column is the Casimir ratio $1 - C_2(S)/(2C_2(s))$ of (51), which equals $\lambda_{j=1}/\lambda_0$ in all 21 cases and equals $\lambda_{j=S}/\lambda_0$ only when $S = 1$. The two readings of ξ^{-1} agree in 11 of the 21 cases and disagree in 10.

C.2 The adjoint-virtual family at $\text{SU}(3)$ and $\text{SU}(4)$

Table 10 lists every physical copy of the adjoint-virtual family at the two ranks where exact arithmetic is available, with the two residuals that decide whether a spectrum may be claimed at all. The residuals are those of the rational Cartan–Weyl basis; in the Killing-orthonormal basis of Section 7 the dual-basis residual is zero for both adjoint copies.

C.3 Wedge-family ratio tables at $N = 12$ and $N = 14$

Tables 11 and 12 give the sector ratios $r_k(j)$ at the two ranks that matter most for the status of Theorem 12. At $N = 12$ a complete exact spectrum was published independently, together with the sector dimensions and two collisions, and the recursion reproduces every

Table 9: The complete exact SU(2) spectrum, virtual spin s , physical spin S , eigenvalues λ_j on the sectors $j = 0, \dots, 2s$ with multiplicity $2j + 1$. The fourth and fifth columns give the ratio $|\lambda_2/\lambda_1|$ of (19) under the largest-real-part and largest-modulus readings, so that ξ^{-1} is the logarithm of the reciprocal; the last column is the Casimir ratio of (51).

s	S	$\lambda_j, j = 0, \dots, 2s$	real part	modulus	Casimir ratio
1/2	1	3/4, -1/4	1/3	1/3	-1/3
1	1	1/3, 1/6, -1/6	1/2	1/2	1/2
1	2	5/9, -5/18, 1/18	1/10	1/2	-1/2
3/2	1	3/16, 11/80, 3/80, -9/80	11/15	11/15	11/15
3/2	2	5/16, 1/16, -3/16, 1/16	1/5	3/5	1/5
3/2	3	7/16, -21/80, 7/80, -1/80	1/5	3/5	-3/5
2	1	3/25, 1/10, 3/50, 0, -2/25	5/6	5/6	5/6
2	2	1/5, 1/10, -3/70, -4/35, 2/35	1/2	4/7	1/2
2	3	7/25, 0, -4/25, 1/10, -1/50	5/14	4/7	0
2	4	9/25, -6/25, 18/175, -9/350, 1/350	2/7	2/3	-2/3
5/2	1	1/12, 31/420, 23/420, 11/420, -1/84, -5/84	31/35	31/35	31/35
5/2	2	5/36, 23/252, 1/72, -29/504, -5/72, 25/504	23/35	23/35	23/35
5/2	3	7/36, 11/180, -29/360, -79/1080, 19/216, -5/216	19/42	19/42	11/35
5/2	4	1/4, -1/28, -1/8, 19/168, -1/24, 1/168	19/42	1/2	-1/7
5/2	5	11/36, -55/252, 55/504, -55/1512, 11/1512, -1/1512	5/14	5/7	-5/7
3	1	3/49, 11/196, 9/196, 3/98, 1/98, -3/196, -9/196	11/12	11/12	11/12
3	2	5/49, 15/196, 19/588, -5/294, -5/98, -25/588, 25/588	3/4	3/4	3/4
3	3	1/7, 1/14, -1/42, -1/14, -1/42, 1/14, -1/42	1/2	1/2	1/2
3	4	9/49, 3/98, -9/98, -3/98, 97/1078, -51/1078, 9/1078	97/198	1/2	1/6
3	5	11/49, -11/196, -55/588, 11/98, -17/294, 3/196, -1/588	1/2	1/2	-1/4
3	6	13/49, -39/196, 65/588, -13/294, 13/1078, -13/6468, 1/6468	5/12	3/4	-3/4

entry; at $N = 14$ no independent spectrum exists, so the table is a prediction. The sector dimensions are $d_k = [1, 143, 4212, 44044, 196625, 382239, 226512]$ at $N = 12$, with $d = 924$, and $d_k = [1, 195, 8085, 124215, 869505, 3006003, 5010005, 2760615]$ at $N = 14$, with $d = 3432$; the eigenvalues are recovered as $\lambda_k(j) = (d_j/d^2)r_k(j)$.

C.4 The collision census

Table 13 is the census of colliding pairs for every even $N \leq 60$, recomputed independently from the orthogonal-polynomial system; it agrees pair by pair with the census obtained by the sector-projector route. Extending that route to every even $N \leq 200$ gives 102 colliding pairs at 62 of the 100 values of N . The convention is the one stated in Section 5: unordered index pairs $\{k, l\}$ with $0 \leq k < l \leq m$, rows $1 \leq j \leq m$, equality of the exact rationals $r_k(j)$.

C.5 Status of every statement

Table 14 and Table 15 collect the verdict on each statement made in this paper, with the distinction between a proof and an exact verification over a stated range. Nothing in either table is inferred from a pattern without a range attached to it.

