

Exact plateau, a tight conserved set and the exact status of the Overhauser-field estimate in the uniform-coupling central spin model

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Abstract

A localized spin one half coupled by Heisenberg exchange to a bath of N spins one half is the standard model of hyperfine decoherence. At infinite temperature its autocorrelation function does not decay to zero: the long-time plateau is the zero-frequency weight that spin-noise and free-induction-decay measurements probe. Two questions were open in the sector where every bath spin carries the same coupling, the exactly solvable endpoint of coupling-distribution space. Can a set of conserved operators reproduce the plateau exactly, and is the semiclassical prescription that divides the conserved part of the Overhauser-field autocorrelation by twelve quantitatively reliable? Both are answered here for every finite bath size, as exact rational numbers. The plateau equals the classical frozen-field value one twelfth plus a correction of order the inverse bath size whose odd- N part is exactly the probability that a simple symmetric random walk of $N + 1$ steps returns to its origin; that probability is bracketed by a two-sided bound in which both additive constants are optimal. A single explicit conserved operator saturates the generalized Mazur bound at every finite bath size, and the smallest set of powers of the Hamiltonian that does so has a top degree that is proved minimal. The semiclassical prescription is asymptotically exact, its relative error falling as the inverse bath size, and about ninety-five per cent of that error is the semiclassical step itself rather than the looseness of the bound fed into it. The uniform limit of the full Gaudin set, by contrast, stops being tight at four bath spins, and the quadratic extensions that repair it at four and five spins stop repairing it at six. No claim is made about the time dependence of the correlation function, and nothing beyond four exact symmetry statements is claimed for inhomogeneous couplings.

1 Introduction

A single localized spin coupled to a large environment of nuclear spins loses its polarization, and the mechanism that dominates in III-V and in many group-IV structures is the contact hyperfine interaction [15, 3]. The model that captures it is old and deceptively simple: a central spin one half $\mathbf{S}_0$ interacting by isotropic exchange with N bath spins one half $\mathbf{S}_k$,

$$H_0 = \sum_{k=1}^N J_k \mathbf{S}_0 \cdot \mathbf{S}_k, \quad (1)$$

with site-dependent couplings J_k set by the envelope of the electron wave function. Dephasing of the central spin in this model has been treated by equation-of-motion methods [6, 7], by high-field diagonalization of the coupling to an unpolarized bath [37], and by cluster and semiclassical expansions of the nuclear dynamics [5]; experiments on GaAs dots measure the resulting free-induction decay and its suppression by narrowed nuclear states [2, 14].

*Corresponding author. All derivations, code, archived verification output and this manuscript are maintained in the Solution repository `scientific-author-77/uniform-csm-exact-plateau-tight-mazur` (project 22793) on the host `git.gewu-lab.ai`; see Section 12 for the full availability statement.

One feature of (1) is not a matter of approximation. At infinite temperature the autocorrelation $S(t) = \langle S_0^z(t) S_0^z(0) \rangle$ does not tend to zero; it approaches a plateau S_∞ equal to the zero-frequency weight of the spectrum, and that weight is what spin-noise spectroscopy sees [36, 27]. Rigorous lower bounds on it follow from Mazur’s inequality [25, 8] in the generalized form that applies to an arbitrary set of linearly independent conserved operators [38]; the same framework underlies exact results for the late-time correlations of fragmented and pair-flip chains [8, 16]. For the central spin model this route gives a persisting fraction of the initial correlations for any coupling distribution whose average does not vanish, with no appeal to integrability [38], and mathematically rigorous bounds on the persisting magnetization with and without a magnetic field, together with a well-defined limit of infinitely many bath spins [35].

Two questions remain, and they are the subject of this paper. The first concerns tightness: a bound is only as good as the conserved set fed into it, and what counts as a complete set is subtle even in models where the conserved quantities are known [8]. Does there exist, for (1), a set of conserved operators whose generalized Mazur bound *equals* the plateau, and how small can such a set be? The second concerns a semiclassical prescription conjectured on numerical evidence in [38]: that the plateau is quantitatively described by the Overhauser field $\mathbf{B} = \sum_k J_k \mathbf{S}_k$ through

$$\hat{S} = \frac{S_{\text{low}}^B}{12 S^B(0)}, \quad (2)$$

where $S^B(t) = \langle B^z(t) B^z(0) \rangle$, $S^B(0) = \langle (B^z)^2 \rangle$, S_{low}^B is the same Mazur bound applied to B^z instead of S_0^z , and the conserved set is the total magnetization $I^z = \sum_{j=0}^N S_j^z$ together with the N quantities $I^z H_l$ built from the Gaudin integrals [12]. The factor twelve is the classical frozen-field value: for a static bath field of fixed length pointing in a uniformly random direction the precessing central spin retains $\langle S_0^z \rangle_t = (\mathbf{n} \cdot \hat{\mathbf{z}})^2 S_0^z(0)$, so that $S_\infty^{\text{cl}} = \frac{1}{4} \langle (\mathbf{n} \cdot \hat{\mathbf{z}})^2 \rangle = \frac{1}{12}$. Estimates of this kind are the working tool of the Overhauser-field picture [29, 7], and their range of validity at finite N is what (2) asks about.

The model has been approached from several directions, and it is worth saying how the present result sits among them. Electron-spin relaxation and decoherence by nuclei in quantum dots were treated in a series of early works [26, 20, 19, 4]. Integrability supplies exact out-of-equilibrium dynamics for time-dependent exchange couplings [11], extensive conserved sets and bright/dark state structures for anisotropic models [39], off-diagonal Bethe-ansatz solutions and a thermodynamic limit for the XXZ central spin model in a field [40], and explicit local dynamics of an inhomogeneous Gaudin magnet from Bethe wave functions [17], with the completeness of the Bethe states established for generic couplings [23] and the integrability structure itself classified in [9, 10]. Large baths are handled either by a quantum mechanical treatment of the bath as a whole [31] or by proving the Gaussian central-limit form of the dephasing for Ising-coupled baths with disorder and long-range couplings [1], and extensions let the bath spins interact among themselves [24]. These routes address dynamics, thermodynamic limits and inhomogeneous couplings; the object here is a different one, an exact finite- N zero-frequency weight together with a conserved set proved to saturate its bound, in the one sector where both are attainable in closed form.

Both questions are answered here in the sector of *uniform* couplings, $J_k = J$ for every k , where (1) becomes $H = J \mathbf{S}_0 \cdot \mathbf{I}$ with $\mathbf{I} = \sum_{k \geq 1} \mathbf{S}_k$. This sector is the exactly solvable endpoint of coupling-distribution space, opposite to the generic strongly inhomogeneous case, and it is also the point at which the standard toolkit degenerates: the Gaudin energies $\varepsilon_l = -1/J_l$ all coincide, so the integrals H_l with $l \geq 1$ are undefined and the $N + 1$ conserved quantities of (2) collapse to the single pair $\{I^z, I^z H\}$. Uniform coupling is where analytic solutions of the model are available and used [14, 28], and it is the sector in which the classical value $1/12$ is expected to emerge, so it is the natural place to ask what the quantum model adds at finite N and which steps of the semiclassical argument that correction actually uses. The price of the restriction is real and is stated wherever it matters: nothing in Sections 3–7 holds for inhomogeneous couplings, and the results say nothing about the shape of $S(t)$, about any time scale, or about the $\ln(x)/x$ asymptotics that rigorous bounds produce for baths dominated by very weak couplings [35].

The answers are as follows, and each is an exact rational statement about every finite $N \geq 1$ rather than an asymptotic or numerical one.

The plateau itself is

$$S_\infty(N) = \frac{1}{12} + \frac{1 - P_N}{6(N+1)}, \quad P_N = \mathbb{P}(S_{N+1} = 0), \quad (3)$$

where $\mathbb{P}(S_{N+1} = 0)$ is the probability that a simple symmetric random walk on $\mathbb{Z}$ returns to the origin after $N+1$ steps (Theorem 1 and Corollary 3). A classical walk-return probability is therefore the exact coefficient of a quantum zero-frequency weight. Section 6 turns the same identity into a sharp two-sided bound: for every odd N , $\sqrt{2/(\pi N + 8 - \pi)} \leq q_N < \sqrt{2/(\pi N + 3\pi/2)}$, with both additive constants optimal (Theorem 12), and the resulting bracket on the three relative errors of the semiclassical estimate is rigorous at every finite odd N (Theorem 14), with widths that fall as $N^{-5/2}$.

The tight conserved set exists and is explicit. The dephased observable has the closed form

$$A_{\text{diag}} = \sum_E \Pi_E S_0^z \Pi_E = J I^z (4H + J)^{-1}, \quad (4)$$

with Π_E the projector onto the eigenspace of H with energy E (Theorem 4); the generalized Mazur bound evaluated on this single conserved operator equals S_∞ at every finite N , so the bound is exactly saturated. Because H has only $N+1$ distinct eigenvalues, (4) is a polynomial in H times I^z , and the smallest such set is $\{I^z H^a : 0 \leq a \leq d\}$ with $d = N$ at even N and $d = N-1$ at odd N ; that d is minimal is a theorem for every N , not a measurement (Theorem 5). The parity dependence of d has a concrete cause: at odd N exactly one eigenspace carries $I^z = 0$ identically, the set $\{I^z H^a\}_{a=0}^N$ becomes linearly dependent, and its Gram matrix is exactly singular (Proposition 6).

The semiclassical estimate (2) is asymptotically exact in this sector, with an error budget that is exact at every finite N (Theorem 10): the relative error of the canonical pair is $14/(5N) - 2q_N/N + O(N^{-2})$, and it splits into a frozen-field part and a Mazur-looseness part in the ratio 20/21 : 1/21. Almost all of the error is therefore the semiclassical step, not the bound. The mechanism is a structural identity rather than an accident: the exact plateau of the Overhauser-field autocorrelation satisfies $S_\infty^B = (N-1)/4 + S_\infty$, because B^z differs from the exactly conserved I^z by the single central spin (Proposition 9). Enriching the conserved set moves $\hat{S}$ monotonically upward through three exactly known levels but changes only the coefficient of the $1/N$ error, from 3 to 14/5 to 8/3 (Section 5.3), and the parity correction $-2q_N/N$ is shared by all three.

Not every natural conserved set is tight, and the negative results are as informative as the positive ones. The canonical pair captures only 3/5 of the plateau in the limit, with a deficit that tends to the constant 1/30 (Proposition 8). The uniform limit of the *full* Gaudin set, the span $V_0 = \text{span}\{I^z, I^z H, I^z X_l\}$ obtained by letting the coupling inhomogeneity go to zero, saturates the bound exactly at $N = 2$ and $N = 3$ and stops doing so from $N = 4$ onwards, with exact deficits 4/195, 25/1728, 4/161, 125/6144 and 8/297 for $N = 4, \dots, 8$ (Section 7). Adjoining $I^z H^2$ does not help: it already lies in V_0 , with an explicit rational dependence at $N = 4$. Two quadratic families do restore saturation at $N = 4$ and $N = 5$ and fail to do so at $N = 6$ and $N = 7$, so the homogeneous point is where the extension problem is hardest and no single small extension is a general route to tightness.

Finally, Section 8 leaves the uniform sector far enough to ask how fragile all of this is, and proves three further exact symmetry statements that hold for every N and every direction in coupling space: invariance under permutations and under nonzero rescaling of the couplings; a reversal theorem that identifies the relative spread as the only variable and makes every zero-frequency quantity a rational function of $u = v^2$; and the vanishing of the first-order coupling-spread correction in *every* direction, not merely along the affine ramp. Together with the parity selection rule of Section 4.2, these four statements explain the shape of the tight set rather than merely exhibiting it, and they locate the uniform point as a singular point of the extension problem: the

quadratic family that is insufficient at zero spread is exactly the one that saturates the bound at every nonzero spread.

2 Model, conventions and the bound

This section fixes what is being computed. It states the Hilbert space, the state and the observable whose infinite-time average is the plateau, records the bound that every result below is measured against, and isolates the one feature of the uniform point that makes it singular: the conserved quantities that the inhomogeneous model is normally built from cease to exist there.

2.1 Hilbert space, state and observable

The bath consists of N spins one half with sites labelled $k = 1, \dots, N$, and site 0 carries the central spin. All results below hold for uniform couplings $J_k = J$ with $J \neq 0$ of either sign, so that

$$H = J \sum_{k=1}^N \mathbf{S}_0 \cdot \mathbf{S}_k = J \mathbf{S}_0 \cdot \mathbf{I}, \quad \mathbf{I} = \sum_{k=1}^N \mathbf{S}_k. \quad (5)$$

The Hilbert space has dimension $D = 2^{N+1}$ and the state is the infinite-temperature one, $\rho = \mathbb{I}/D$. Write $\langle X \rangle = \text{Tr}(\rho X)$ and, for Hermitian X, Y ,

$$\langle X|Y \rangle = \text{Tr}(\rho X^\dagger Y) = \langle XY \rangle. \quad (6)$$

The observable is $A = S_0^z$, so that $\langle A^2 \rangle = 1/4$, and the three further operators that appear throughout are the total magnetization, its bath part, and the Overhauser field,

$$I^z = \sum_{j=0}^N S_j^z = S_0^z + Z, \quad Z = \sum_{k=1}^N S_k^z, \quad B^z = JZ. \quad (7)$$

The correlation function of interest is $S(t) = \langle A(t)A(0) \rangle$ with $A(t) = e^{iHt} A e^{-iHt}$ in units $\hbar = 1$, and its long-time average is the quantity this paper computes,

$$S_\infty = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T S(t) dt. \quad (8)$$

Inserting the spectral resolution of H and writing Π_E for the orthogonal projector onto the eigenspace of energy E , the time average kills every oscillating term and leaves the *dephased second moment*

$$S_\infty = \frac{1}{D} \sum_E \text{Tr}[A \Pi_E A \Pi_E] = \langle A_{\text{diag}}^2 \rangle, \quad A_{\text{diag}} := \sum_E \Pi_E A \Pi_E. \quad (9)$$

The two forms are equal because $\text{Tr}[A \Pi_E A \Pi_E] = \text{Tr}[(\Pi_E A \Pi_E)^2]$. The operator A_{diag} is conserved by construction and satisfies $\langle A_{\text{diag}} A \rangle = \langle A_{\text{diag}}^2 \rangle = S_\infty$, and (9) is the reason it is the natural object to look for: everything below is a study of A_{diag} and of how much of it a given conserved set can see. Because H scales linearly with J , (8) is independent of J for $J \neq 0$; the coupling sets the time scale and nothing else.

2.2 The generalized Mazur bound

For a finite set of conserved operators $\mathcal{C} = \{C_1, \dots, C_n\}$ with $[C_i, H] = 0$, the generalized Mazur inequality [38] bounds the long-time average of any observable X from below by the squared Hilbert–Schmidt projection of X onto the span of $\mathcal{C}$,

$$S_{\text{low}}(X; \mathcal{C}) = \mathbf{a}^\dagger G^+ \mathbf{a} \leq \liminf_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle X(t)X(0) \rangle dt, \quad G_{ij} = (C_i|C_j), \quad a_i = (C_i|X), \quad (10)$$

where G^+ is the Moore–Penrose pseudoinverse of the Gram matrix G . If the C_i are linearly independent, G is positive definite and $G^+ = G^{-1}$; the pseudoinverse is what makes (10) well defined on a set that is not, and one such set appears below at every odd N . The bound is invariant under rescaling any C_i by a nonzero number, and it is monotone in the span: enlarging $\mathcal{C}$ can only increase S_{low} . Equality in (10) holds if and only if the projection of X onto the conserved algebra equals A_{diag} ; this is the content of the Suzuki-type completeness statements discussed in [8], and it is the sense in which a conserved set is called *tight* here.

Two specializations are used. With $X = A = S_0^z$ the right-hand side of (10) is S_∞ and the bound is written $S_{\text{low}}(N)$; with $X = B^z$ it is S_∞^B , the plateau of the Overhauser-field autocorrelation, and the bound is written $S_{\text{low}}^B(N)$. The semiclassical estimate (2) is then

$$\hat{S}(N) = \frac{S_{\text{low}}^B(N)}{12 S^B(0)}, \quad S^B(0) = \langle (B^z)^2 \rangle = \frac{J^2 N}{4}, \quad (11)$$

where the last equality uses (7) and $\langle S_k^z S_l^z \rangle = \delta_{kl}/4$. Since S_{low}^B scales as J^2 and $S^B(0)$ as J^2 , $\hat{S}$ is J -independent, exactly like S_∞ and S_{low} . The estimate is a statement about a ratio of zero-frequency weights: it asks whether the fraction of the field’s own plateau that is conserved, divided by the classical factor twelve, reproduces the fraction of the spin’s plateau that is conserved.

2.3 What degenerates at uniform coupling

For inhomogeneous couplings the model (1) is a Gaudin magnet: with $\varepsilon_l = -1/J_l$ the operators

$$H_l = \sum_{m \neq l} \frac{\mathbf{S}_l \cdot \mathbf{S}_m}{\varepsilon_l - \varepsilon_m}, \quad l = 0, \dots, N, \quad (12)$$

commute with H_0 and with each other [12, 17], and the conserved set prescribed for (2) is $\{I^z\} \cup \{I^z H_l\}$ [38], the same integrable structure that is exploited in [9, 10]. At uniform coupling every ε_l coincides, the denominators in (12) vanish, and the integrals H_l for $l \geq 1$ are undefined. Two exact relations control what survives. First $\sum_{l \geq 0} H_l = 0$, so that $I^z H = -\sum_{l \geq 1} I^z H_l$ remains meaningful as the uniform limit of a combination of the integrals. Second, the quantity $\sum_l \varepsilon_l H_l^z$ that plays the role of the magnetization-like integral is proportional to $I^z H$ when all ε_l are equal. The conserved set of the prescription therefore collapses to the *canonical uniform pair*

$$\mathcal{C}_{\text{can}} = \{I^z, I^z H\}, \quad (13)$$

which is the object whose exact Mazur bound is computed in Section 5. Both entries are conserved: H is a scalar under the global rotation generated by $\mathbf{S}_0 + \mathbf{I}$, hence $[I^z, H] = 0$, and $I^z H$ is a product of two commuting conserved operators. The larger span obtained from the full Gaudin set by taking the uniform limit exists as well, and is the subject of Section 7.

2.4 The collective-spin reduction

Everything exact in this paper rests on one reduction, so it is stated once and in full. The bath space decomposes into irreducible representations of the global bath $\text{SU}(2)$,

$$(\mathbb{C}^2)^{\otimes N} = \bigoplus_k g(N, k) V_k, \quad g(N, k) = \binom{N}{\frac{N}{2} - k} - \binom{N}{\frac{N}{2} - k - 1}, \quad (14)$$

where k runs over $\frac{N \bmod 2}{2}, \frac{N \bmod 2}{2} + 1, \dots, \frac{N}{2}$, V_k is the spin- k irrep of dimension $2k + 1$, and $g(N, k)$ is its multiplicity; a binomial coefficient with a negative lower index is zero. The dimension identity $\sum_k g(N, k)(2k + 1) = 2^N$ holds by construction. Because H in (5) acts on the bath only

through the collective operator $\mathbf{I}$, every one of the $g(N, k)$ copies behaves identically and the model reduces to one central spin one half coupled to a spin k ,

$$\frac{1}{2} \otimes k \longrightarrow \begin{cases} j = k + \frac{1}{2} & \text{with energy } h_+ = Jk/2, \\ j = k - \frac{1}{2} & \text{with energy } h_- = -J(k+1)/2 \quad (k \geq \frac{1}{2}). \end{cases} \quad (15)$$

The two families never coincide since $h_+ \geq 0 > h_-$, and within a family different k give different energies, so H has exactly $N+1$ distinct eigenvalues. Standard Clebsch–Gordan coefficients give, inside a copy of bath spin k ,

$$\langle k, j=k \pm \frac{1}{2}, m | S_0^z | k, j=k \pm \frac{1}{2}, m \rangle = \pm \frac{m}{2k+1}, \quad I^z \text{ eigenvalue} = m, \quad (16)$$

and the only off-diagonal matrix elements of S_0^z connect $j = k + \frac{1}{2}$ with $j = k - \frac{1}{2}$ at fixed m , that is, connect different eigenvalues of H . They therefore drop out of every dephased trace, which is what makes (16) sufficient for (9).

3 The exact plateau

Theorem 1 (exact plateau). *For every $N \geq 1$ and every $J \neq 0$, with*

$$q_N = \begin{cases} \binom{N}{(N-1)/2} / 2^N, & N \text{ odd}, \\ 0, & N \text{ even}, \end{cases} \quad (17)$$

the plateau of the uniform-coupling central spin model is

$$S_\infty(N) = \frac{N+3}{12(N+1)} - \frac{q_N}{6(N+1)} = \frac{N+3-2q_N}{12(N+1)}. \quad (18)$$

Proof. By (9) and (16), each eigenspace contributes the squares of the diagonal elements $\pm m/(2k+1)$ summed over its states. With $M_2(j) = \sum_{m=-j}^j m^2 = j(j+1)(2j+1)/3$ and $D = 2^{N+1}$,

$$S_\infty = \frac{1}{2^{N+1}} \sum_k g(N, k) \frac{M_2(k + \frac{1}{2}) + M_2(k - \frac{1}{2})}{(2k+1)^2}. \quad (19)$$

The numerator simplifies: $M_2(k + \frac{1}{2}) + M_2(k - \frac{1}{2}) = (k + \frac{1}{2})(4k^2 + 4k + 3)/3$, and dividing by $(2k+1)^2 = 4(k + \frac{1}{2})^2$ gives $(4k^2 + 4k + 3)/(6(2k+1))$. Since $4k^2 + 4k + 3 = (2k+1)^2 + 2$,

$$S_\infty = \frac{1}{6} \left[\frac{1}{2} + \frac{1}{2^N} \sum_k \frac{g(N, k)}{2k+1} \right], \quad (20)$$

where $\sum_k g(N, k)(2k+1) = 2^N$ was used for the first term. Only the inverse sum remains, and it is where the two parities separate.

Even N . The $\text{SU}(2)$ character of V_k is $\chi_k(t) = \sin((2k+1)t)/\sin t$, and the character of a tensor power is its N th power, so $\sum_k g(N, k)\chi_k(t) = (2\cos t)^N$ and hence

$$\sum_k g(N, k) \sin((2k+1)t) = \sin t (2\cos t)^N. \quad (21)$$

For even N every $2k+1$ is odd, and $\int_0^{\pi/2} \sin(nt) dt = 1/n$ for odd n . Integrating (21) term by term and substituting $u = \cos t$,

$$\sum_k \frac{g(N, k)}{2k+1} = \int_0^{\pi/2} \sin t (2\cos t)^N dt = 2^N \int_0^1 u^N du = \frac{2^N}{N+1}. \quad (22)$$

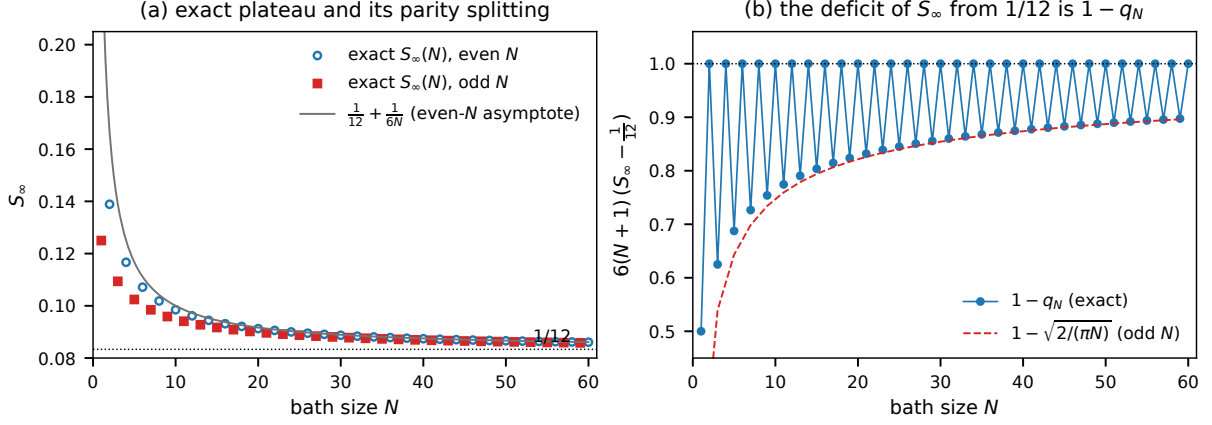


Figure 1: The exact plateau as a function of bath size. (a) $S_\infty(N)$ from (18) for bath sizes up to 60, with the even- N branch above and the odd- N branch below the horizontal line at the classical frozen-field value $1/12$; the smooth upper curve is the even- N asymptote $1/12 + 1/(6N)$. Both branches are exact rational numbers evaluated on an integer grid, not measurements. (b) The same data rescaled as $6(N+1)(S_\infty - 1/12)$, which equals $1 - q_N$ identically by (27); the exact value is plotted against the naive asymptotic surrogate $1 - \sqrt{2/(\pi N)}$, which is visibly too small at odd N and which Section 6 replaces by a proved two-sided bracket.

Inserting (22) into (20) gives $S_\infty = (N+3)/(12(N+1))$, which is (18) with $q_N = 0$.

Odd N . Now $2k+1 = 2q$ is even and the integral above no longer produces $1/(2k+1)$. Write $N = 2m+1$ and $k = j + \frac{1}{2}$, so that $g(2m+1, j + \frac{1}{2}) = \binom{2m+1}{m-j} - \binom{2m+1}{m-j-1}$. Abel summation of the telescoping difference, followed by the substitutions $p = m-j$ and $r = m-p$, gives

$$\begin{aligned} \sum_k \frac{g(N, k)}{2k+1} &= \frac{1}{2} \sum_{r=0}^m \frac{\binom{2m+1}{r}}{2m+2-r} = \frac{1}{2m+2} \sum_{r=0}^m \binom{2m+2}{r} \\ &= \frac{2^{2m+2} - \binom{2m+2}{m+1}}{2(2m+2)} = \frac{2^N - \binom{N}{(N-1)/2}}{N+1}, \end{aligned} \quad (23)$$

where the second equality uses $\binom{n}{r}/(n+1-r) = \binom{n+1}{r}/(n+1)$, the third uses the symmetry $\sum_{r>m+1} \binom{2m+2}{r} = \sum_{r \leq m} \binom{2m+2}{r}$ together with the middle term $\binom{2m+2}{m+1}$, and the last uses $\binom{N+1}{(N+1)/2} = 2\binom{N}{(N-1)/2}$. Substituting into (20),

$$S_\infty = \frac{1}{6} \left[\frac{1}{2} + \frac{1}{N+1} - \frac{\binom{N}{(N-1)/2}}{2^N(N+1)} \right] = \frac{N+3}{12(N+1)} - \frac{q_N}{6(N+1)}, \quad (24)$$

which is (18). Neither branch used J beyond the ordering of the eigenvalues, so the result holds for every $J \neq 0$. $\square$

The first values are exact rationals and alternate with parity,

$$S_\infty : \frac{1}{8}, \frac{5}{36}, \frac{7}{64}, \frac{7}{60}, \frac{59}{576}, \frac{3}{28}, \frac{605}{6144}, \frac{11}{108}, \frac{491}{5120}, \frac{13}{132}, \quad N = 1, \dots, 10, \quad (25)$$

and the alternation is a property of the formula, not noise: at even N the correction is absent, at odd N it lowers the plateau. Table 1 collects these values together with the two bounds and the estimate computed in Sections 5 and 5.3, and Figure 1(a) shows both branches against the classical value.

3.1 Asymptotics and the classical limit

The central binomial coefficient gives $\binom{N}{(N-1)/2}/2^N = \sqrt{2/(\pi N)} (1 + O(N^{-1}))$, hence

$$S_\infty(N) = \frac{1}{12} + \frac{1}{6N} + O(N^{-3/2}), \quad \frac{S_\infty(N)}{\langle (S_0^z)^2 \rangle} = 4S_\infty(N) \rightarrow \frac{1}{3}, \quad (26)$$

with the parity term entering at order $N^{-3/2}$ and therefore larger than the $O(N^{-2})$ remainder. The approach is from above and monotone along each parity branch: $N(S_\infty - \frac{1}{12})$ rises through 0.148148 at $N = 8$, 0.161616 at $N = 32$, 0.165375 at $N = 128$ and 0.166666500 at $N = 10^6$ towards its limit $1/6$. The limit $1/12$ is exactly the frozen-field value derived in Section 1, so the uniform quantum model reproduces the semiclassical answer at infinite bath size and the whole content of (18) is the finite- N correction to it.

3.2 The walk-return form

The parity term has an exact probabilistic reading, and stating it removes the case distinction from (18).

Lemma 2 (walk-return identity). *Let S_n be the position after n steps of a simple symmetric random walk on $\mathbb{Z}$ and let $P_N = \mathbb{P}(S_{N+1} = 0)$, that is $P_N = [N+1 \text{ even}] \binom{N+1}{(N+1)/2} / 2^{N+1}$. Then $q_N = P_N$ for every $N \geq 1$.*

Proof. At even N the walk has $N+1$ odd length and cannot return, so both sides vanish. At odd $N = 2k+1$, $\binom{N+1}{(N+1)/2} = \binom{2k+2}{k+1} = 2\binom{2k+1}{k} = 2\binom{N}{(N-1)/2}$, whence $P_N = 2\binom{N}{(N-1)/2} / 2^{N+1} = q_N$. $\square$

Corollary 3 (parity-free plateau). *For every $N \geq 1$,*

$$S_\infty(N) = \frac{1}{12} + \frac{1 - P_N}{6(N+1)}, \quad (27)$$

with no remainder term at even N , where P_N is the return probability of a classical simple symmetric walk of $N+1$ steps.

Two remarks bound what this says. First, (27) is an identity between a quantum zero-frequency weight and a classical combinatorial quantity; it is established here, not explained, and no mechanism is claimed for why a walk-return probability should govern the plateau. Combinatorial return probabilities of one-dimensional walks are otherwise elementary [21], and the appearance of one here is what makes the correction sharply bracketable in Section 6. Second, (27) replaces a piecewise formula by a single all- N one, which is what allows the error budgets of Sections 5 and 6 to be written as exact rational functions of one variable.

3.3 Relation to the published two-bath-spin result

For two bath spins with arbitrary couplings the plateau is known in closed form [32],

$$S_\infty = \frac{7J_1^2 - 4J_1J_2 + 7J_2^2}{72(J_1^2 - J_1J_2 + J_2^2)}, \quad (28)$$

and at $J_1 = J_2$ this gives $5/36$, in exact agreement with the $N = 2$ entry of (25). The agreement is a genuine cross-check of two different routes rather than a consistency of notation: (28) is a statement about an inhomogeneous model and reduces to the uniform sector only on the diagonal, while Theorem 1 holds for every N but only there.

4 A tight conserved operator and a minimal tight set

Theorem 4 (closed form of the dephased observable). *For every $N \geq 1$ and $J \neq 0$ the operator $4H + J$ is invertible and*

$$A_{\text{diag}} = \sum_E \Pi_E S_0^z \Pi_E = J I^z (4H + J)^{-1} =: C^*. \quad (29)$$

Consequently C^* is conserved, and the generalized Mazur bound (10) evaluated on the single operator $\mathcal{C} = \{C^*\}$ equals the plateau exactly,

$$S_{\text{low}}(A; \{C^*\}) = \frac{(C^*|A)^2}{(C^*|C^*)} = \frac{\langle C^* A \rangle^2}{\langle (C^*)^2 \rangle} = S_{\infty}(N) \quad \text{for every } N. \quad (30)$$

Proof. The eigenspaces of H are labelled by (k, branch) with energies $h_{\pm}$ from (15), all distinct. On such an eigenspace I^z acts as the scalar m and S_0^z as $\text{sgn} \cdot m/(2k+1)$ by (16), with $\text{sgn} = +1$ on the upper branch and -1 on the lower one, so A_{diag} acts as $\text{sgn} \cdot m/(2k+1)$ there; the off-diagonal Clebsch–Gordan elements connect different energies and are removed by the dephasing. On the same eigenspace,

$$\frac{J}{4h_+ + J} = \frac{J}{2Jk + J} = \frac{1}{2k + 1}, \quad \frac{J}{4h_- + J} = \frac{J}{-2J(k+1) + J} = -\frac{1}{2k + 1}, \quad (31)$$

so $J(4H + J)^{-1}$ acts as $\text{sgn}/(2k+1)$ and multiplying by I^z reproduces A_{diag} eigenspace by eigenspace. The eigenvalues of $4H + J$ are $\pm J(2k+1) \neq 0$, so the inverse exists. For the second claim, C^* is a function of H and of the conserved I^z and is therefore conserved, and $\langle C^* A \rangle = \langle A_{\text{diag}} A \rangle = S_{\infty}$ together with $\langle (C^*)^2 \rangle = \langle A_{\text{diag}}^2 \rangle = S_{\infty}$ from (9) gives (30). $\square$

Equation (29) is the answer to the tightness question, and it is worth being precise about what kind of answer it is. The operator A_{diag} saturates (10) for any model, since it is by construction the projection of A onto the algebra of conserved quantities; the content of Theorem 4 is that in the uniform sector this projection is a *closed-form rational function of H and I^z* rather than an object defined through the spectral resolution, so that the saturating operator is computable without diagonalizing anything. That is exactly the completeness issue discussed in general terms in [8]: the bound is saturated by the right conserved quantity, and the question is whether that quantity lies in a tractable class.