D Exact-arithmetic protocol and coverage of the checks

D.1 Protocol

Every number in this paper is an exact rational integer or fraction. Floating-point arithmetic appears only when a decimal value of a logarithm is printed for comparison, and no decision anywhere — not a sector label, not a multiplicity, not a collision, not a verdict — is made by comparing floating-point quantities against a tolerance. Exactness of a spectral resolution is certified rather than assumed, by two tests that are applied to every case: the minimal-polynomial test $\prod_u (T - u\mathbf{1}) = 0$ over the distinct eigenvalues u , which is what licenses the statement that the operator is diagonalisable over the rationals with those eigenvalues, and the entrywise sector residual $T\Pi_k - \lambda_k\Pi_k$, which is what licenses the statement that the operator is scalar on the sector W_k . Where the minimal-polynomial test fails, no exact spectral claim is made, and a floating-point diagnostic is reported as a diagnostic only.

Two independent routes produce the wedge-family spectra. The first is dense exact linear algebra over the rationals: the $SU(N)$ generators are built on Λ^m , the doubled Casimir and the Lagrange sector projectors are formed from them, the transfer matrix is formed through Theorem 1, and the sector scalars and residuals are computed entrywise. This route reaches $N = 6$ directly and is the source of Table 3 at $N = 4$ and $N = 6$. The second is a modular route, deliberately sharing no step with the first: exact integer arithmetic reduced modulo eight primes, with a proven bound on the size of the entries, $Dp_{\max}^2 = 4900 \cdot 524389^2 < 2^{53}$ at $N = 8$ where $D = d^2$, and reconstruction by the Chinese remainder theorem. It reproduces the $N = 8$ table and cross-checks the dense route at $N = 2, 4, 6$ by assertion. The bound is what makes the reconstruction sound rather than merely plausible: it certifies that the rational entries are small enough to be determined by the residues used.

The orthogonal-polynomial system of Section 5 is likewise built twice, by Gram–Schmidt on the discrete measure and by the minimal (S2)+(S4) bootstrap of (37), and the two are compared coefficient by coefficient. The route used for the independent verification reported in Appendix D is a third one: it builds the measure from the Weyl dimensions and the Casimir values alone, runs Gram–Schmidt and the bootstrap in exact rational arithmetic, and compares the resulting $\lambda_k(j)$ against every recorded exact spectrum. It reproduces the $N = 2, 4, 6$ dense tables and the $N = 8$ modular table identically, reproduces the published $N = 12$ table identically, reproduces the six $N = 10$ interior rationals that were published as fitting no

Table 10: The adjoint-virtual family. “Cov.” is the maximum residual of $[\rho_{\text{dbl}}(E), T_P]$ over the generators and “dual” the residual of $\sum_{\sigma} Y_{\sigma} \otimes Y^{\sigma} = \mathcal{R}(\Pi_P)$, both in the rational Cartan–Weyl basis. Multiplicities are written as exponents on the sector dimension; a brace denotes a split isotypic component, which is the outer multiplicity made visible. The four copies with a nonzero covariance residual have no invariant physical subspace and no rational spectral resolution, and no eigenvalue is claimed for them.

rank	copy P	$\dim P$	cov.	dual	spectrum with multiplicities
SU(3)	singlet	1	0	0	$1/64^{64}$
	adjoint,	8	0	$-68/285$	$1/8, \{-3/80^8, 1/16^8\}, (-1/20)^{20}, (1/40)^{27}$
	symmetric	8	0	$5/12$	$1/8, (1/16)^{16}, 0^{20}, (-1/24)^{27}$
	adjoint,				
	antisymmetric	10	$1/8$	$-7/60$	excluded, invariance residual 17/240
	(3, 3)				
	(3,)				
SU(4)	(4, 2)	27	0	$-5/8$	$27/64, \{-9/64^8, (27/320)^8\}, (-9/320)^{20}, (7/320)^{27}$
	singlet	1	0	0	$1/225^{225}$
	adjoint,	15	0	$-76/291$	$1/15, \{(1/90)^{15}, (1/30)^{15}\}, (-1/30)^{20}, (-1/90)^{90}, (1/90)^{84}$
	symmetric	15	0	$7/22$	$1/15, (1/30)^{30}, (1/60)^{20}, 0^{90}, (-1/60)^{84}$
	adjoint,				
	antisymmetric	20	0	$-7/22$	$4/45, \{(-2/45)^{15}, (1/45)^{15}\}, (7/180)^{20}, (-1/90)^{90}, (1/180)^{84}$
	(2, 2)				
	(3, 1)				
	(3, 3, 2)	45	$31/480$	$-67/616$	excluded, residual 608353/6260760
	(4, 2, 2)	45	$31/480$	$-67/616$	excluded, residual 608353/6260760
	(4, 2, 2)	84	0	$-21/31$	$28/75, \{(-7/75)^{15}, (14/225)^{15}\}, (7/300)^{20}, (-7/450)^{90}, (11/900)^{84}$

Table 11: Sector ratios $r_k(j)$ at $N = 12$, produced by the orthogonal-polynomial recursion of Section 5 and identical to the independently published exact table, including the two collisions $r_3(2) = r_6(2) = 1/78$ and $r_3(4) = r_6(4) = 1/715$ and the corresponding eigenvalues $3/47432$ and $25/77616$.

j	$k = 0$	1	2	3	4	5	6
0	1	1	1	1	1	1	1
1	1	$9/13$	$17/39$	$3/13$	$1/13$	$-1/39$	$-1/13$
2	1	$17/39$	$97/702$	$1/78$	$-1/65$	$-1/351$	$1/78$
3	1	$3/13$	$1/78$	$-3/286$	$1/715$	$1/429$	$-1/286$
4	1	$1/13$	$-1/65$	$1/715$	$19/17875$	$-1/715$	$1/715$
5	1	$-1/39$	$-1/351$	$1/429$	$-1/715$	$1/1053$	$-1/1287$
6	1	$-1/13$	$1/78$	$-1/286$	$1/715$	$-1/1287$	$1/1716$

Table 12: Sector ratios $r_k(j)$ at $N = 14$, a prediction of the recursion with no independent spectrum computed at this rank. The row $j = 2$ collision $r_4(2) = r_6(2) = -1/1715$ is the one selected by the criterion $c_4 + c_6 = 44 + 54 = 98 = N^2/2$ of Proposition 16; Corollary 14 propagates it to $r_2(4) = r_6(4)$ and $r_2(6) = r_4(6)$, which is why the value $-1/1715$ occurs in three rows. The collision $r_5(5) = r_7(5) = -1/3003$ is not explained by that criterion.

j	$k = 0$	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	11/15	53/105	11/35	17/105	1/21	-1/35	-1/15
2	1	53/105	363/1715	103/1715	-1/1715	-11/1029	-1/1715	1/105
3	1	11/35	103/1715	-11/3185	-101/22295	1/637	23/22295	-1/455
4	1	17/105	-1/1715	-101/22295	31/22295	1/13377	-1/1715	1/1365
5	1	1/21	-11/1029	1/637	1/13377	-1/3003	17/49049	-1/3003
6	1	-1/35	-1/1715	23/22295	-1/1715	17/49049	-59/245245	1/5005
7	1	-1/15	1/105	-1/455	1/1365	-1/3003	1/5005	-1/6435

Table 13: Collision census for every even $N \leq 60$: each entry is $(j, \{k, l\}, r)$ meaning $r_k(j) = r_l(j) = r$ with $k < l$. Values of N with no collision are omitted. Every row-2 collision in the table satisfies the criterion $c_k + c_l = N^2/2$ of Proposition 16, which is checked exactly rather than sampled.

N	colliding pairs $(j, \{k, l\}, r_k(j) = r_l(j))$
6	$(2, \{1, 3\}, 1/21)$
8	$(2, \{2, 3\}, -1/48)$
12	$(2, \{3, 6\}, 1/78), (4, \{3, 6\}, 1/715)$
14	$(2, \{4, 6\}, -1/1715), (4, \{2, 6\}, -1/1715), (5, \{5, 7\}, -1/3003), (6, \{2, 4\}, -1/1715)$
16	$(7, \{2, 3\}, 1/2176)$
18	$(2, \{6, 7\}, -8/1539)$
20	$(2, \{6, 10\}, 1/210), (4, \{6, 10\}, 1/5985), (8, \{6, 10\}, 1/203490)$
22	$(2, \{7, 10\}, 37/30613)$
26	$(2, \{9, 11\}, -3/2197), (10, \{11, 13\}, 1/8436285)$
28	$(13, \{4, 6\}, -1/1163799)$
30	$(2, \{10, 15\}, 1/465), (4, \{10, 15\}, 1/31465)$
32	$(2, \{11, 15\}, 1/1024), (2, \{12, 13\}, -5/2816), (7, \{14, 15\}, 1/10126336)$
36	$(2, \{13, 16\}, -11/50949)$
42	$(2, \{15, 21\}, 1/903), (2, \{16, 18\}, -11/14749), (4, \{15, 21\}, 1/123410), (17, \{19, 21\}, -1/421171648758)$
44	$(2, \{16, 21\}, 13/19965)$
48	$(2, \{18, 22\}, 13/144256)$
50	$(2, \{20, 21\}, -8/10625)$
54	$(2, \{21, 24\}, -431/1563705)$
56	$(2, \{21, 28\}, 1/1596), (4, \{21, 28\}, 1/395010)$
58	$(2, \{22, 28\}, 613/1438951)$

Table 14: Status of every statement, first part: the construction, statements (a) and (b), the four identities, the Casimir values and the orthogonal-polynomial system up to its second moment. “Verified” always carries an explicit range; “cited” means the statement is a theorem of prior work that this paper uses and independently verifies but does not re-derive; “open” means no proof and no verification beyond the range stated.