4.1 The interpolant and the minimal saturating degree

Since H has $N+1$ distinct eigenvalues, the function $J/(4h+J)$ is reproduced on the spectrum by a polynomial of degree at most N , so $C^* = I^z p_N(H)$ lies in the span of $\{I^z, I^z H, \dots, I^z H^N\}$. The polynomial is explicit. Introduce the interpolation variable

$$y = \frac{4H + J}{J}, \quad \text{so that} \quad C^* = I^z \frac{1}{y}, \quad (32)$$

and let y_i be the $N+1$ distinct eigenvalues of y . They form a step-2 arithmetic progression with exactly one point removed,

$$\begin{aligned} \text{even } N : & \text{ the odd integers } -(N+1), \dots, N+1 \text{ with } y = -1 \text{ removed,} \\ \text{odd } N : & \text{ the even integers } -(N+1), \dots, N+1 \text{ with } y = 0 \text{ removed,} \end{aligned} \quad (33)$$

and the removed point is never an eigenvalue, which is a second proof of the invertibility of $4H + J$. With $W_N(y) = \prod_i (1 - y/y_i)$ the interpolant of $1/y$ on these nodes is unique and closed,

$$p_N(y) = \frac{1 - W_N(y)}{y}, \quad W_N(y) = \begin{cases} (1-y) \prod_{j=1}^{N/2} \left(1 - \frac{y^2}{(2j+1)^2}\right), & N \text{ even,} \\ \prod_{j=1}^{(N+1)/2} \left(1 - \frac{y^2}{4j^2}\right), & N \text{ odd,} \end{cases} \quad (34)$$

so p_N has degree N , is an odd polynomial in y at odd N and a general one at even N , and its leading coefficient is $(-1)^{N/2}/((N+1)!!)^2$ at even N and $(-1)^{(N-1)/2}/(2^{N+1}(((N+1)/2)!)^2)$ at odd N . The coefficients in the variable y are bounded, $|c_a| \leq 1$, and converge coefficient-wise as $N \rightarrow \infty$; what grows super-factorially is the exact common denominator $\prod_i y_i$, whose decimal logarithm is 6.0, 201.9 and 378.3 at $N = 8, 120, 200$. The reason to prefer the compact form (29) over the monomial basis is therefore conditioning, not magnitude: the Gram matrix of (10) on the monomials $\{I^z H^a\}$ is a Hankel moment matrix and is exponentially ill-conditioned.

Theorem 5 (minimal saturating degree). *Let*

$$d = \begin{cases} N, & N \text{ even}, \\ N-1, & N \text{ odd}. \end{cases} \quad (35)$$

Then $\{I^z H^a\}_{a=0}^d$ is a set of $d+1$ mutually commuting conserved operators built from I^z and H alone whose generalized Mazur bound equals $S_\infty(N)$ exactly, and no set $\{I^z H^a\}_{a=0}^{d'}$ with $d' < d$ saturates the bound.

Proof. Saturation follows from Theorem 4 once $C^* = I^z p_N(H)$ is shown to lie in the span of $\{I^z H^a\}_{a \leq d}$, and commutativity is clear since all entries are functions of the commuting pair (I^z, H) . For the span and for minimality, let $T_y = \text{Tr}(\Pi_y \hat{I}^z)$ be the Mazur weight of the y -eigenspace, where $\hat{I}^z = 2I^z$, and let $C_N = \{y_i : T_{y_i} \neq 0\}$ be the set of nodes that carry weight. Because I^z and H commute, $I^z r(H) = I^z p(H)$ if and only if $r = p$ on C_N , and the Mazur bound depends on the interpolant only through its values on C_N ; hence the minimal saturating degree is exactly $|C_N| - 1$. From (16) the weight of an eigenspace is proportional to $g(N, k) j(j+1)(2j+1)$ with $j = k \pm \frac{1}{2}$, so $|C_N| = N+1$ at even N , while at odd N exactly one eigenspace carries $I^z \equiv 0$: for $k = \frac{1}{2}$ the lower branch has $j = k - \frac{1}{2} = 0$, so its whole eigenspace, at $y = -2$, has $T_{-2} = 0$. Thus $|C_N| = N$ at odd N , which is (35). The degree is exactly $|C_N| - 1$ and not less because the top divided difference of $1/y$ on C_N is $(-1)^{|C_N|-1} / \prod_{C_N} y_i \neq 0$, so the interpolant on $|C_N|$ nodes has a nonzero leading coefficient. $\square$

The node sets are small enough to list, and they display the parity mechanism directly:

$$\begin{aligned} N=1 &: \{-2, 2\}, & N=2 &: \{-3, 1, 3\}, & N=3 &: \{-4, -2, 2, 4\}, \\ N=4 &: \{-5, -3, 1, 3, 5\}, & N=5 &: \{-6, -4, -2, 2, 4, 6\}, & N=6 &: \{-7, -5, -3, 1, 3, 5, 7\}, \end{aligned} \quad (36)$$

in every case a step-2 progression on $-(N+1), \dots, N+1$ with one point removed, and $y=0$ is never a node. At even N the removed point is $y=-1$ and the node at $y=-2$ survives; at odd N it is $y=0$ and $y=-2$ survives as a node of W_N while carrying no weight.

Proposition 6 (the odd- N linear dependence). *At odd N the $N+1$ operators $\{I^z H^a\}_{a=0}^N$ are linearly dependent, and their Gram matrix in (10) is exactly singular. The single relation is*

$$I^z q_N^*(y(H)) = 0, \quad q_N^*(y) = \prod_{y_i \in C_N} (y - y_i), \quad (37)$$

a monic polynomial of degree N . At even N the set is independent and $\det G \neq 0$ exactly. On the reduced set $\{a \leq d\}$ the Gram determinant is nonzero at both parities.

Proof sketch. The operator I^z vanishes identically on the eigenspace at $y = -2$, so the monic polynomial vanishing on C_N annihilates I^z after multiplication; this is an exact integer matrix identity in the product basis. Since $G_{ab} = 2^{-(N+3)} \mu_{a+b}$ with $\mu_r = \sum_y T_y y^r$ a moment sequence, the Gram determinant factorizes as $\det(\sum_y T_y y^{a+b}) = (\prod_i T_i) V^2$ with $V = \prod_{i < j} (y_j - y_i)$ a Vandermonde product, so it vanishes exactly when some $T_i = 0$, that is, exactly at odd N . $\square$

This corrects an earlier form of the tight-set statement, which quoted $\{I^z H^a\}_{a=0}^N$ as the tight set at every N ; that set is dependent at odd N and its Gram matrix is singular there, so S_{low} is

not defined on it as written [33]. The value of the bound is unaffected, because the redundant direction is an operator that is identically zero and carries no Mazur weight. The two equivalent repairs are the parity-dependent top degree (35) used here and the pseudoinverse in (10).

Remark (Casimir form). The identity $(4H + J)^2 = J^2(1 + 4\mathbf{I}^2)$, both sides diagonal on the joint (k, branch) basis with value $J^2(2k + 1)^2$, rewrites (29) as

$$C^* = I^z \frac{\text{sgn}^+(H)}{\sqrt{1 + 4\mathbf{I}^2}}, \quad \text{sgn}^+(x) = \begin{cases} +1, & x \geq 0, \\ 0, & x < 0, \end{cases} \quad (38)$$

in which the bath Casimir $\mathbf{I}^2$ appears instead of H and the factor J cancels. The two forms are the same operator; (29) is the one that is rational in H and therefore the one that yields Theorem 5.

4.2 A parity selection rule for conserved sets

The shape of the tight set is not a choice. Let $U = \prod_{q=0}^N (2S_q^x)$ be the global π -rotation about x , so that $US^xU = S^x$, $US^yU = -S^y$ and $US^zU = -S^z$.

Theorem 7 (parity selection rule). *For every N , every coupling vector, every polynomial M in $\{H, X_l\}$ and every integer $q \geq 0$,*

$$((I^z)^q M | A) = 0 \quad \text{whenever } q \text{ is even}, \quad (39)$$

where the X_l are the uniform-limit Gaudin operators of (76) below. In particular every conserved operator that carries any of the plateau is U -odd.

Proof. Every scalar product $\mathbf{S}_a \cdot \mathbf{S}_b$ is U -even, hence so are H , the X_l and every polynomial M in them, while I^z and A are U -odd. For $C = (I^z)^q M$ one has $UCU = (-1)^q C$ and $UAU = -A$, and the trace is cyclic, so

$$(C|A) = \text{Tr}[\rho CA] = \text{Tr}[\rho (UCU)(UAU)] = (-1)^{q+1} (C|A), \quad (40)$$

which forces $(C|A) = 0$ for even q . $\square$

Three consequences matter for the search problem. Every conserved operator that can carry any of the plateau must be U -odd, and since the natural conserved densities are U -even, the only way to build one is to multiply by an odd power of I^z ; the single factor of I^z in the tight set $\{I^z H^a\}$ and in the candidate extensions of Section 7 is forced, not chosen. A search for a saturating set can therefore be restricted to odd powers of I^z with no loss, and any ansatz containing $(I^z)^{2r} M$ can be deleted before a single Gram matrix is computed. The rule is exact, independent of N and of the couplings.

5 The canonical pair and the semiclassical estimate

The second question of the introduction asks whether the semiclassical prescription (2) is quantitatively reliable at uniform coupling. This section answers it exactly. The traces that enter the bound are computed as rational numbers, the bound and the prescription are evaluated in closed form, and their difference from the plateau is decomposed into two non-negative pieces whose sizes are then known for every N .

5.1 Exact traces

Only the diagonal part of $H = \sum_k [S_0^z S_k^z + (\frac{1}{2})(S_0^+ S_k^- + S_0^- S_k^+)]$ contributes to a trace against a diagonal operator, and any product with a single spin component at one site has zero trace. Writing

$n_{11} = (I^z|I^z)$, $n_{12} = (I^z|I^z H)$, $n_{22} = (I^z H|I^z H)$, $a_1 = (I^z|A)$, $a_2 = (I^z H|A)$, $b_1 = (I^z|B^z)$ and $b_2 = (I^z H|B^z)$ at $J = 1$,

$$n_{11} = \frac{N+1}{4}, \quad n_{12} = \frac{N}{8}, \quad n_{22} = \frac{N(5N-3)}{64}, \quad a_1 = \frac{1}{4}, \quad a_2 = \frac{N}{16}, \quad b_1 = \frac{N}{4}, \quad b_2 = \frac{N}{16}. \quad (41)$$

The last of these uses $B^z = Z = I^z - S_0^z$ at $J = 1$, so that $b_1 = n_{11} - a_1 = N/4$ and $b_2 = n_{12} - a_2 = N/16$. The only non-obvious entry is n_{22} , and it follows from an exact operator identity worth stating on its own because it returns in Section 7:

$$H^2 = \frac{\mathbf{I}^2}{4} - \frac{H}{2}. \quad (42)$$

Indeed $(\mathbf{S}_0 \cdot \mathbf{S}_k)^2 = \frac{3}{16} - \frac{1}{2}\mathbf{S}_0 \cdot \mathbf{S}_k$, both sides agreeing on the two eigenvalues $-3/4$ and $1/4$ of $\mathbf{S}_0 \cdot \mathbf{S}_k$, while for $k \neq l$ $(\mathbf{S}_0 \cdot \mathbf{S}_k)(\mathbf{S}_0 \cdot \mathbf{S}_l) = \frac{1}{4}\mathbf{S}_k \cdot \mathbf{S}_l - \frac{1}{2}\mathbf{S}_0 \cdot (\mathbf{S}_k \times \mathbf{S}_l)$, and $\sum_{k \neq l} \mathbf{S}_k \times \mathbf{S}_l = \mathbf{I} \times \mathbf{I} - \sum_k \mathbf{S}_k \times \mathbf{S}_k = \mathbf{iI} - \mathbf{iI} = 0$. The remaining $\frac{1}{4}\sum_{k \neq l} \mathbf{S}_k \cdot \mathbf{S}_l = \mathbf{I}^2/4 - 3N/16$ cancels $\sum_k 3/16$. In the product basis $\mathbf{I}^2$ is diagonal with $Z^2 + N/2$, and with $\langle Z^2 \rangle = N/4$ and $\langle Z^4 \rangle = (3N^2 - 2N)/16$ this gives $\langle \mathbf{I}^2 \rangle = 3N/4$, $\langle Z^2 \mathbf{I}^2 \rangle = (5N^2 - 2N)/16$, $\langle (I^z)^2 \mathbf{I}^2 \rangle = N(5N+1)/16$ and hence $n_{22} = \frac{1}{4}N(5N+1)/16 - \frac{1}{2} \cdot \frac{N}{8} = N(5N-3)/64$.

Proposition 8 (canonical bound and estimate). *For every $N \geq 1$ the Gram determinant and the two Mazur numerators of the canonical pair (13) factor completely,*

$$\det G = \frac{N(N-1)(5N+3)}{256}, \quad \text{num}_A = \frac{N(N-1)(N+3)}{1024}, \quad \text{num}_B = \frac{N^2(N-1)(5N-1)}{1024}, \quad (43)$$

and therefore

$$S_{\text{low}}(N) = \frac{N+3}{4(5N+3)}, \quad S_{\text{low}}^B(N) = \frac{N(5N-1)}{4(5N+3)} J^2, \quad \hat{S}(N) = \frac{5N-1}{12(5N+3)}. \quad (44)$$

In the limit

$$\frac{S_{\text{low}}}{S_{\infty}} \longrightarrow \frac{3}{5}, \quad S_{\infty} - S_{\text{low}} \longrightarrow \frac{1}{30}, \quad \hat{S} - \frac{1}{12} = -\frac{1}{3(5N+3)} \text{ exactly for every } N. \quad (45)$$

Proof. Substituting (41) into $\det G = n_{11}n_{22} - n_{12}^2$ and into $\text{num}_A = a_1^2 n_{22} - 2a_1 a_2 n_{12} + a_2^2 n_{11}$, $\text{num}_B = b_1^2 n_{22} - 2b_1 b_2 n_{12} + b_2^2 n_{11}$ gives (43) as polynomial identities in N , and (44) is their ratio. The limits follow from (18) and (44) by expansion; the last identity in (45) is exact because $(5N-1)/(12(5N+3)) - 1/12 = -4/(12(5N+3))$. $\square$

At $N = 1$ the factor $(N-1)$ makes $\det G$ vanish: the Gram matrix is exactly singular, the pair collapses to the single operator I^z , and the closed forms still give $S_{\text{low}}(1) = 1/8 = S_{\infty}(1)$. For $N \geq 2$ the pair has rank two and $S_{\text{low}} < S_{\infty}$ strictly.

The content of (45) is negative and it is not a finite-size effect. The canonical pair, which is what the prescribed conserved set becomes at uniform coupling, misses $2/5$ of the plateau asymptotically, with a deficit that tends to the constant $1/30$. What is missing is exactly the higher powers of H that Theorem 5 shows to be necessary.