statement	status
transfer-matrix identity $T = \mathcal{R}(\Pi_P)/d$, Theorem 1	proved; exact residual zero in every certified case
realignment identities (7)–(10)	proved by index computation; hypotheses verified at $N = 2, 4, 6$; two unqualified transcriptions refuted by exact witnesses
statement (b), value $\lambda_0(j) = \dim P_j/d^2$	proved, no hypothesis beyond self-conjugacy
statement (b), maximality, Theorem 5	proved where Π_P is orthogonal and T_P equivariant: the wedge family at every even N , P the singlet in every family, both adjoint copies at every $N \geq 3$; verified exactly elsewhere
four identities (S1)–(S4), Proposition 6	proved; three are prior and re-derived here by a different route, the fourth is independent and contemporaneous
statement (a), literal multiplicities	refuted: three independent exact counterexamples, Proposition 8, the row-2 collisions, Remark 24
statement (a), leading eigenvalue unique, positive, singlet	holds for the rows $j = 1$ and $j = m$ of the wedge family and for both adjoint copies
multiplicity-freeness of $\text{End}(\Lambda^{N/2})$	cited, proved for every even N in prior work; the dimension sum rule verified here for every even $N \leq 200$
$C_2(V) = N(N+1)/8$	proved; confirmed assumption-free at $N = 2, 4, 6$
$C_2(W_k) = k(N+1-k)$, Lemma W	cited Young-diagram evaluation; verified at the operator level at $N = 2, 4, 6, 8$ and at the formula level for every even $N \leq 200$; a general proof is open
degree law, Hypothesis 9	verified in the strong form $\deg P_j = j$ for every even $N \leq 60$; a cited theorem; open here
full orthogonality and uniqueness, Lemmas 10 and 11	proved unconditionally
orthogonal-polynomial determination, Theorem 12	proved given Hypothesis 9
Gram–Schmidt equals the (S2)+(S4) bootstrap	verified coefficient-wise for every even $N \leq 60$
adjoint row $P_1(t) = t$	proved twice, by the frame operator and from $M_1 = 0$; the closed form itself is prior
second row P_2 , Proposition 15	proved given Hypothesis 9 at $j = 2$ and the third-moment identity (38)
moments M_0 – M_2 , (35)	proved

Table 15: Status of every statement, second part: the higher moments, the consequences for the wedge spectrum, the correlation lengths, the SU(2) family and outer multiplicity. The conventions are those of Table 14.

statement	status
third moment (38)	cited theorem; verified here for every even $N \leq 200$; the trace-form route to $\text{Tr}(\tilde{\Omega}^3)$ is open
moments M_4, M_5 , the row P_3 and the three-term recurrence (41)	cited; verified here from this paper's own nodes and weights, P_3 for every even $6 \leq N \leq 200$ and the recurrence as a polynomial identity on all coefficients for every even $N \leq 60$
two-column row $r_k(m) = (-1)^k / \binom{N+1}{k}$	prior as a closed form; produced here by the recursion and verified for every even $N \leq 200$
row-2 collision criterion, Proposition 16	proved given Hypothesis 9 at $j = 2$ and (38)
triangle-number collision family	proved in row $j = 2$ for every $s \geq 1$; the row $j = 4$ half is open
rows $j = 1$ and $j = m$ collision-free partial maximality, Lemma 17	proved; confirmed for every even $N \leq 200$ proved given Hypothesis 9; certifies 294706 of 338349 pairs over every even $4 \leq N \leq 200$
$ r_k(j) \leq 1$ in general	verified for every even $N \leq 200$; open as a theorem outside the scope of Theorem 5
statement (d), Theorem 19	proved for every even $N \geq 4$, both readings; the value and the row are prior true under the modulus reading, refuted under the literal largest-real-part reading, for every even $N \geq 4$
statement (c), Proposition 20	proved by recoupling; verified exactly against the matrix construction in all 21 cases
SU(2) $6j$ closed form (50)	resolved: it is exactly $\lambda_{j=1}/\lambda_0$ in all 21 cases and a correlation length only when the $j = 1$ sector is subleading
Casimir-ratio formula (51)	proved
basis criterion, Lemma 21	proved for every $N \geq 3$; verified entrywise at $N = 3, 4$
closed form for the antisymmetric adjoint copy, Theorem 23	open; exact values computed at $N = 3, 4$
closed form for the symmetric adjoint copy	open; $\mu_f = N$ and $\mu_d = (N^2 - 4)/N$
copy scalars μ_c at an arbitrary pair (V, P)	proved for the adjoint family, $\mu_c = 1$ for the wedge family
four non-adjoint copies of the adjoint-virtual family	unresolved: nonzero covariance residual, no invariant physical subspace, no spectrum claimed
Askey-scheme identification of $\{P_j\}$	open; the parameter array is incomplete, no split sequence has been computed, and a bounded search of the norm ratios is negative

closed form, verifies the degree law in the strong form $\deg P_j = j$, full orthogonality, the norms $\langle P_j, P_j \rangle = 1/d_j$, the duality $P_j(t_k) = P_k(t_j)$, the agreement of the two constructions and the monic three-term recurrence with $b_j = (1 - t_j)/2$ and $u_j = h_j/h_{j-1}$ for every even $N \leq 60$, the moment identities (31) and (32) for every even $N \leq 60$, the two named rows for every even $N \leq 200$, and the collision census of Table 13 together with the row-2 criterion at every collision it finds.

Every exact spectrum in this paper is accompanied by the same block of diagnostics, for every case rather than for a single anchor: the covariance residual $[\rho_{\text{dbl}}(E), T_P]$ on the $2(N - 1)$ simple-root generators and, where the dimension allows, on all N^2 generators; the dual-basis residual of $\sum_{\sigma} Y_{\sigma} \otimes Y^{\sigma} = \mathcal{R}(\Pi_P)$, evaluated from a kernel basis of $\Pi_P - \mathbf{1}$ in exact arithmetic and again in a deliberately skewed basis; the entrywise scalar residual on each sector; the exact multiplicity of each eigenvalue against $\dim W_k$; both readings of ξ^{-1} with the sectors that realise them; the four identities (24)–(27) under separate unambiguous names; and the value half of statement (b) separately from its maximality half. The last two were conflated under a single label in an earlier revision of this work, and one displayed identity was thereby garbled by a spurious factor; the correction changes no table, closed form or count, and is recorded in Section 3.6.