5.2 Two structural identities and the exact error budget

Proposition 9 (structural identities). *For every $N \geq 1$ and any conserved set $\mathcal{C}$ containing I^z ,*

$$S_{\text{low}}^B(\mathcal{C}) = \frac{N-1}{4} J^2 + S_{\text{low}}(\mathcal{C}), \quad (46)$$

and for the exact dephasing,

$$S_{\infty}^B = \frac{N-1}{4} J^2 + S_{\infty}, \quad \frac{S_{\infty}^B}{S^B(0)} = 1 - \frac{1}{N} + \frac{4S_{\infty}}{N} \longrightarrow 1. \quad (47)$$

Consequently the estimate decomposes exactly as

$$\hat{S} = \frac{1}{12} \left(1 - \frac{1}{N}\right) + \frac{S_{\text{low}}}{3N} = \frac{5N - 1}{12(5N + 3)}. \quad (48)$$

Proof. Since $B^z = J(I^z - S_0^z)$ and $P_C I^z = I^z$ whenever $I^z \in \mathcal{C}$, the Mazur-optimal approximation of B^z is $J(I^z - P_C S_0^z)$, and

$$\langle (I^z - P_C S_0^z)^2 \rangle = n_{11} - 2\langle I^z P_C S_0^z \rangle + \langle (P_C S_0^z)^2 \rangle = n_{11} - 2a_1 + S_{\text{low}}(\mathcal{C}), \quad (49)$$

using $\langle I^z P_C S_0^z \rangle = \langle P_C I^z S_0^z \rangle = \langle I^z S_0^z \rangle = a_1$ and $n_{11} - 2a_1 = (N - 1)/4$. The same computation with P_C replaced by the exact dephasing $\sum_E \Pi_E(\cdot) \Pi_E$ gives (47), and dividing (46) by $12S^B(0) = 3NJ^2$ gives (48). $\square$

Identity (47) says that at uniform coupling the Overhauser-field autocorrelation is asymptotically entirely non-decaying, and it needs no semiclassical assumption: B^z differs from the exactly conserved I^z by the single central spin, which is a $1/N$ effect. This is where the uniform sector differs qualitatively from the inhomogeneous one, where the persisting fraction of the field's own correlation depends on how much of the bath is very weakly coupled [35].

Define now the same estimator fed with the exact quantum plateau of the field instead of its bound,

$$\hat{S}_{\text{ex}} := \frac{S_{\infty}^B}{12S^B(0)} = \frac{(N - 1)/4 + S_{\infty}}{3N}, \quad (50)$$

so that the total error splits exactly into a part caused by the semiclassical step and a part caused by the looseness of the bound,

$$S_{\infty} - \hat{S} = \underbrace{(S_{\infty} - \hat{S}_{\text{ex}})}_{E_{\text{fr}}} + \underbrace{(\hat{S}_{\text{ex}} - \hat{S})}_{E_{\text{Mz}}}, \quad E_{\text{Mz}} = \frac{S_{\infty} - S_{\text{low}}}{3N}, \quad E_{\text{fr}} = S_{\infty} \left(1 - \frac{1}{3N}\right) - \frac{N - 1}{12N}. \quad (51)$$

Theorem 10 (exact error budget). *For even N ,*

$$E_{\text{fr}} = \frac{2}{9(N + 1)}, \quad E_{\text{Mz}} = \frac{N + 3}{18(N + 1)(5N + 3)}, \quad S_{\infty} - \hat{S} = \frac{7N + 5}{6(N + 1)(5N + 3)}, \quad (52)$$

and the relative error is

$$\frac{S_{\infty} - \hat{S}}{S_{\infty}} = \frac{14N + 10}{5N^2 + 18N + 9} = \frac{14}{5N} + O(N^{-2}), \quad \frac{E_{\text{fr}}}{S_{\infty} - \hat{S}} \rightarrow \frac{20}{21}, \quad \frac{E_{\text{Mz}}}{S_{\infty} - \hat{S}} \rightarrow \frac{1}{21}. \quad (53)$$

For every $N \geq 1$, both parities included,

$$\frac{S_{\infty} - \hat{S}}{S_{\infty}} = \frac{14N + 10 - 2q_N(5N + 3)}{(5N + 3)(N + 3 - 2q_N)} = \frac{14}{5N} - \frac{2q_N}{N} + O(N^{-2}). \quad (54)$$

Proof. Substituting (18) with $q_N = 0$ and (44) into (51) gives (52) as rational identities in N , and dividing by $S_{\infty} = (N + 3)/(12(N + 1))$ gives (53). For odd N each of E_{fr} , E_{Mz} and their sum is reduced by the parity term times the appropriate factor, and collecting those factors produces (54); expanding at large N with $q_N = O(N^{-1/2})$ gives the second form. $\square$

Equation (53) is an even- N identity, and quoting it as an all- N one is wrong in a way that matters: because $q_N \sim \sqrt{2/(\pi N)}$ is a power law rather than an exponential, the parity term enters at order $N^{-3/2}$, which is larger than the $O(N^{-2})$ remainder it is usually absorbed into. At $N = 1$ the exact value is $2/3$ while the parity-free form would give $3/4$. The leading $1/N$ coefficient, the ordering $\hat{S} < S_{\infty}$ and the $20/21 : 1/21$ split are unaffected. The correction, and the exact all- N form (54), were supplied in [32] and are re-derived here from (41).

Table 1: Uniform-sector quantities as exact rational numbers, with the decimal value of the last three columns. S_∞ is the plateau of Theorem 1, S_{low} the canonical-pair bound of Proposition 8, $\hat{S}$ the estimate (2), and the frozen-field share is $E_{\text{fr}}/(S_\infty - \hat{S})$ from (51), whose limit is $20/21 = 0.952381$. At $N = 1$ the canonical pair has rank one and the share is exactly 1. The $N = 10^6$ row is decimal only; every other entry is exact.

N	S_∞	S_{low}	$\hat{S}$	$S_\infty - \hat{S}$	$(S_\infty - \hat{S})/S_\infty$	$E_{\text{fr}}/(S_\infty - \hat{S})$
1	1/8	1/8	1/24	0.08333333	0.66666667	1.00000000
2	5/36	5/52	3/52	0.08119658	0.58461538	0.91228070
3	7/64	1/12	7/108	0.04456019	0.40740741	0.93506494
4	7/60	7/92	19/276	0.04782609	0.40993789	0.92929293
5	59/576	1/14	1/14	0.03100198	0.30266344	0.93333333
6	3/28	3/44	29/396	0.03391053	0.31649832	0.93617021
7	605/6144	5/76	17/228	0.02390865	0.24280122	0.93490983
8	11/108	11/172	13/172	0.02627046	0.25792812	0.93989071
32	35/396	35/652	53/652	0.00709549	0.08028046	0.94905386
128	131/1548	131/2572	213/2572	0.00181039	0.02139304	0.95153533
10^6	0.08333350	0.05000012	0.08333327	0.00000023	0.00000280	0.95238084

The interpretation of (53) is the sharpest statement of this section. The estimator is asymptotically exact in the uniform sector, with a relative error of $14/(5N)$, and about 95 per cent of that error is the semiclassical step $\hat{S}_{\text{ex}} \rightarrow S_\infty$ rather than the looseness of the conserved set fed into it. Accuracy and tightness have come apart: the canonical pair captures only 3/5 of the plateau, and the estimate built on it is nevertheless correct to $O(1/N)$, because the quantity being estimated, S_∞^B , is itself almost entirely conserved by (47). In the uniform sector the accuracy of the prescription is driven entirely by the fact that B^z lies within $1/N$ of the exactly conserved I^z , and not at all by how much of the plateau the conserved set captures.

5.3 A three-level hierarchy of uniform estimates

The prescription (2) is defined only once a conserved set is fixed, and at uniform coupling there are three natural choices, all of them exactly solvable:

$$\begin{array}{lll}
\text{level 1} & \{I^z\} & S_{\text{low}} = \frac{1}{4(N+1)} \quad \hat{S}_1 = \frac{N}{12(N+1)}, \\
\text{level 2} & \{I^z, I^z H\} & S_{\text{low}} = \frac{N+3}{4(5N+3)} \quad \hat{S}_2 = \frac{5N-1}{12(5N+3)}, \\
\text{level 3} & \text{projection} = \text{exact dephasing} & S_{\text{low}} = S_\infty \quad \hat{S}_3 = \frac{3N+1}{36(N+1)} - \frac{q_N}{18N(N+1)}.
\end{array} \tag{55}$$

Level 2 is the canonical pair of Section 5 and is what the prescription of [38] reduces to here. Level 3 is what any set whose projection equals the exact dephasing gives, for instance the joint spectral projectors of the commuting family $\{H, I^z, \mathbf{I}^2\}$, or the tight operator C^* of Theorem 4; there $\hat{S}_3 = \hat{S}_{\text{ex}}$ of (50). Level 1 is included because it is what a single-quantity bound gives, and because its plateau bound $1/(4(N+1))$ *vanishes* as $N \rightarrow \infty$ while its estimate still converges to S_∞ at rate $3/N$.

Proposition 11 (ordering, exact gaps, and the three relative errors). *For every $N \geq 1$,*

$$\hat{S}_1 \leq \hat{S}_2 \leq \hat{S}_3 \leq S_\infty, \tag{56}$$

with exact gaps

$$\hat{S}_2 - \hat{S}_1 = \frac{N-1}{12(N+1)(5N+3)}, \quad \hat{S}_3 - \hat{S}_2 = E_{\text{Mz}}, \quad S_\infty - \hat{S}_3 = E_{\text{fr}}, \tag{57}$$

Table 2: The three levels of (55) for small bath sizes, as exact rational numbers. $\hat{S}_3$ is the estimate obtained from a conserved set whose projection is the exact dephasing, and the last three columns are the relative errors (58); their limits $NE_i \rightarrow c_i$ are 3, 14/5 and 8/3 respectively. At $N = 1$ all three levels coincide because the canonical pair has rank one there.

N	$\hat{S}_1$	$\hat{S}_2$	$\hat{S}_3$	S_∞	E_1	E_2	E_3
1	1/24	1/24	1/24	1/8	2/3	2/3	2/3
2	1/18	3/52	7/108	5/36	3/5	38/65	8/15
3	1/16	7/108	13/192	7/64	3/7	11/27	8/21
4	1/15	19/276	13/180	7/60	3/7	66/161	8/21
5	5/72	1/14	127/1728	59/576	19/59	125/413	50/177
6	1/14	29/396	19/252	3/28	1/3	94/297	8/27
7	7/96	17/228	1403/18432	605/6144	157/605	2791/11495	412/1815
8	2/27	13/172	25/324	11/108	3/11	122/473	8/33

and exact relative errors, as rational functions of q_N ,

$$E_1 = \frac{3 - 2q_N}{N + 3 - 2q_N}, \quad E_2 = \frac{14N + 10 - 2q_N(5N + 3)}{(5N + 3)(N + 3 - 2q_N)}, \quad E_3 = \frac{2[4N - q_N(3N - 1)]}{3N(N + 3 - 2q_N)}, \quad (58)$$

where $E_i = (S_\infty - \hat{S}_i)/S_\infty$. Each satisfies $E_i = c_i/N - 2q_N/N + O(N^{-2})$ with

$$c_1 = 3, \quad c_2 = \frac{14}{5}, \quad c_3 = \frac{8}{3}. \quad (59)$$

Proof. The ordering is the monotonicity of (10) in the span together with $\hat{S}_3 \leq S_\infty$ from $E_{\text{fr}} \geq 0$; the gaps are the definitions (51) together with the identity $\hat{S}_3 = ((N - 1)/4 + S_\infty)/(3N)$. Dividing by $S_\infty = (N + 3 - 2q_N)/(12(N + 1))$ gives (58), and expanding each at large N with $q_N = O(N^{-1/2})$ gives (59). $\square$

All three levels share the same parity subleading term $-2q_N/N$, and the reason is structural rather than a coincidence of coefficients. At levels 1 and 2 the estimates are parity-free, so writing $S_\infty = S_e(1 - \delta)$ with $S_e = (N + 3)/(12(N + 1))$ and $\delta = 2q_N/(N + 3)$ gives $E_i = E_{i,e} - (1 - E_{i,e})\delta + O(\delta^2)$, and since $E_{i,e} \rightarrow 0$ while $\delta = 2q_N/N + O(q_N/N^2)$ the shift is $-2q_N/N$ for any parity-free estimate. At level 3 the parity part of $\hat{S}_3$ enters as well, and the same coefficient results. Enriching the conserved set therefore changes the $1/N$ coefficient, from 3 to 14/5 to 8/3, but never the parity correction. Table 2 lists the three levels together with the plateau and the three relative errors as exact rationals for $N \leq 8$, including $N = 1$, where the canonical pair has rank one and all three levels collapse to the single estimate 1/24. The parity-free forms are strictly too large at every odd $N \geq 3$ for all three levels. Levels 1 and 3 agree with the values obtained independently in [32], which is a consistency check of three routes rather than the source of either claim. Figure 2 plots the four uniform quantities and the three relative errors against N up to 40: the ordering (56) appears as the stacking of the estimates, the common $1/N$ envelope as the parallel descent of the error curves, and the parity term as the splitting of those curves into two interleaved branches.

6 The parity term: optimal bounds and rigorous finite- N brackets

Everything in Sections 3–5 is an exact rational number once q_N is known, and q_N is a central binomial coefficient. Its asymptotics $q_N \sim \sqrt{2/(\pi N)}$ are classical, but an asymptotic is not a bound, and the error budgets above are only as rigorous as the control on q_N . This section replaces the asymptotic by a proved two-sided bracket with optimal constants and pushes it through to

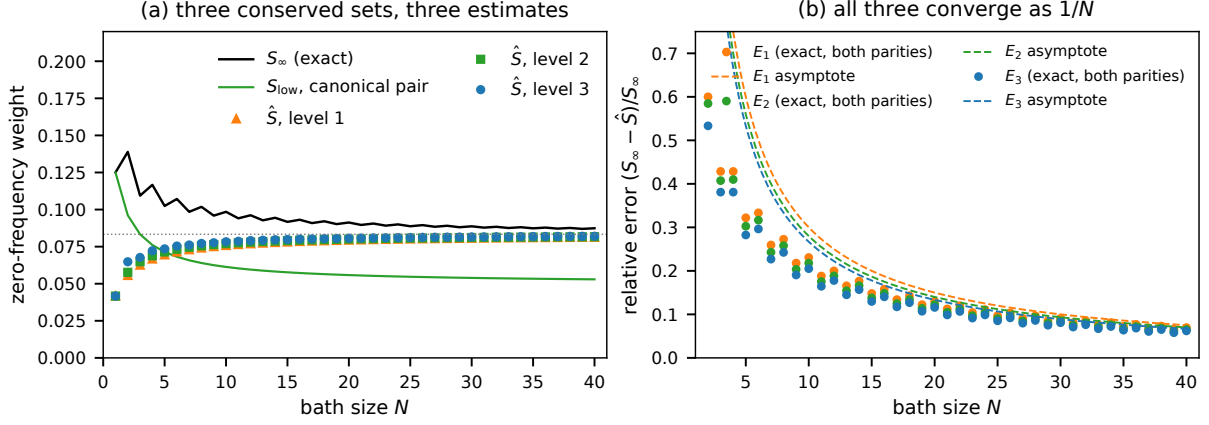


Figure 2: The three-level hierarchy of uniform estimates. (a) Zero-frequency weights against bath size N for $N \leq 40$: the exact plateau S_∞ from (18), the three estimates $\hat{S}_1, \hat{S}_2, \hat{S}_3$ from (55), and the canonical-pair plateau bound S_{low} from (44), which is the only curve that falls towards zero rather than towards $1/12$. The ordering (56) is visible as the stacking of the three estimates below S_∞ . (b) The same data as the relative errors $E_i = (S_\infty - \hat{S}_i)/S_\infty$ of (58), evaluated exactly at both parities, together with their asymptotes c_i/N for $c_i = 3, 14/5, 8/3$; the parity splitting of the points is the shared $-2q_N/N$ term of Proposition 11. All entries are exact rational numbers on an integer grid.

the relative errors. Bounds on the Wallis sequence of this shape have a long history [18, 22]; the argument below uses none of it and rests only on the Wallis limit and three polynomial identities.

Write $q = (N + 1)/2$, $b_q = \binom{2q}{q}/4^q$ and

$$g(q) = \frac{1}{b_q^2} = \frac{16^q}{\binom{2q}{q}^2}, \quad (60)$$

an exact rational for every integer $q \geq 1$. Lemma 2 says $q_N = b_q$ at odd N , and Wallis says $g(q)/q \rightarrow \pi$.

Theorem 12 (optimal two-sided bracket). *For every integer $q \geq 1$,*

$$\frac{1}{\sqrt{\pi(q-1)+4}} \leq b_q < \frac{1}{\sqrt{\pi(q+\frac{1}{4})}}, \quad (61)$$

with equality on the left if and only if $q = 1$. Equivalently, for every odd $N \geq 1$ with $q = (N+1)/2$,

$$\sqrt{\frac{2}{\pi N + 8 - \pi}} \leq q_N < \sqrt{\frac{2}{\pi N + 3\pi/2}}, \quad (62)$$

and the relative width of the bracket is $(16-5\pi)/(4\pi N) = 0.0232395\dots/N$, so (62) is asymptotically sharp. Both additive constants are optimal: the left one cannot be increased and the right one cannot be decreased.