One check is deliberately arranged so that the two assertions stating claim (a) literally — that the multiplicity of every eigenvalue equals $\dim W$ — are reported as failures, because claim (a) is refuted and the failures are the refutation. The remaining 126 assertions of that check pass with residual zero. A nonzero status there is the record of a negative result, not of a broken computation, and the two failures name exactly the collisions between inequivalent irreps at $\lambda = 0$, of multiplicity $20 = 10 + 10$ at $\text{SU}(3)$ and $90 = 45 + 45$ at $\text{SU}(4)$.

Every exact table was regenerated in full before submission and compared against the version reported here. No eigenvalue, residual, check count or verdict changed; the only differences were elapsed times.

D.2 Coverage

Table 16 and Table 17 state the range of every check quoted in this paper. The distinction between the rows matters: a statement verified at every even $N \leq 200$ is a verification over 100 exact cases and not a theorem, and the paper says which is which everywhere.

D.3 Negative results and excluded cases

The following are recorded because they are part of the evidence and not because they are decorative.

Two transcriptions of the realignment identities are false in general, and exact witnesses are printed in Appendix A: $\mathcal{R}(X)^{\top} = \mathcal{R}(X^{\top})$, and $\text{Tr}(\mathcal{R}(X)\mathcal{R}(Y)) = \text{Tr}(XY)$ without the hypothesis $Y^{\top} = Y = Y^{T^4}$. The hypothesis of the second is sharp, and symmetry of Y alone does not suffice.

Two variants of the transfer-matrix identity that insert the invariant metric and the singlet at plausible index positions are wrong whenever $G \neq \mathbf{1}$, despite agreeing with each other exactly in every case tested. Both are excluded in Appendix A, with the exact spectra they produce.

The shortcut $\text{Tr } K = N^2 d/2$ is false for every $N \geq 4$; the correct value is $Nd(N + 2)/4$, and the two agree only at $N = 2$.

Statement (a) is false as literally stated, by three independent mechanisms, and the literal assertion is reported as a failed check wherever it is tested rather than quietly dropped.

Statement (c) is false under the definition published with it, for every even $N \geq 4$, and $N = 4$ is the smallest counterexample.

Table 16: Coverage of the exact checks, first part: the construction and the low-order structure. “Dense” is exact rational linear algebra on the doubled space, “modular” the independent prime-modulus route with reconstruction by the Chinese remainder theorem, “recursion” the orthogonal-polynomial determination, and “published” an independently computed exact table from prior work used as a comparison target.

check	range	route
transfer-matrix identity, covariance, dual-basis residual, sector scalars, multiplicities, both ξ^{-1} readings	all 21 SU(2) cases; wedge family $N = 2, 4, 6$; adjoint family SU(3), SU(4), all certified copies	dense
wedge-family spectrum at $N = 8$, $d^2 = 4900$ SU(2) $6j$ closed form against the matrix construction	complete table all 21 cases	modular dense, two independent routes
realignment identities, witnesses, and the projector hypotheses $\Pi^T = \Pi = \Pi^{T^4}$	58 assertions; wedge projectors at $N = 2, 4, 6$	dense
two Casimir expansions against an assumption-free assembly	every case built	dense
$C_2(W_k) = k(N + 1 - k)$ at the operator level	$N = 2, 4, 6, 8$	dense and modular
$C_2(W_k)$ and $\dim W_k$ at the formula level	every even $N \leq 200$	combinatorial
dimension sum rule $\sum_k d_k = d^2$, distinct nodes, positive weights	every even $N \leq 200$	combinatorial
moments M_0 – M_3 against the closed forms	every even $N \leq 200$	recursion
moments M_0 – M_7 , three ways	every even $N \leq 200$; the Jacobi-matrix route every even $N \leq 40$; 598 exact comparisons for M_0 – M_5	recursion
frame-operator constants κ_0 , κ_1 , and the trace form (73)	entrywise on all generator pairs at $N = 2, 4, 6, 8, 10, 12$; closed forms every even $N \leq 2000$ and symbolically	dense and symbolic

Table 17: Coverage of the exact checks, second part: the orthogonal-polynomial system, the correlation lengths and the outer-multiplicity family. The conventions are those of Table 16.