Proof. Three exact polynomial identities, each side differing by identically one, are the whole input:

$$(2q+1)^2(4q+5) - 4(q+1)^2(4q+1) = 1, \quad (63)$$

$$4(q+1)^2 - (2q+1)(2q+3) = 1, \quad (64)$$

$$4(q+1)^2(4q+7) - (2q+3)^2(4q+3) = 1, \quad (65)$$

together with the exact ratio $g(q+1)/g(q) = 4(q+1)^2/(2q+1)^2$. Dividing (63) by the positive product of the two terms shows that $A(q) := g(q)/(q + \frac{1}{4})$ is strictly decreasing; (64) gives the same for $B(q) := g(q)/(q + \frac{1}{2})$ strictly increasing; and (65) gives $\text{incr}(q) := g(q+1) - g(q) = g(q)(4q+3)/(2q+1)^2$ strictly increasing. All three sequences have limit π by Wallis, and a monotone sequence with limit π stays on one side of it, so $A > \pi > B$ and $\text{incr} < \pi$.

The right inequality of (61) is $A(q) > \pi$, that is $g(q) > \pi(q + \frac{1}{4})$. For the left one, $\text{incr}(q) < \pi$ says that $\delta(q) := g(q) - \pi q$ is strictly decreasing, so $\delta(q) \leq \delta(1) = 4 - \pi$ with equality only at $q = 1$; that is $g(q) \leq \pi(q-1) + 4$. Substituting $q = (N+1)/2$ gives (62), since $\pi(q-1) + 4 = (\pi N + 8 - \pi)/2$ and $\pi(q + \frac{1}{4}) = (\pi N + 3\pi/2)/2$, and the relative width follows from $q_{\text{hi}}/q_{\text{lo}} = \sqrt{1 + (8 - 5\pi/2)/(\pi N + 3\pi/2)}$. The left constant is attained at $q = 1$, where $b_1 = 1/2$ exactly, so it cannot be increased; the right one cannot be decreased because $A(q)$ decreases strictly to π , hence $\inf_q [g(q) - \pi(q + \frac{1}{4})] = 0$ and no smaller offset works for all q . The only analytic input is the Wallis limit. $\square$

Remark (certification without floating point). Both inequalities of (62) are certified exactly, with no floating point in the decision, for every odd $N \leq 2001$ by replacing π with a rigorous rational sandwich $\pi_{\text{lo}} < \pi < \pi_{\text{hi}}$ obtained from Machin's formula and using it in the direction that makes each implication valid,

$$g(q) \leq \pi_{\text{lo}}(q-1) + 4 \implies g(q) \leq \pi(q-1) + 4, \quad g(q) > \pi_{\text{hi}}(q + \frac{1}{4}) \implies g(q) > \pi(q + \frac{1}{4}), \quad (66)$$

with $g(q)$ an exact rational. The sandwich used has width 5.4×10^{-128} . The margins have exact limits: the lower one, $g(q) - \pi(q-1) - 4$, tends to $5\pi/4 - 4 = -0.0730092\dots$, while the upper one, $g(q) - \pi(q + \frac{1}{4})$, tends to zero from above like $\pi/(32q)$, that is relatively like $1/(32q^2)$. Beyond $N = 2001$ the check is numerical at 60 digits only, and there the upper endpoint exceeds the exact value by $(q_{\text{hi}} - q_N)/q_{\text{hi}} = 6.25 \times 10^{-14}$ at $N = 10^6 + 1$ and 6.25×10^{-22} at $N = 10^{10} + 1$, both reproduced here; the second is a direct numerical illustration of why $3\pi/2$ cannot be decreased.

6.1 Rigorous brackets on the three relative errors

The relative errors (58) are rational functions of $q \equiv q_N$ on the common denominator $N + 3 - 2q$, and each is monotone in q .

Lemma 13 (monotonicity in q). *Symbolically,*

$$\frac{dE_1}{dq} = -\frac{2N}{(N+3-2q)^2}, \quad \frac{dE_2}{dq} = -\frac{2(5N^2+4N-1)}{(5N+3)(N+3-2q)^2}, \quad \frac{dE_3}{dq} = \frac{2(1-N^2)}{N(N+3-2q)^2}. \quad (67)$$

Hence E_1 and E_2 are strictly decreasing in q for every $N \geq 1$, since $5N^2 + 4N - 1 = (5N - 1)(N + 1) > 0$, while E_3 is strictly decreasing for $N \geq 2$ and identically constant at $N = 1$, where $E_3 = 2/3$ for every q .

Theorem 14 (rigorous finite- N brackets). *For every odd $N \geq 1$, with $q_{\text{lo}} = \sqrt{2/(\pi N + 8 - \pi)}$ and $q_{\text{hi}} = \sqrt{2/(\pi N + 3\pi/2)}$ from (62),*

$$E_i(q_{\text{hi}}) < E_i(N) \leq E_i(q_{\text{lo}}) \quad \text{for } i = 1, 2 \text{ at every odd } N, \quad (68)$$

and the same for $i = 3$ at every odd $N \geq 3$. At $N = 1$ the level-3 bracket collapses to the single value $2/3$, and the right inequality is an equality for all three levels because $q_1 = 1/2$ attains the lower endpoint of (62) exactly. No numerics beyond (62) enter.

Proof. E_i is strictly decreasing in q by Lemma 13 and $q_{\text{lo}} \leq q_N < q_{\text{hi}}$ by Theorem 12, so $E_i(q_{\text{hi}}) < E_i(q_N) \leq E_i(q_{\text{lo}})$, with the two stated exceptions following from the constancy of E_3 at $N = 1$ and from equality in the left bound of (62) at $N = 1$. $\square$

Table 3: Widths $W_i = E_i(q_{lo}) - E_i(q_{hi})$ of the rigorous brackets (68) at three odd bath sizes, computed at 60 decimal digits from the proved endpoints of Theorem 12; the exact q_N lies strictly inside the endpoint pair at each size. The column headed ‘rel. width’ is the width of the bracket on q_N itself, $(q_{hi} - q_{lo})/q_{lo}$, and the last column is the asymptotic law $(16 - 5\pi)/(4\pi N)$ proved in Theorem 12. Fitting local exponents over $N = 99$ to 401 gives 2.4576 for the absolute widths W_i (expected 5/2) and 1.4924 for W_i/E_i (expected 3/2).

N	W_1	W_2	W_3	W_2/E_2	rel. width	law
3	4.1965×10^{-4}	4.3519×10^{-4}	3.7302×10^{-4}	1.068×10^{-3}	0.005151076480	0.007746514912
99	3.5123×10^{-7}	3.5193×10^{-7}	3.5119×10^{-7}	1.356×10^{-5}	0.000231212521	0.000234742876
401	1.1287×10^{-8}	1.1292×10^{-8}	1.1287×10^{-8}	1.676×10^{-6}	0.000057736333	0.000057953977

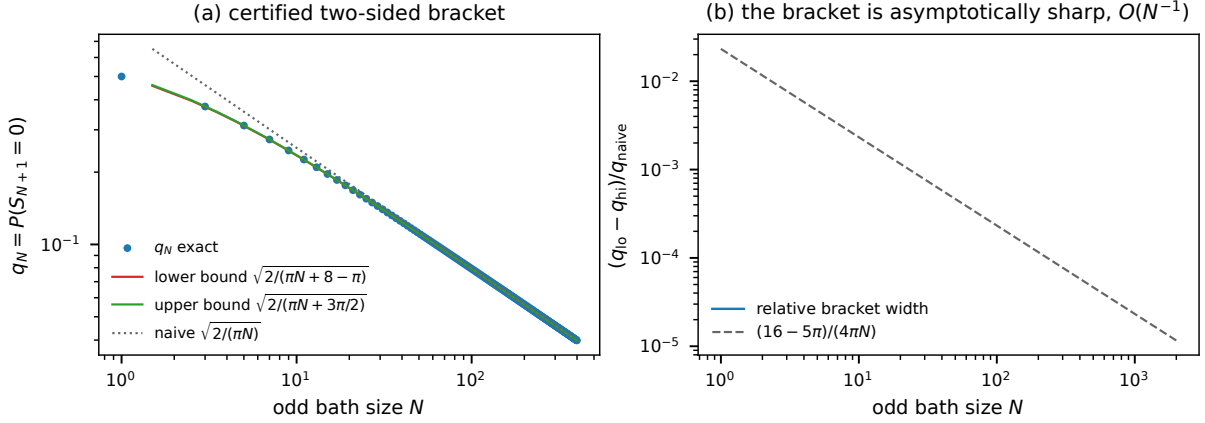


Figure 3: The parity term and its proved bracket. (a) q_N against odd bath size N on logarithmic axes, from $N = 1$ to 10^2 , with the two proved bounds of Theorem 12, $\sqrt{2/(\pi N + 8 - \pi)}$ below and $\sqrt{2/(\pi N + 3\pi/2)}$ above, and the naive asymptotic surrogate $\sqrt{2/(\pi N)}$ for comparison; the exact values sit inside the two bounds at every plotted size, and the surrogate is not a bound on either side. (b) The width of the same bracket relative to the naive asymptotic value, $|q_{lo} - q_{hi}|/\sqrt{2/(\pi N)}$, against odd N on logarithmic axes from 1 to 10^3 , with the proved asymptotic law $(16 - 5\pi)/(4\pi N)$; the two curves are indistinguishable beyond $N \approx 10^2$, which is the content of the sharpness statement in Theorem 12. Normalizing the width by q_{lo} instead, as Table 3 does, changes it only at relative order N^{-1} . The curves are bounds on exact rational numbers, not error bars on measurements.

The left inequality in (68) cannot be made non-strict uniformly in N : at $N = 1$, $i = 3$ it is an equality. The widths are small and fall fast. With $W_i(N) = E_i(q_{lo}) - E_i(q_{hi})$,

$$W_i(3) \approx (4.20, 4.35, 3.73) \times 10^{-4}, \quad W_i(99) \approx 3.51 \times 10^{-7}, \quad W_i(401) \approx 1.13 \times 10^{-8}, \quad (69)$$

and Table 3 gives them separately by level together with the width relative to E_i itself.

The two exponents are exact rather than fitted, and the reason is the point of the table. From (67) the three leading terms give $|dE_i/dq| \rightarrow 2/N$ for every level, while $q_{hi} - q_{lo} = \frac{16-5\pi}{4\pi} \sqrt{2/\pi} N^{-3/2} + O(N^{-5/2})$; the product is $O(N^{-5/2})$ absolute. Relative to $E_i = O(1/N)$ the width is therefore $O(N^{-3/2})$, which is exactly the order of the universal parity shift $-2q_N/N$ in (58). The bracket resolves each relative error precisely down to the order at which the parity term must be retained, and it converts the asymptotic budget of Section 5.2 into a rigorous one at every finite odd N . Figure 3 shows the exact q_N inside the two proved endpoints over four decades of N , and the relative width of the bracket against the law $(16 - 5\pi)/(4\pi N)$ that Theorem 12 gives it.

6.2 Consequences for the plateau asymptotics

The bracket also pins two coefficients that would otherwise be fitted. The Stirling expansion of the central binomial coefficient, with $b_q = (\pi q)^{-1/2}(1 - 1/(8q) + 1/(128q^2) + O(q^{-3}))$ and $q = (N + 1)/2$, gives

$$q_N = \sqrt{\frac{2}{\pi N}} \left(1 - \frac{3}{4N} + \frac{25}{32N^2} + O(N^{-3}) \right), \quad (70)$$

where the coefficient $3/4$ is precisely what the optimal upper constant $3\pi/2$ of Theorem 12 delivers and $25/32$ is the next one. Defining $s(N) = (1 - q_N/\sqrt{2/(\pi N)})N$, so that (70) reads $s(N) = 3/4 - 25/(32N) + O(N^{-2})$, three statements follow.

First, substituting the two proved endpoints of (62) into s , which is strictly decreasing in q_N , gives for every odd $N \geq 1$

$$f_{\text{lo}}(N) < s(N) \leq f_{\text{hi}}(N), \quad f_{\text{lo}}(N) = N \left(1 - \sqrt{\frac{N}{N+3/2}} \right), \quad f_{\text{hi}}(N) = N \left(1 - \sqrt{\frac{N}{N+(8-\pi)/\pi}} \right), \quad (71)$$

and both endpoints approach their limits $3/4$ and $(8 - \pi)/(2\pi) = 0.7732395 \dots$ from below. So the interval $[3/4, (8 - \pi)/(2\pi)]$ is the bracket of the two limits, not a finite- N bracket on $s(N)$; at every finite odd N the value of s lies strictly below both, and a claim to the contrary in [33] is false at the lower end, as its own tabulated values 0.5578, 0.6723, 0.7422, 0.74922 and 0.7499992 at $N = 3, 9, 99, 999, 10^6 + 1$ show; all five were recomputed here, exactly from (27) up to $N = 999$ and at 60 digits beyond.

Second, the strict inequality $s(N) < 3/4$ holds for every odd $N \geq 1$ and is certified exactly on the same range as Remark 6: squaring, which is legitimate since both sides are positive for $N \geq 1$, it is equivalent to

$$g(q) < \frac{8\pi(2q-1)^3}{(8q-7)^2}, \quad q = \frac{N+1}{2}, \quad (72)$$

and the gap is positive at every $q \geq 1$ but tends to zero: the Stirling expansion gives $g(q) = \pi q + \pi/4 + \pi/(32q) + O(q^{-2})$ while a long division of the right-hand side gives

$$\frac{8\pi(2q-1)^3}{(8q-7)^2} = \pi q + \frac{\pi}{4} + \frac{27\pi}{64q} + O(q^{-2}), \quad (73)$$

so that the gap equals $25\pi/(64q) + O(q^{-2}) = 25\pi/(32(N+1)) + O(N^{-2})$. A parenthetical in the source record asserts that the limit of this gap is $\pi/2$; that is contradicted both by the margins tabulated in the same record, which decay from -2.1×10^1 at $N = 1$ to -6.1×10^{-4} at $N = 4001$, and by the recomputation of (73), which reproduces the measured gap to four significant digits at $q = 10^4, \dots, 10^7$. Nothing in the argument depends on the gap having a positive limit: what is needed is that it stays positive, which it does, exactly for all odd $N \leq 2001$ and at 60 digits beyond.

Third, the next coefficient is exact: $(3/4 - s(N))N \rightarrow 25/32 = 0.78125$, approached from below with values 0.7730458891, 0.7811679686, 0.7812491797 and 0.7812499992 at $N = 99, 9999, 10^6 + 1$ and $10^9 + 1$, each reproduced here to the last digit quoted.

The same substitution pins the half-power coefficient of the plateau itself. Writing (27) as

$$S_\infty(N) - \frac{1}{12} = \frac{1}{6(N+1)} - \frac{\kappa_q}{3\sqrt{2\pi}} (N+1)^{-3/2}, \quad \kappa_q := b_q \sqrt{\pi q} \longrightarrow 1, \quad (74)$$

the coefficient $c = 1/(3\sqrt{2\pi}) = 0.13298076013381088 \dots$ is exact rather than fitted, and Theorem 12 transfers verbatim into a bracket on κ_q ,

$$\sqrt{\frac{q}{q+4/\pi-1}} < \kappa_q \leq \sqrt{\frac{q}{q+\frac{1}{4}}}, \quad \text{relative error bar } \frac{4/\pi-5/4}{2q} = \frac{0.0232}{N+1}. \quad (75)$$

Table 4: The Gaudin-limit span V_0 of (77) at uniform coupling, as exact rational numbers computed by two independent exact engines: dense rational matrices in the full 2^{N+1} -dimensional product space for $N \leq 4$, and an integer sector-decomposed engine for $N \leq 8$. The deficit is $S_\infty - S_{\text{low}}(A; V_0)$ with S_∞ from Theorem 1. The span saturates the bound exactly at $N = 2$ and $N = 3$ and fails from $N = 4$ onwards.

N	operators	rank	$S_{\text{low}}(A; V_0)$	S_∞	deficit	tight?
2	4	3	5/36	5/36	0	yes
3	5	4	7/64	7/64	0	yes
4	6	5	5/52	7/60	4/195	no
5	7	6	19/216	59/576	25/1728	no
6	8	7	53/644	3/28	4/161	no
7	9	8	5/64	605/6144	125/6144	no
8	10	9	89/1188	11/108	8/297	no

At even N one has $q_N = 0$, $\kappa_q = 0$, and (74) reduces to the exact identity $S_\infty - 1/12 = 1/(6(N+1))$ with no remainder. The content of (74) is therefore entirely the odd- N parity correction, and it upgrades the $O(N^{-3/2})$ remainder in the estimator budget $S_\infty - \hat{S} = 7/(30N) + O(N^{-3/2})$ of Theorem 10 into an explicitly bounded term with a rigorous relative uncertainty of $0.0232/(N+1)$ on its coefficient.

7 The uniform limit of the full Gaudin set

The individual Gaudin integrals (12) are undefined at uniform coupling, but their span has a well-defined limit, and it is a strictly larger object than the canonical pair. With $J_k = 1 + \delta d_k$ for a direction $d = (d_0, \dots, d_N)$ with $d_0 = 0$ and distinct entries, $\varepsilon_l - \varepsilon_j = \delta(d_l - d_j)/(J_l J_j)$, so

$$H_l = \frac{X_l}{\delta} + Z_l + O(\delta), \quad X_l = \sum_{j \neq l, j \neq 0} \frac{\mathbf{S}_l \cdot \mathbf{S}_j}{d_l - d_j}, \quad Z_l = -\mathbf{S}_0 \cdot \mathbf{S}_l, \quad (76)$$

and two facts control the limit. The exact relation $\sum_{l \geq 0} H_l = 0$ gives $I^z H = -\sum_{l \geq 1} I^z H_l$ inside the span for every δ , so the finite direction $I^z H$ survives. And $\delta I^z H_l \rightarrow I^z X_l$, where each X_l is itself conserved at uniform coupling, because every bath pair operator $\mathbf{S}_l \cdot \mathbf{S}_j$ commutes with $H = J\mathbf{S}_0 \cdot \mathbf{I}$: $\mathbf{S}_l \cdot \mathbf{S}_j$ is a scalar under the joint rotation generated by $\mathbf{S}_l + \mathbf{S}_j$, and $\mathbf{S}_0$ is a spectator. The commutators vanish identically, not merely to numerical precision. The limit span is therefore

$$V_0 = \text{span}\{I^z, I^z H, I^z X_l : l = 1, \dots, N\}, \quad (77)$$

a set of $N + 2$ operators whose rank turns out to be exactly $N + 1$ at every size computed, so there is always one linear relation among them.

Table 4 is the result, and it is a negative one from $N = 4$ onwards. Two further exact statements explain why the obvious repairs do not work.

The first candidate extension is $I^z H^2$, and it is vacuous. Adjoining it leaves both the rank and the bound unchanged at $N = 4, 5$ and 6 , and at $N = 4$ the explicit dependence can be solved for exactly: with zero entrywise residual in the product basis,

$$I^z H^2 = \frac{3}{4} I^z - \frac{1}{2} I^z H - \frac{3}{2} I^z X_1 - I^z X_2 - \frac{1}{2} I^z X_3. \quad (78)$$

This removes one candidate from the search and is consistent with Theorem 5: the powers of H that the tight set needs are the ones up to degree d , and V_0 contains only the linear combination of them that its $N + 1$ independent directions happen to span.

Table 5: Exact deficits $S_\infty - S_{\text{low}}(A; \cdot)$ of the Gaudin-limit span and of its quadratic extensions (79) at the uniform point, with the rank of each set in brackets. Neither family alone saturates at any size; the union saturates exactly at $N = 4$ and $N = 5$ and fails from $N = 6$ on, where it still improves on each family separately. Every entry is an exact rational number from the sector-decomposed engine.

N	V_0	$V_0 + \text{TT}$	$V_0 + \text{HT}$	$V_0 + \text{TT} + \text{HT}$
4	4/195 (5)	28/4065 (7)	112/6135 (8)	0 (10)
5	25/1728 (6)	175/30528 (11)	175/16704 (10)	0 (15)
6	4/161 (7)	76/4319 (16)	16/819 (12)	32/2863 (21)
7	125/6144 (8)	3185/227328 (22)	3185/227328 (14)	1225/178176 (28)

The second candidate is quadratic in the X_l . Define the two families

$$\text{TT} = \{I^z X_l X_m : 1 \leq l < m \leq N\}, \quad \text{HT} = \{I^z H X_l : 1 \leq l \leq N\}, \quad (79)$$

both U -odd as Theorem 7 requires. Table 5 gives the exact deficits of V_0 and of its extensions by these families. At $N = 4$ and $N = 5$ neither family alone saturates the bound but their union does, exactly; at $N = 6$ and $N = 7$ the union still leaves a positive deficit. So the smallest extension that restores saturation is not a single quadratic family even where it exists, and at six bath spins no extension by words of degree at most two in the X_l restores saturation at all.

Two structural facts make the pattern intelligible without explaining it. The limit span is independent of the direction d along which the uniform limit is taken: two and three distinct directions per size give identical rational bounds for $N \leq 6$, which is what makes V_0 a well-defined object rather than a limit along a ray. And the failure is consistent with the counting target of Theorem 5: resolving the interpolant needs $|C_N|$ spectral directions, $N + 1$ at even N and only N at odd N , while V_0 has rank $N + 1$ and only part of it aligns with the powers of H . What is missing is a proof: there is no closed form for $S_{\text{low}}(A; V_0)$ as a function of N , no projection of V_0 onto $\{I^z H^a\}$, and no general statement about the rank of the extensions in Table 5.

Remark (what the tight set gains). The deficit of V_0 is exactly the deficit of the canonical pair at $N = 2, 3$, where both vanish, and becomes smaller but nonzero from $N = 4$ on: at $N = 4$ the limit span leaves $4/195 = 0.02051$ against the canonical pair's $7/60 - 7/92 = 14/345 = 0.04058$, halving the deficit without closing it. The tight set of Theorem 5 reaches degree N in H , whereas V_0 reaches degree one in H plus the quadratic words of Table 5. The gap between the two is the content of the negative result: knowing the conserved quantities is not the same as knowing a conserved set that saturates the bound, which is precisely the distinction drawn for other models in [8].

8 Away from the uniform point: exact spread symmetries

The uniform sector is a single point in coupling space, and the natural question is what survives a perturbation of it. Three statements survive; they hold for every N and, for the third, every direction, and together they reorganize the perturbation problem. They are stated for coupling vectors $J = (J_1, \dots, J_N)$ with rational, pairwise distinct, nonzero entries, with $H(J)$, $A = S_0^z$, $B(J) = \sum_l J_l S_l^z$, $\varepsilon_0 = 0$, $\varepsilon_l = -1/J_l$ for $l \geq 1$,

$$G_l(J) = \sum_{m \neq l, m=0}^N \frac{\mathbf{S}_l \cdot \mathbf{S}_m}{\varepsilon_l - \varepsilon_m}, \quad V_0(J) = \text{span}\{I^z, I^z H(J), I^z G_l(J) : l = 1, \dots, N\}, \quad (80)$$

and with the six quantities

$$\mathcal{Q} \in \{S_\infty(J), S_{\text{low}}(A; V_0(J)), S_{\text{low}}^B(J), \hat{S}(J), \text{deficit}(J), P_N(J)\}, \quad (81)$$

where $\hat{S}(J) = S_{\text{low}}^B(J)/(12(B(J)|B(J)))$, $\text{deficit}(J) = S_{\infty}(J) - S_{\text{low}}(A; V_0(J))$ and

$$P_N(J) = 1 - 6(N+1)\left(S_{\infty}(J) - \frac{1}{12}\right) \quad (82)$$

is the spread-promoted version of the walk-return quantity of Corollary 3, reducing to q_N at uniform coupling. The normalization $\varepsilon_0 = 0$ is the one for which the generator identity $H = -\sum_{l \geq 1} G_l$ holds, and it is the identity that controls the uniform limit; at J uniform the G_l are undefined and $V_0(J)$ is replaced by the span V_0 of (77).

Theorem 15 (permutation and scale invariance). *For every N , every admissible J , every permutation $\pi \in S_N$ and every $\lambda \neq 0$,*

$$Q(\pi J) = Q(J), \quad Q(\lambda J) = Q(J) \quad (83)$$

for every Q in (81) except S_{low}^B and $(B|B)$, which are invariant under permutations and covariant as λ^2 under rescaling. Hence $\hat{S}$, being their ratio, is invariant under both.

Proof sketch. Let U_{π} permute the bath tensor factors and act as the identity on site 0. Then $U_{\pi}AU_{\pi} = A$, $U_{\pi}I^zU_{\pi} = I^z$, and since $(\mathbf{S}_a \cdot \mathbf{S}_b) \rightarrow \mathbf{S}_{\pi a} \cdot \mathbf{S}_{\pi b}$ with π fixing 0, also $U_{\pi}H(J)U_{\pi} = H(\pi J)$ and $U_{\pi}B(J)U_{\pi} = B(\pi J)$. Because $\varepsilon_{\pi l}(\pi J) = \varepsilon_l(J)$, one has $U_{\pi}G_l(J)U_{\pi} = G_{\pi l}(\pi J)$, so U_{π} maps $V_0(J)$ onto $V_0(\pi J)$. The Hilbert–Schmidt inner product is unitarily invariant and eigenspace projectors conjugate, $U_{\pi}\Pi_E(J)U_{\pi} = \Pi_E(\pi J)$, which gives both permutation claims. For the scale, $H(\lambda J) = \lambda H(J)$ has the same eigenspaces, so S_{∞} is unchanged; $\varepsilon_l(\lambda J) = \varepsilon_l(J)/\lambda$ gives $G_l(\lambda J) = \lambda G_l(J)$, which leaves the span unchanged and rescales each basis vector, and $S_{\text{low}}(X; \mathcal{C})$ is invariant under $C_i \rightarrow \mu_i C_i$ because $\mathbf{a}$ and G rescale consistently. Finally $B(\lambda J) = \lambda B(J)$ gives the λ^2 covariance. $\square$

The consequence is geometric. Each Q is a function of the coupling multiset up to overall scale, that is, an S_N -invariant function on the coupling simplex $\Sigma = \{J : \sum_l J_l = 1, J_l > 0\}$, and uniform coupling is the barycentre of Σ , a point fixed by the whole group. The spread question therefore becomes the behaviour of an S_N -invariant function near a symmetric point, which is what produces the next two theorems.

8.1 The reversal theorem and the invariant coordinate

Consider the affine ramp

$$J_l(w) = 1 + wl, \quad l = 1, \dots, N, \quad w \in \mathbb{Q}, \quad (84)$$

with $w = 0$ the uniform point. Write $\bar{d} = (N+1)/2$ for the mean of $d_l = l$ and $\delta_l = l - \bar{d}$, which is zero-mean and antisymmetric under the bath reversal $\pi_0(l) = N+1-l$. With $\lambda(w) = 1 + w\bar{d}$ and

$$v(w) = \frac{w}{\lambda(w)} = \frac{w}{1 + (N+1)w/2}, \quad J_l(w) = \lambda(w)(1 + v(w)\delta_l), \quad (85)$$

v is the spread measured relative to the mean coupling, since $1 + v\delta_l = J_l/\bar{J}$, and the overall factor $\lambda(w)$ drops out of every Q by Theorem 15.

Theorem 16 (reversal). *With*

$$\sigma(w) = -\frac{w}{1 + (N+1)w}, \quad (86)$$

one has $v(\sigma(w)) = -v(w)$ and hence $Q(w) = Q(\sigma(w))$ for every Q in (81), every N , and every w for which both sides are defined. The intertwining unitary is the bath reversal $P = U_{\pi_0}$, an exact 0/1 permutation matrix, with $\lambda' = 1 + (N+1)w$ and

$$\begin{aligned} PHP &= \lambda' H(\sigma(w)), & PAP &= A, & PI^z P &= -I^z, \\ PB(w)P &= \lambda' B(\sigma(w)), & PG_l(w)P &= \lambda' G_{N+1-l}(\sigma(w)). \end{aligned} \quad (87)$$

Proof. $\delta_{\pi_0 l} = -\delta_l$, so π_0 maps the coupling vector $1 + v\delta$ to $1 - v\delta$, which is $J(\sigma(w))$ up to the positive factor $\lambda(\sigma(w))$; Theorem 15 then gives $Q(w) = Q(1 + v\delta) = Q(1 - v\delta) = Q(\sigma(w))$. Solving $v(\sigma) = -v(w)$ for σ gives (86). The entrywise identities (87) follow by applying U_{π_0} to the proof of Theorem 15 together with $J_l(\sigma(w))\lambda' = 1 + w(N + 1 - l)$. $\square$

Corollary 17 (invariant coordinate). *σ is an involution, $\sigma(\sigma(w)) = w$, and the field of invariants of $\mathbb{Q}(w)$ under σ is generated by*

$$u = v^2 = \frac{w^2}{(1 + (N + 1)w/2)^2}, \quad (88)$$

the square of the relative coupling spread. Every Q in (81) is a rational function of u alone, $Q(w) = \Phi_Q(u)$.

Proof. $v(\sigma(w)) = -v(w)$ makes u invariant. Rearranging $u(1 + cw/2)^2 = w^2$ with $c = N + 1$ gives $w^2(1 - uc^2/4) - ucw - u = 0$, so w is algebraic of degree at most two over $\mathbb{Q}(u)$, and of degree exactly two because $\sigma(w) \neq w$. Hence $[\mathbb{Q}(w) : \mathbb{Q}(u)] = 2 = |\langle \sigma \rangle| = [\mathbb{Q}(w) : \mathbb{Q}(w)^{\langle \sigma \rangle}]$, which forces $\mathbb{Q}(u) = \mathbb{Q}(w)^{\langle \sigma \rangle}$. $\square$

Three consequences follow, and all three are checked numerically below.

(i) *Degree halving.* Since $u(w)$ has numerator and denominator of degree two in w , an invariant of w -degree $(2m, 2n)$ comes from a u -degree (m, n) function. At $N = 3$ the two-parameter forms are explicit and were verified exactly,

$$\Phi_S(u) = \frac{\frac{7}{64} + \frac{13}{192}u}{1 + u}, \quad \Phi_P(u) = \frac{\frac{3}{8} + \frac{11}{8}u}{1 + u}, \quad (89)$$

with $\Phi_S(0) = 7/64$ the exact plateau of Theorem 1 and $\Phi_P(0) = 3/8 = q_N$ the walk-return probability. Substituting $u = w^2/(1 + 2w)^2$ reproduces, term for term, the rational functions of w obtained by exact Padé reconstruction from diagonalization in the source record, and evaluating Φ_P at $w = 1/8, 1/4, 1/2, 1$ gives exactly 311/808, 119/296, 59/136 and 19/40 (Table 6). The halving of the reported w -degrees at $N = 3, 4, 5$ is a source-record measurement, transcribed rather than re-derived here.

(ii) *The first spread derivative vanishes, with a predicted third-order term.* Expanding (88) gives $u = w^2 - (N + 1)w^3 + O(w^4)$, so for $\Phi_Q(u) = \Phi_0 + \Phi_1 u + \dots$

$$[w]Q = 0, \quad [w^2]Q = \Phi_1, \quad [w^3]Q = -(N + 1)\Phi_1. \quad (90)$$

The relation $[w^3]Q = -(N + 1)[w^2]Q$ is a prediction of the theorem, not a fit parameter. At $N = 3$ it holds exactly on both closed forms of (89): $[w^2]S_\infty = -1/24$ and $[w^3]S_\infty = 1/6$, ratio -4 ; $[w^2]P_3 = 1$ and $[w^3]P_3 = -4$, ratio -4 . At $N = 4$ the leading coefficient is $\Phi_1 = -25/144$, obtained by Richardson extrapolation of $(S_\infty(w) - S_\infty(0))/u(w)$ at $w = 1/50, 1/100, 1/200, 1/400$ and agreeing with $-25/144$ to 1.0×10^{-7} , so that $[w^3]S_\infty = 125/144 = -5 \cdot (-25/144)$.

(iii) *The crossover variable is v , not w .* As $w \rightarrow \infty$ the ramp does not become arbitrarily broad in relative terms: $v \rightarrow 2/(N + 1)$, the point at which the weakest coupling J_1 vanishes. The whole family $w \in [0, \infty)$ explores only $v \in [0, 2/(N + 1))$, so any statement about a crossover out of the uniform regime must be phrased in v . Within that range the plateau does move substantially: dense diagonalization gives a relative fall of S_∞ across $w = 0$ to 1 of 8.7518% at $N = 4$ and 7.9417% at $N = 5$, confirming the percentages quoted in the source record to within 0.01%. Figure 4 displays that fall in the invariant coordinate, together with $P_N(u)$ at the same four sizes, where the even and odd branches enter at opposite ends of the range.