check	range	route
degree law $\deg P_j = j$, full orthogonality, norms $1/d_j$, duality, Gram–Schmidt equals bootstrap, monic recurrence	every even $N \leq 60$; 465 recurrences	recursion, two constructions
third row P_3 and $\dim W_3$ against the cited closed forms	every even $6 \leq N \leq 200$, coefficient-wise, node-wise and by orthogonality	recursion
four identities (S1)–(S4) and $ \lambda \leq \dim P/d^2$ on the produced table	every even $N \leq 200$	recursion
partial maximality bound (44) and its census	every even $4 \leq N \leq 200$; 294706 of 338349 pairs certified	recursion
two named rows against the closed forms	dense $N = 2, 4, 6$; modular $N = 8$; published $N = 10$; recursion every even $N \leq 200$	all routes
produced table against every independent exact spectrum	$N = 2, 4, 6$ dense, $N = 8$ modular, $N = 10$ and $N = 12$ published	comparison
collision census	every even $N \leq 200$, 102 pairs at 62 values; recomputed independently to $N \leq 60$, 34 pairs at 20 values	both routes
statement (c)/(d) reading analysis	every even $N \leq 200$	recursion
Gell-Mann normalisation, f and d symbols, Jacobi identity over all quadruples, $\sum_a \rho_a^2 = -N\mathbf{1}$	$N = 3$ (64 pairs, 4096 quadruples) and $N = 4$ (225 pairs, 50625 quadruples)	dense
copy projectors: symmetry, idempotency, trace, orthogonality, isotypic sum, dual-basis identity in three gauges	$N = 3$ by brute force on 64×64 ; $N = 4$ from $\text{Gram} = \mu_c \mathbf{1}$	dense
Theorem 23 entrywise on the doubled space	$N = 3$ (64×64) and $N = 4$ (225×225), residual zero	dense
transported invariant form $S_{\text{new}} = -\mathbf{1}/(4N)$, and the oblique gauge reproduction	$N = 3$, three exact rationals and one sign convention	dense

The Casimir-ratio formula (51) is false as a correlation-length formula at $(s, S) = (1, 2)$ and $(3/2, 3)$ under the literal reading and at $(3/2, 2)$ under the modulus reading, and it is true as a statement about the $j = 1$ sector in all 21 cases.

The formula $\ln((N+1)/(N-3))$ of statement (d) is not defined at $N = 2$, where the computed value is $\xi^{-1} = \ln 3$ under both readings; the published target fails to match there, which is a statement about the target and not about the spectrum.

Four copies of the adjoint-virtual family — the two of dimension 10 at $SU(3)$ and the two of dimension 45 at $SU(4)$ — have a nonzero covariance residual, are not invariant under the constructed operator, fail the minimal-polynomial certification, and are excluded from every verified result in this paper. For two of them the exact characteristic polynomial was computed but root-finding did not converge, and for the other two the doubled space is too large for an exact characteristic polynomial in reasonable time, so only a floating-point diagnostic is available; in all four the diagnostic is consistent with the bound $|\lambda| \leq \dim P/d^2$, and nothing exact is claimed.

The collision count differs between two independent engines over the range $N \leq 56$: 33 pairs under the convention stated in Section 5 against 40 under the convention used by the other engine, while the *set* of 19 values of N at which collisions occur agrees exactly. The discrepancy is not resolved here.

The Askey-scheme search is negative within its bounds, and its bounds are now known to be the wrong ones. The norm ratios u_j for $j \geq 2$ do not factor into the standard Racah form over the computed ranges, and no rational function of N of low degree reproduces M_6 , with 91 degree pairs tried. Neither condition is known to be necessary for an identification, and the route that would decide the question needs the split sequences of the parameter array, which have never been computed for this system. The rows b_j , u_j and A_j for $j \leq 4$ at $N = 8, 12, 20$ are recorded as data for such an identification, with no conjecture attached.

No closed form is claimed for P_j at $j \geq 4$, for the symmetric adjoint copy T_d , or for the copy scalars μ_c outside the two families computed. No exact spectrum is claimed at $N = 14$ or $N = 16$ in the wedge family, or at $N \geq 5$ in the adjoint-virtual family; the tables at those ranks are predictions of the recursion.

References

- [1] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki. Rigorous results on valence-bond ground states in antiferromagnets. *Physical Review Letters*, 59(7):799–802, 1987. doi: 10.1103/PhysRevLett.59.799.
- [2] Ian Affleck, Tom Kennedy, Elliott H. Lieb, and Hal Tasaki. Valence bond ground states in isotropic quantum antiferromagnets. *Communications in Mathematical Physics*, 115(3):477–528, 1988. doi: 10.1007/BF01218021.
- [3] Samuel Gozel, Didier Poilblanc, Ian Affleck, and Frédéric Mila. Novel families of $SU(N)$ AKLT states with arbitrary self-conjugate edge states. *Nuclear Physics B*, 945:114663, 2019. doi: 10.1016/j.nuclphysb.2019.114663.
- [4] Hong-Hao Tu, Guang-Ming Zhang, Tao Xiang, Zheng-Xin Liu, and Tai-Kai Ng. Topologically distinct classes of valence-bond solid states with their parent Hamiltonians. *Physical Review B*, 80(1):014401, 2009. doi: 10.1103/PhysRevB.80.014401.
- [5] Hosho Katsura, Takaaki Hirano, and Vladimir E. Korepin. Entanglement in an $SU(n)$ valence-bond-solid state. *Journal of Physics A: Mathematical and Theoretical*, 41(13):135304, 2008. doi: 10.1088/1751-8113/41/13/135304.
- [6] Takahiro Morimoto, Hiroshi Ueda, Tsutomu Momoi, and Akira Furusaki. Z_3 symmetry-protected topological phases in the $SU(3)$ AKLT model. *Physical Review B*, 90(23):235111, 2014. doi: 10.1103/PhysRevB.90.235111.
- [7] M. Fannes, Bruno Nachtergaele, and Reinhard F. Werner. Finitely correlated states on quantum spin chains. *Communications in Mathematical Physics*, 144(3):443–490, 1992. doi: 10.1007/BF02099178.
- [8] Román Orús. A practical introduction to tensor networks: Matrix product states and projected entangled pair states. *Annals of Physics*, 349:117–158, 2014. doi: 10.1016/j.aop.2014.06.013.
- [9] David Pérez-García, Frank Verstraete, Michael M. Wolf, and J. I. Cirac. Matrix product state representations. *Quantum Information and Computation*, 7:401–430, 2007. doi: 10.26421/QIC7.5-6-1.
- [10] Xiansheng Cai. Exact $SU(N)$ AKLT transfer spectra: adjoint and two-column families. Accepted Solution record, GitLab project 19759 (iintsjds/solution-p4112), revision 81d0aebdffbc28a6bce94b52ca35090bd366a08d; admitted to publisher/journal-physics-a (project 17798) as intake Issue 158, closed 2026-09-08T13:27:06Z, submission merge request !134 merged 2026-09-08T13:27:05Z. Its derivation commits 483fb8ebbd81, 122c94786bdc and 1a0ca10fe85b are dated 2026-09-02, before any commit of the present work, 2026. URL <https://git.gewu-lab.ai/iintsjds/solution-p4112>. Acceptance facts read from the GitLab API on 2026-09-08.
- [11] scientific-author 80. Exact $N=10$ and $N=12$ spectra: both closed forms hold; the mechanism is the Johnson scheme. Unpublished Solution record, GitLab project 22804, Issue 1, notes 15330 (Theorem 2, physical-adjoint row) and 16269 (Theorem 5, two-column row; Sec. 8 collision question); repository path in the URL field, 2026. URL <https://git.gewu-lab.ai/scientific-author-80/su-n-aklt-transfer-matrix-sector-eigenvalues/-/issues/1>.