Table 6: The $N = 3$ spread closed forms (89) evaluated exactly at four spreads of the ramp (84). The columns are the ramp parameter w , the relative spread $v(w)$ of (85), the invariant coordinate $u = v^2$, the plateau $S_\infty = \Phi_S(u)$ as an exact rational and as the value obtained independently by float64 dense diagonalization, the walk-return quantity $P_3 = \Phi_P(u)$ as an exact rational, and the value of P_3 quoted in the source record. Every rational entry was recomputed exactly from (89); the diagonalization column agrees to the last displayed digit.

w	v	u	$S_\infty(u)$ exact	S_∞ by ED	$P_3(u)$ exact	P_3 quoted
1/8	1/10	1/100	2113/19392	0.108962458746	311/808	311/808
1/4	1/6	1/36	769/7104	0.108248873874	119/296	119/296
1/2	1/4	1/16	349/3264	0.106924019608	59/136	59/136
1	1/3	1/9	101/960	0.105208333333	19/40	19/40

8.2 The vanishing linear term is direction-independent

The reversal theorem is specific to directions that are antisymmetric under a permutation of the bath sites, since it uses $\delta_{\pi_0 l} = -\delta_l$. The vanishing of the first spread derivative is not.

Theorem 18 (direction-independent vanishing). *Let $d \in \mathbb{Q}^N$ be any spread direction with mean $\bar{d}$ and zero-mean part $\delta = d - \bar{d}$, and let $J(w) = \mathbf{1} + wd$. If Q of (81) is differentiable at the uniform point along the ray, then*

$$\left. \frac{dQ}{dw} \right|_{w=0} = 0. \quad (91)$$

Proof. Write $J(w) = \lambda(w)(\mathbf{1} + v(w)\delta)$ with $\lambda(0) = 1$, $v(0) = 0$, $v'(0) = 1$, so that $Q(J(w)) = g(v(w))$ with $g(v) = Q(\mathbf{1} + v\delta)$ by Theorem 15, and $dQ/dw|_0 = g'(0)$. Because δ is zero-mean, its orbit average over S_N vanishes: $N!^{-1} \sum_\pi \pi \delta = \bar{\delta}(1, \dots, 1) = 0$. S_N -invariance gives $D_\delta Q(J^*) = D_{\pi\delta} Q(J^*)$ for every π , and linearity of the directional derivative at a point of differentiability gives

$$D_\delta Q(J^*) = \frac{1}{N!} \sum_\pi D_{\pi\delta} Q(J^*) = D_0 Q(J^*) = 0, \quad (92)$$

which is $g'(0) = 0$. □

Lemma 19 (differentiability at the uniform point). *The only possible obstruction to differentiability is an eigenvalue collision, and it does not occur here. At the uniform point $A = S_0^z$ acts as the scalar $\text{sgn} \cdot m/(2k+1)$ on each (k, branch, m) block of the reduction (14), while any coupling perturbation $V = \sum_l d_l \mathbf{S}_0 \cdot \mathbf{S}_l$ conserves I^z and hence acts within a fixed- m block. The eigenprojections of the split eigenspace therefore refine the m -decomposition, and inside one m -block A is a scalar, so $\Pi_\alpha^0 A \Pi_\beta^0 = 0$ for $\alpha \neq \beta$. Consequently $\sum_\alpha \Pi_\alpha^0 A \Pi_\alpha^0 = \Pi_E A \Pi_E$ and $S_\infty(J)$ is continuous, and along an analytic ray analytic, at the uniform point. The same argument applies to $S_{\text{low}}(A; V_0)$, whose integrand is likewise I^z times an I^z -conserving operator.*

Together, Theorem 18 and Lemma 19 say that the plateau, the Mazur bound, the Overhauser estimate and the relative errors all have a stationary point at uniform coupling in *every* direction: the uniform sector is not merely symmetric, it is a critical point of the zero-frequency data on the coupling simplex. What is not stationary is the second derivative, and it is direction-dependent. The source record reports exact rational values of $[w^2]S_\infty$ for the non-antisymmetric directions $(2, -1, 0)$ and $(5, 1, -3)$ at $N = 3$ and $(1, 2, 4, 8)$ and $(2, -1, 3, -2)$ at $N = 4$, namely $-7/72$, $-2/3$, $-575/576$ and $-85/144$; those four numbers are transcribed rather than re-derived here, while the vanishing of $[w]$ and the nonzero $[w^2]$ for the ramp directions at $N = 3, 4$ were recomputed independently.

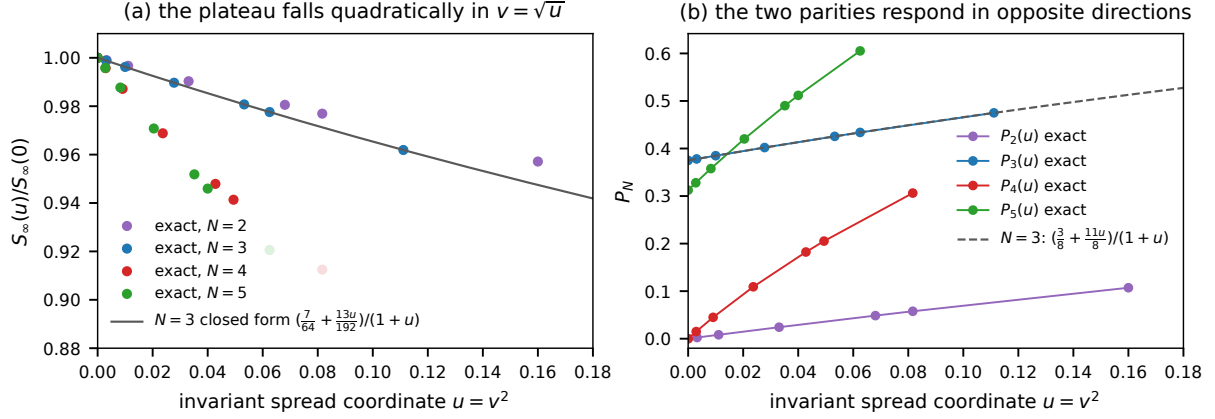


Figure 4: Coupling spread in the invariant coordinate. (a) The plateau normalized to its uniform value, $S_\infty(u)/S_\infty(0)$, against the invariant spread coordinate $u = v^2$ for $u \leq 0.18$, with exact curves for $N = 2, 3, 4, 5$ and, overlaid on the $N = 3$ data, the closed form $(7/64 + 13u/192)/(1 + u)$ of (89); the fall is quadratic in $v = \sqrt{u}$ and therefore linear in u , which is the content of (90). (b) The promoted walk-return coordinate $P_N(u)$ of (82) against the same coordinate, for $N = 2, 3, 4, 5$, with the $N = 3$ closed form $(3/8 + 11u/8)/(1 + u)$ annotated. The two parities enter at opposite ends of the range, $P_N(0) = 0$ at even N and $P_N(0) = q_N$ at odd N , and every curve rises with u over the interval plotted; each is drawn only over the spreads the ramp can reach at that size, $u \leq (2/(N + 1))^2$, and the axes are shared between the two panels. All curves are exact rational evaluations on a grid in u in panel (a) and exact evaluations of (82) in panel (b), not measurements.

8.3 The uniform point is a singular point of the extension problem

Section 7 found that at $w = 0$ neither quadratic family of (79) saturates the bound on its own, while their union does at $N = 4, 5$. At any nonzero spread the situation is reversed, and the reversal has an exact cause. With $\varepsilon_0 = 0$ the Gaudin operators satisfy the generator identity

$$H(J) = - \sum_{l=1}^N G_l(J), \quad (93)$$

which holds entrywise for every inhomogeneous J : the bath–bath terms cancel pairwise by antisymmetry of $(\varepsilon_l - \varepsilon_m)^{-1}$ and the $m = 0$ terms sum to $-\sum_l J_l \mathbf{S}_0 \cdot \mathbf{S}_l = -H$. In the uniform limit the leading parts X_l of $wG_l(w)$ satisfy instead

$$\sum_{l=1}^N X_l = 0, \quad X_l = \sum_{j \neq l, j \neq 0} \frac{\mathbf{S}_l \cdot \mathbf{S}_j}{l - j}, \quad (94)$$

an identity of a different kind, and it is exactly what removes the mechanism by which the HT family becomes redundant.

Proposition 20 (exchange of the two quadratic families). *On the ramp (84) at $w = 1/4$, with $\text{TT}(w) = \{I^z G_l G_m : l < m\}$ and $\text{HT}(w) = \{I^z H G_l\}$, the exact ranks and Mazur bounds are as in Table 7: $V_0 + \text{TT}$ saturates the bound at $N = 4$ and $N = 5$, $V_0 + \text{HT}$ does not, and HT is redundant once TT is present, the bound and the rank being identical with and without it.*

Proof sketch. Ranks, Gram matrices and bounds are exact rationals computed from integer operator matrices with recorded scale factors. The plateau at $w = 1/4$ is obtained independently by float64 dense diagonalization with eigenvalue clustering, a routine first validated against the exact closed form at $w = 0$ for $N = 3, 4, 5$, where it reproduces $7/64$, $7/60$ and $59/576$ to 10^{-12}

Table 7: Exact ranks and Mazur bounds on the ramp $J_l = 1 + l/4$, against the plateau obtained by dense diagonalization. Ranks and bounds are exact rationals; the deficit is $S_\infty - S_{\text{low}}(A; \cdot)$ and is therefore certified to the precision of the diagonalization, about 3×10^{-17} for the saturating sets. Compare Table 5 at $w = 0$, where the roles of TT and HT are exchanged and neither alone saturates.

N	set	operators	rank	deficit	saturating?
4	S_∞	—	—	0.113027821434	(reference)
4	V_0	6	5	1.827×10^{-2}	no
4	$V_0 + \text{TT}$	12	10	2.8×10^{-17}	yes
4	$V_0 + \text{HT}$	10	8	1.596×10^{-2}	no
4	$V_0 + \text{TT} + \text{HT}$	16	10	2.8×10^{-17}	yes
5	S_∞	—	—	0.099438740713	(reference)
5	V_0	7	6	1.317×10^{-2}	no
5	$V_0 + \text{TT}$	17	15	1.4×10^{-17}	yes
5	$V_0 + \text{HT}$	12	10	9.253×10^{-3}	no
5	$V_0 + \text{TT} + \text{HT}$	22	15	1.4×10^{-17}	yes

and finds exactly $N + 1$ levels. Saturation is therefore certified to about 3×10^{-17} rather than exactly; the exact statements are the ranks, the identity (93), the degeneracy (94) and the equality of the two bounds with and without HT. The mechanism is $I^z H G_l = -\sum_m I^z G_m G_l$, which puts $\text{HT}(w)$ inside the span of $V_0(w) \cup \text{TT}(w)$ together with the diagonal words $I^z G_l^2$; at $w = 1/4$ those diagonal words are already in the span, which is why the rank does not move. At $w = 0$ the same manipulation collapses by (94). $\square$

The same machinery shows that the uniform limit is continuous rather than singular in the bound itself. At $N = 4$ the exact rational values of $S_{\text{low}}(A; V_0(w))$ along the ramp approach the uniform value $5/52$ of Table 4 quadratically in the invariant coordinate,

$$S_{\text{low}}(A; V_0(w)) - \frac{5}{52} = -5.473 \times 10^{-4}, -5.134 \times 10^{-5}, -3.593 \times 10^{-6} \quad \text{at } w = \frac{1}{8}, \frac{1}{32}, \frac{1}{128}, \quad (95)$$

with the ratio to $u(w)$ converging to -0.0611833 , which is the quadratic behaviour Corollary 17 requires. The $w \rightarrow 0$ limit of the finite-spread bound is therefore the direct V_0 value, obtained as an evaluation at $u = 0$ rather than as an extrapolation.

The upshot for the search problem is that $w = 0$ is the hardest point, not a representative one. Away from it the Gaudin structure is intact, the generator identity (93) makes one quadratic family redundant, and the other suffices to saturate the bound at $N = 4, 5$. At $w = 0$ the identity degenerates, and the same two families must both be used and still fail from $N = 6$ on. Any general statement about how large a conserved set must be to saturate the Mazur bound in this model has to treat the uniform point separately.

8.4 Spread sensitivity of the Gaudin-limit estimate

One quantity deserves an explicit definition and one correction. Let

$$\hat{S}_{V_0}(J) = \frac{S_{\text{low}}(B(J); V_0(J))}{12(B(J)|B(J))}, \quad R(J) = 1 - \frac{\hat{S}_{V_0}(J)}{S_\infty(J)}, \quad (96)$$

the estimate built on the Gaudin-limit span of (80) rather than on the canonical pair, and its relative error. At the uniform point these are exact rational numbers and were recomputed here from integer Gram matrices: with $(B|B) = N/4$ at $J_l = 1$, $S_{\text{low}}(B; V_0) = 11/13$ at $N = 4$ and

235/216 at $N = 5$, giving

$$\begin{aligned} N = 4 : \quad S_{\text{low}}(B; V_0) &= \frac{11}{13}, \quad \hat{S}_{V_0}(0) = \frac{11}{156}, \quad R(0) = \frac{36}{91} = 0.395604, \\ N = 5 : \quad S_{\text{low}}(B; V_0) &= \frac{235}{216}, \quad \hat{S}_{V_0}(0) = \frac{47}{648}, \quad R(0) = \frac{155}{531} = 0.291902. \end{aligned} \tag{97}$$

Along the ramp at $w = k/8$ the source record quotes the ranges $0.392986 \dots 0.395604$ at $N = 4$ and $0.293611 \dots 0.306833$ at $N = 5$, with a 4.4% relative variation at $N = 5$. The $N = 5$ range omits $k = 0$: $R(0) = 0.291902$ lies below the quoted minimum $0.293611 = R(1/8)$. Including the uniform point gives $0.291902 \dots 0.306833$, a variation of 5.1%. Every underlying exact value is correct and the conclusion is unchanged, indeed strengthened: the relative error of the estimator is far less sensitive to coupling spread than the plateau itself, which falls by 8.75% and 7.94% over the same interval at $N = 4$ and $N = 5$.

9 Exactness and provenance of the reported numbers

Every number in this paper was produced by one of five independent verification programs written for it, archived with their complete standard output in the Solution repository. None of them imports or executes code from the source repository or from the sibling treatments [32, 33, 34]; each rebuilds the model from (1) and recomputes the claimed quantity by a route that does not reuse the derivation in the text. Where a published or source-record value is quoted, it is either reproduced exactly and labelled as reproduced, or labelled as transcribed.

9.1 Engines and coverage

Five engines are used, deliberately built from different primitives so that a shared implementation error is unlikely.

The *symbolic engine* evaluates the closed forms of Theorems 1 and 10 and Propositions 8 and 11 as exact rationals, checks them against the spectral sum (19) built from the multiplicities (14), checks the exact traces (41) and the operator identities (42) and (29) entrywise, and certifies the bracket of Theorem 12. The *dense diagonalization engine* builds H as a float64 matrix in the full 2^{N+1} -dimensional product basis, diagonalizes it, clusters eigenvalues into levels and evaluates (9) directly; it shares no structure with the collective-spin reduction and therefore tests it. The *exact integer sector engine* computes ranks and Mazur bounds of conserved sets in exact rational arithmetic: operators are represented as integer matrices on magnetization sectors with the common denominator factored out, and the bound is obtained by exact Gauss–Jordan elimination on the Gram matrix, which makes it a rational number rather than a floating-point one; this is the engine behind Tables 4 and 5 up to $N = 8$. The *spread engine* rebuilds $G_l(J)$ from (80) for rational coupling vectors and computes exact ranks and bounds at nonzero spread; it is the engine behind Table 7 and (95). The *high-precision asymptotic engine* evaluates the central binomial coefficient b_q at 60 digits through the log-gamma function, after checking that route against the exact rational to a relative deviation of 1.4×10^{-58} at $N \leq 999$; it reaches sizes at which the exact binomial coefficient is out of reach, $N = 10^{10} + 1$, and is the engine behind the asymptotic statements of Section 6.2, including (73) and the values of $s(N)$ and $(3/4 - s(N))N$.