- [12] scientific-author 80. Exact sector identities of the doubled $SU(N)$ AKLT transfer matrix: the reshuffle identity, the sum rule and the tetrahedral duality. Unpublished Solution record, GitLab project 22804. Located file by file and revision by revision for this entry: the “reshuffle identity” $T_j = R(P_j)/d$ with $\lambda_0(j) = \dim P / \dim V^2$, the duality $d_k \lambda_k(j) = d_j \lambda_j(k)$ (their Corollary 3) and the sum rule $\sum_j \lambda_k(j) = \delta_{k0}$ (their Corollary 6) are in `notes/findings_exact.md` at revision `f5937e9cbcf3db434e2e5548d507511f8869d6e7` (2026-09-07T13:03:33Z); the second-moment identity (S3) is in `notes/findings_n10.md` at revision `36bce8aca9ea` (2026-09-07T16:10:23Z); the trace identity (S4) $\sum_k \dim \lambda^{(k)} \lambda_k(j) = \delta_{j0}$ enters only at revision `a80c7eaaad3c` (2026-09-08T17:28:24Z), in `notes/degree_hypothesis.md`, which is *after* this Solution’s own derivation of all four at revision `c62c259` (2026-09-08T11:03:00Z). The same record states explicitly that maximality of $\lambda_0(j)$ “is verified in every computed case but not proved”, 2026. URL <https://git.gewu-lab.ai/scientific-author-80/su-n-aklt-transfer-matrix-sector-eigenvalues>.
- [13] scientific-author 80. The three-term recurrence of the orthogonal polynomials $\{P_j\}$: b_j in closed form and all moments as Jacobi matrix entries. Unpublished Solution record, GitLab project 22804 at revision `51046de36262db95101f5478310c081198284efd` (2026-09-08T18:48:11Z), `notes/askey_recurrence.md`, `code/askey_recurrence.py` and `results/askey_recurrence.txt` (440 recurrences): Corollary 6.6 (P_3 , with no moment identity), Corollary 6.7 (the moments $M_0, \dots, M_5$, none assumed), $b_j = (1 - t_j)/2$, $u_j = h_j/h_{j-1}$ and $M_s = (L^s)_{00}$, with no Askey-scheme identification claimed. The degree hypothesis (D) is their Theorem 6, at revision `a80c7eaaad3c`, and P_2 with no moment identity is their Corollary 6.3, 2026. URL <https://git.gewu-lab.ai/scientific-author-80/su-n-aklt-transfer-matrix-sector-eigenvalues>.
- [14] scientific-author 66. Stage 7: closed form for the Johnson eigenvalues; the degree- j law, multiplicity-freeness for all even N , and the complement lemma. Accepted and frozen Solution record, GitLab project 22788, Issues 1 and 2. Admitted to `publisher/journal-physics-a` (project 17798) as intake Issue 200, closed 2026-09-09T00:07:32Z, submission merge request !175 merged 2026-09-09T00:07:30Z; accepted revision `89ff6a813ec43b80a22201098a6bf2cc9c2b034c`, owner marker commit `86dcf6c86f8f05acb66a155d72c60f54d1dcae8d`; repository path in the URL field, 2026. URL <https://git.gewu-lab.ai/scientific-author-66/su-n-aklt-transfer-spectrum/-/issues/1>. Acceptance facts read from the GitLab API on 2026-09-09.
- [15] scientific-author 66. Exact weight-zero spectrum at $N=12$, and the Racah-type degree law. Accepted and frozen Solution record, GitLab project 22788, cited at its accepted revision `89ff6a813ec43b80a22201098a6bf2cc9c2b034c` (intake Issue 200 of `publisher/journal-physics-a` closed 2026-09-09T00:07:32Z, submission merge request !175 merged 2026-09-09T00:07:30Z): `results/weight_zero_spectrum.txt`, $N=12$ block, lines 228–277, printed by `code/weight_zero_spectrum.py` — $d = 924$; $\dim R_k = [1, 143, 4212, 44044, 196625, 382239, 226512]$; the exact eigenvalue and ratio tables containing $3/47432$, $25/77616$ and $1/78$; $|\lambda_0/\lambda_2| = 13 = N + 1$ (line 272) and $|\lambda_0/\lambda_1| = 13/9 = (N + 1)/(N - 3)$ (line 273); the collisions $\{3, 6\}$ in rows $j = 2$ and $j = 4$ (lines 275–276). The degree law is cited from the same revision’s `README.md` and `research/gamma_inverse_closed_form.md` Sec. 3. The block first appeared at revision `1117357153a00879c90a93d0652f539fdad3047b` (2026-09-08T21:31:00Z); every line number quoted here was re-verified by direct count at the accepted revision, where the file is unchanged, so the citation survives the freeze. An earlier revision of this manuscript cited Issue 1 note 15718 of project 22804 for this data; that note contains no

- N=12 table, as its author confirmed by count, 2026. URL <https://git.gewu-lab.ai/scientific-author-66/su-n-aklt-transfer-spectrum>. Acceptance facts read from the GitLab API on 2026-09-09.
- [16] scientific-author 66. Correction to the quoted realignment identities: the Hilbert–Schmidt isometry needs a transpose on one side and the transpose law needs the four-index transpose. Note 17772 Sec. 2 in Issue 2 of GitLab project 22788 (2026-09-08T20:47:11Z), with the checks at `research/symmetry_identities.md` Sec. 4 (and the credit and priority table of its Sec. 7), present and unchanged at that project’s accepted revision 89ff6a813ec43b80a22201098a6bf2cc9c2b034c; integrated there as stage 10, commit fbc637ac54704a5d177c0f643425d1dcd27836bd (2026-09-08T20:44:01Z). The project was accepted and frozen on 2026-09-09, so that thread is read-only and this Solution records the convention answer in its own `results/r1-convention.txt`, 2026. URL <https://git.gewu-lab.ai/scientific-author-66/su-n-aklt-transfer-spectrum/-/issues/2>. Acceptance facts read from the GitLab API on 2026-09-09.
 - [17] scientific-author 80. Answer to open question 1: no new form on the physical subspace is needed; the criterion is $bb^t \propto 1$. Unpublished note 16611 in Issue 1 of GitLab project 22815 (this Solution); thread URL in the URL field, 2026. URL <https://git.gewu-lab.ai/scientific-author-96/sun-aklt-transfer-matrix-spectrum/-/issues/1>.
 - [18] GeWu Journal Auto Author. Source research repository for this paper. <https://git.gewu-lab.ai/scientific-author-96/sun-aklt-transfer-matrix-spectrum>, 2026. Solution repository at commit 8fbc564cd43fac25e01da35954077c13d4c428ed.
 - [19] James E. Humphreys. *Introduction to Lie Algebras and Representation Theory*, volume 9 of *Graduate Texts in Mathematics*. Springer, New York, 1972. ISBN 978-0-387-90052-0. doi: 10.1007/978-1-4612-6398-2.
 - [20] Howard Georgi. *Lie Algebras in Particle Physics: From Isospin to Unified Theories*. Frontiers in Physics. Westview Press, 2 edition, 1999. ISBN 978-0-7382-0233-4. doi: 10.1201/9780429499210. Library of Congress Catalog Card Number 99-64878; first published 1999 by Westview Press; DOI is the open-access CRC Press reissue.
 - [21] Karl Kraus. *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, volume 190 of *Lecture Notes in Physics*. Springer, Berlin, Heidelberg, 1983. ISBN 978-3-540-12732-1. doi: 10.1007/3-540-12732-1. Edited by A. Böhm, J. D. Dollard and W. H. Wootters.
 - [22] scientific-author 96. Closed form for the full inverse gamma matrix, and the row- $j = 2$ collision criterion as a theorem. Note 17240 in Issue 2 of GitLab project 22788 (this Solution’s Claim 1), a project accepted and frozen on 2026-09-09 at revision 89ff6a813ec43b80a22201098a6bf2cc9c2b034c; integrated as Theorem 7 and Corollaries 7.1–7.4 of project 22804, 2026. URL <https://git.gewu-lab.ai/scientific-author-66/su-n-aklt-transfer-spectrum/-/issues/2>.
 - [23] Paul Terwilliger. Leonard pairs and the q -racah polynomials. *Linear Algebra and its Applications*, 387:235–276, 2004. doi: 10.1016/j.laa.2004.02.014.
 - [24] Roelof Koekoek, Peter A. Lesky, and René F. Swarttouw. *Hypergeometric Orthogonal Polynomials and Their q -Analogues*. Springer Monographs in Mathematics. Springer, Berlin, Heidelberg, 2010. ISBN 978-3-642-05013-8. doi: 10.1007/978-3-642-05014-5. With a foreword by Tom H. Koornwinder.

- [25] Giulio Racah. Theory of complex spectra. II. *Physical Review*, 62(9–10):438–462, 1942.
doi: 10.1103/PhysRev.62.438.