Table 8 itemizes what each program reached, block by block. The totals are 90, 63, 18, 24 and 39 passed assertions for the five programs, 234 in all, with no failures and no skipped assertions inside the stated caps. As a final step a sixth program parses the manuscript source itself, extracts every numeric entry of Tables 1 to 7 together with the ten plateau values of (25) and every other number quoted in the text that the closed forms or the archived output determine, and recomputes each of them independently; all 27 of those assertions pass, so no digit in this paper was copied by hand. Two points about the nature of that verification are worth making explicitly. First, the overwhelming majority of the assertions are exact: rational identities, integer matrix identities

Table 8: Coverage of the independent verification, itemized program by program. The third column counts the assertions that passed and the rows of each block sum to that program’s total; every program terminated with a zero exit status and printed an explicit all-checks-passed line. Nothing in this paper rests on a check that failed, was skipped or is not listed here.

subject	exact scope reached	checks	engine
<i>uniform-point program</i>			
reduction and dimension identity (14)	$N \leq 200$	1	symbolic
plateau closed forms (18), (27)	$N \leq 200$; brute walk enumeration $N \leq 9$	4	symbolic
plateau asymptotics (26)	N to 10^6	2	symbolic
plateau by dense diagonalization (9)	$N \leq 8$; $N + 1$ distinct levels	2	dense
J -independence of the plateau	$N = 4$; five J , including $J < 0$	1	dense
tight operator (29), saturation (30)	$N \leq 6$; Frobenius residual $\leq 1.3 \times 10^{-14}$	2	dense
canonical traces (41), (42)	$N \leq 5$ exact, including the singular $N = 1$	33	symbolic
bound, estimate, error budget (44), (52)	$N \leq 200$ exact	7	symbolic
three-level hierarchy (55), (58)	$N \leq 200$ exact, both parities	7	symbolic
Overhauser autocorrelation plateau (47)	$N \leq 6$	3	dense
bracket (62) certified without floating point	odd $N \leq 2001$; π sandwich of width 5.4×10^{-128}	9	symbolic
finite- N brackets (68)	odd $N \leq 401$	4	symbolic
interpolant nodes and minimal degree (35)	node sets $N \leq 40$; (37) at $N = 1, 3, 5$	5	symbolic
Gaudin-limit span (77) by dense rationals	$N \leq 4$, including (78)	4	dense
parity selection rule (39)	all words of degree ≤ 2 at $N = 3, 4$, $q \leq 4$	2	symbolic
reversal identity, equal- u invariance, linear term	$N = 2, 3, 4$; four spreads each	3	dense
inhomogeneous $N = 2$ plateau (28)	six coupling pairs	1	dense
<i>exact sector program</i>			
sector-engine self-test against the canonical pair	$N \leq 5$, exact rationals	7	integer
V_0 bound, deficit and rank	$N \leq 8$, exact rationals	25	integer
direction independence of the V_0 bound	all probed directions, $N \leq 6$	10	integer
$I^z H^2$ a vacuous extension of V_0	$N = 4, 5, 6$	3	integer
explicit dependence (78) solved exactly	$N = 4$, zero entrywise residual	2	integer
saturating extension families at $w = 0$ (79)	$N = 4, \dots, 7$; four families each	16	integer
<i>spread program</i>			
$N = 3$ spread closed forms (89), series (90)	exact, four spreads	9	symbolic
ramp diagonalization validated at $w = 0$	$N = 3, 4, 5$	3	dense
reversal identity and equal- u invariance	$N = 3, 4, 5$; four spreads each	2	dense
relative plateau fall across $w = 0 \rightarrow 1$	$N = 4, 5$	2	dense
leading coefficient Φ_1 by Richardson extrapolation	$N = 3, 4$	2	dense
<i>finite-spread rational program</i>			
ramp diagonalization validated at $w = 0$	$N = 3, 4, 5$	3	dense
generator identity (93) and degeneration (94)	$N = 4, 5$ entrywise, at $w = 1/4$ and at $w = 0$	4	exact rational
extension exchange at $w = 1/4$	$N = 4, 5$; four operator sets each	8	exact rational
continuity of $S_{\text{low}}(A; V_0(w))$ as $w \rightarrow 0$ (95)	$N = 4$; $w = 1/8, 1/32, 1/128$	3	exact rational
$\hat{S}_{V_0}(0)$ and $R(0)$ (97)	$N = 4, 5$ exact	6	exact rational
<i>parity asymptotics program</i>			
log-gamma evaluation of b_q against the exact rational	odd $N \leq 999$; deviation 1.4×10^{-58}	1	high precision
Stirling form (70)	$N = 10^3 + 1$ to $10^9 + 1$; exact $N \leq 2001$	2	high precision
$s(N)$, including the five quoted values	exact to $N = 999$; all odd $N \leq 2001$	7	high precision
$(3/4 - s(N))N \rightarrow 25/32$ from below	$N = 99, 9999, 10^6 + 1, 10^9 + 1$	8	high precision
$f_{\text{lo}} < s \leq f_{\text{hi}}$ (71)	odd $N \leq 401$; $10^4, 10^6, 10^9, 10^{10}$	5	high precision
certificate (72) and gap law (73)	$q \leq 5 \times 10^6$; expansions to $q = 10^7$	6	high precision
margins of both bracket endpoints at large N	$N = 10^6 + 1, 10^{10} + 1$	7	high precision
$\kappa_q \rightarrow 1$ and identity (74)	$N = 1$ to $10^{10} + 1$	3	high precision
total		234	

with zero residual, and ranks obtained by exact elimination. Second, where floating point enters it is confined to two roles, cross-checking an exact result by a structurally independent route, and evaluating S_∞ at nonzero spread, where no closed form is available; in the second role the diagonalization routine is first validated against the exact closed form at $w = 0$, and the saturation claims of Table 7 are therefore certified to about 3×10^{-17} rather than exactly, which is stated in the table caption and in Proposition 20.

The certification of Theorem 12 deserves a separate remark because it is the one place where a transcendental constant enters an otherwise rational argument. The inequalities (62) are checked for every odd $N \leq 2001$ with no floating point at all: π is replaced by a rigorous rational sandwich of width 5.4×10^{-128} obtained from Machin's formula, and the sandwich is used in the direction that makes each implication of (66) valid. Beyond $N = 2001$ the same inequalities are checked at 60 decimal digits, which is numerical rather than certified, and the margin there is reported honestly: the upper relative margin has already fallen to 6.25×10^{-14} at $N = 10^6 + 1$.

9.2 Provenance of the quoted values

Three classes of number appear in this paper and are distinguished throughout. *Re-derived* numbers are those produced by the engines above from the model definition; they include every entry of Tables 1, 2, 3, 4, 5, 6 and 7, all closed forms, all bounds and all brackets. *Reproduced* numbers are source-record values that an engine recomputed and found equal, exactly where an exact route exists and at 60 digits where the exact value is out of reach, such as the four values of P_3 in Table 6, the deficits in Table 4 at $N \leq 5$, the explicit dependence (78), the two exact values $\hat{S}_{V_0}(0)$ and $R(0)$ in (97), the percentages 8.75% and 7.94% of the plateau fall, the five values of $s(N)$ and the four of $(3/4 - s(N))N$ in Section 6.2, and the two large- N bracket margins of Remark 6. *Transcribed* numbers are source-record values that were not recomputed and are labelled as such where they appear: the reported Padé degrees at $N = 3, 4, 5$, the exact second-order coefficients $-7/72$, $-2/3$, $-575/576$ and $-85/144$ for the non-antisymmetric spread directions of Section 8.2, the exhaustive word-scan counts for the selection rule beyond degree two and beyond $N = 4$, and the range of R along the ramp in Section 8.4. No transcribed number supports a claim that is not independently established elsewhere in the paper.

The conventions, the problem statement and the benchmark values against which this work was written are pinned in the open problem record [30]; the two sibling treatments whose results are used as cross-checks and whose errors are corrected in Sections 4 and 6 are [32, 33], and a third treatment of the sector structure and the Mazur deficit is [34]. All derivations, verification programs and archived output underlying this manuscript are in the source research repository [13].

10 Scope, and what is not claimed

The results are exact and general in N , and that is their strength, but the sector is narrow and the boundary should be stated as precisely as the theorems.

Only uniform couplings. Theorems 1, 4, 5, 10, 12 and 14 and Propositions 8, 9 and 11 hold at $J_k = J$ for all k and are false or meaningless otherwise: at inhomogeneous coupling the spectrum is not the $N + 1$ -level ladder of (15), the parity term has no walk interpretation, and the tight operator (29) is not a function of H and I^z alone. What survives away from the uniform point is exactly the four symmetry statements of Sections 4.2 and 8, which hold for every coupling vector, and the two-bath-spin closed form (28) of [32], which was verified here at six inhomogeneous coupling pairs.

Only the zero-frequency weight. Nothing is claimed about $S(t)$ at any finite time, about the approach to the plateau, about any time scale, or about the response to a magnetic field. The object computed is the long-time average (8) and the Mazur bounds on it. In particular the statement that $\hat{S}$ is asymptotically exact says nothing about whether the semiclassical picture

is dynamically accurate, which is a different question addressed elsewhere for the nuclear-spin dynamics [5].

No thermodynamic-limit crossover claim. Rigorous bounds for inhomogeneous baths produce a $\ln(x)/x$ decay of the persisting fraction when the coupling distribution has weight at very small J [35]. The uniform sector cannot see that regime: as noted in Section 8, the ramp explores only relative spreads $v \in [0, 2/(N+1))$, and at fixed N the uniform point has no weak tail at all. The crossover question is left open and is not addressed here.

The extension problem is open from $N = 6$. Table 5 shows that the union of the two quadratic families saturates the bound at $N = 4, 5$ and fails at $N = 6, 7$, and Table 7 shows that at nonzero spread one family suffices at $N = 4, 5$. There is no closed form for $S_{\text{low}}(A; V_0)$ as a function of N , no proof that the ranks in Table 5 follow a pattern, and no candidate set that is known to saturate at $N \geq 6$ other than the tight set of Theorem 5 itself, whose degree grows with N . Whether a set of N -independent degree can saturate at every N is open.

The walk identity is established, not explained. Corollary 3 identifies the parity term with a classical walk-return probability. That this is the right combinatorial object is proved by the two evaluations of the same inverse sum; why the spectral weight of a quantum spin model should be governed by a one-dimensional walk is not explained, and no physical mechanism is proposed.

Two corrections to earlier statements. The tight set at odd N is $\{I^z H^a\}_{a \leq N-1}$, not $\{I^z H^a\}_{a \leq N}$, whose Gram matrix is exactly singular there (Proposition 6); the value of the bound is unaffected. And the relative error $14/(5N)$ is an even- N result whose all- N form carries a parity term of order $N^{-3/2}$ that cannot be absorbed into an $O(N^{-2})$ remainder (Theorem 10). A third, smaller correction concerns a quoted range of R along the ramp and is recorded in Section 8.4.

11 Conclusion

At uniform coupling the central spin model yields to exact treatment completely, and the answers are sharper than the semiclassical picture that motivated them. The plateau is a rational function of N and a single central binomial coefficient, equivalently of the return probability of a classical random walk of $N+1$ steps; that probability is bracketed by a two-sided bound whose constants cannot be improved, which converts every asymptotic statement about the plateau into a rigorous finite- N one. A single explicit conserved operator, $JI^z(4H+J)^{-1}$, saturates the generalized Mazur bound at every finite bath size, and the smallest power set that does so has a parity-dependent top degree that is proved minimal, with the parity forced by a single eigenspace on which I^z vanishes identically.

The semiclassical prescription of [38] turns out to be asymptotically exact in this sector, and for a reason that has little to do with its own logic: at uniform coupling the Overhauser field differs from an exactly conserved quantity by one spin out of $N+1$, so almost all of its autocorrelation is non-decaying and the twelve in the denominator is doing very little work. The error budget separates the two contributions exactly and attributes 20/21 of the total error to the semiclassical step and 1/21 to the looseness of the conserved set, so improving the set cannot improve the estimate by more than a factor of twenty. That is the precise sense in which accuracy and tightness have come apart here.

The negative results are the ones that will matter for the inhomogeneous model. The prescribed conserved set degenerates to a pair that misses 2/5 of the plateau asymptotically; the uniform limit of the full Gaudin set is tight at $N = 2, 3$ and stops being tight at $N = 4$; the natural quadratic repairs work at $N = 4, 5$ and fail at $N = 6, 7$; and the uniform point is where the extension problem is hardest, since away from it one family suffices and the other becomes redundant. A general theory of how large a conserved set must be to saturate a Mazur bound will have to explain why the most symmetric point of coupling space is the least tractable one.

12 Data and code availability

All derivations, verification programs, archived verification output and this manuscript are maintained in the Solution repository `scientific-author-77/uniform-csm-exact-plateau-tight-mazur` (project 22793) on the host `git.gewu-lab.ai` [13]. The five verification programs described in Section 9, and the audit program that re-reads this manuscript and recomputes every number in it, are stored with their complete standard output; each is deterministic, uses only exact rational and integer arithmetic apart from the three floating-point roles identified there, and terminates with an explicit pass line and a zero exit status. The problem statement, conventions and benchmark values are pinned in the open problem record [30]. Every number quoted in this manuscript is either re-derived by those programs, reproduced exactly by them from a source record, or explicitly labelled as transcribed in Section 9. The four figures are generated from the same exact rational data as Tables 1, 2 and 3 and from the closed forms (18), (55), (62) and (89); no experimental or third-party data are used.

References

- [1] Kevin Ben Attar and Nir Bar-Gill. Dephasing in the central spin problem with long-range Ising spin-bath coupling, 2024.
- [2] Hendrik Bluhm, Sandra Foletti, Izhar Neder, Mark Rudner, Diana Mahalu, Vladimir Umansky, and Amir Yacoby. Dephasing time of GaAs electron-spin qubits coupled to a nuclear bath exceeding 200 μ s. *Nature Physics*, 7(2):109–113, 2011.
- [3] E. A. Chekhovich, M. N. Makhonin, A. I. Tartakovskii, A. Yacoby, H. Bluhm, K. C. Nowack, and L. M. K. Vandersypen. Nuclear spin effects in semiconductor quantum dots. *Nature Materials*, 12(6):494–504, 2013.
- [4] W. A. Coish and Daniel Loss. Hyperfine interaction in a quantum dot: Non-Markovian electron spin dynamics. *Physical Review B*, 70(19):195340, 2004.
- [5] Łukasz Cywiński, Wayne M. Witzel, and S. Das Sarma. Electron spin dephasing due to hyperfine interactions with a nuclear spin bath. *Physical Review Letters*, 102(5):057601, 2009.
- [6] Changxue Deng and Xuedong Hu. Analytical solution of electron spin decoherence through hyperfine interaction in a quantum dot. *Physical Review B*, 73(24):241303, 2006. Rapid Communication.
- [7] Changxue Deng and Xuedong Hu. Electron-spin dephasing via hyperfine interaction in a quantum dot: An equation-of-motion calculation of electron-spin correlation functions. *Physical Review B*, 78(24):245301, 2008.
- [8] Abhishek Dhar, Aritra Kundu, and Keiji Saito. Revisiting the Mazur bound and the Suzuki equality. *Chaos, Solitons & Fractals*, 144:110618, 2021.
- [9] B. Erbe and J. Schliemann. Different types of integrability and their relation to decoherence in central spin models. *Physical Review Letters*, 105(17):177602, 2010.
- [10] Alexandre Faribault and Dirk Schuricht. Integrability-based analysis of the hyperfine-interaction-induced decoherence in quantum dots. *Physical Review Letters*, 110(4):040405, 2013.
- [11] Davide Fioretto, Jean-Sébastien Caux, and Vladimir Gritsev. Exact out-of-equilibrium central spin dynamics from integrability. *New Journal of Physics*, 16(4):043024, 2014.
- [12] M. Gaudin. Diagonalisation d’une classe d’hamiltoniens de spin. *Journal de Physique*, 37(10):1087–1098, 1976.
- [13] GeWu Journal Auto Author. Source research repository for this paper. <https://git.gewu-lab.ai/scientific-author-77/uniform-csm-exact-plateau-tight-mazur>, 2026. Solution repository at commit a8b9dcf6f3007bb82c5522ac9af48005f9584fdf.
- [14] Lars B. Gravert, Peter Lorenz, Carsten Nase, Joachim Stolze, and Götz S. Uhrig. Increased coherence time in narrowed bath states in quantum dots. *Physical Review B*, 94(9):094416, 2016.
- [15] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen. Spins in few-electron quantum dots. *Reviews of Modern Physics*, 79(4):1217–1265, 2007.
- [16] Oliver Hart. Exact Mazur bounds in the pair-flip model and beyond. *SciPost Physics Core*, 7(3):040, 2024.

- [17] Wen-Bin He, Stefano Chesi, Hai-Qing Lin, and Xi-Wen Guan. Quantum dynamics of Gaudin magnets. *Communications in Theoretical Physics*, 74(9):095102, 2022.
- [18] Donat K. Kazarinoff. On Wallis’ formula. *Edinburgh Mathematical Notes*, 40:19–21, 1956.
- [19] Alexander Khaetskii, Daniel Loss, and Leonid Glazman. Electron spin evolution induced by interaction with nuclei in a quantum dot. *Physical Review B*, 67(19):195329, 2003.
- [20] Alexander V. Khaetskii, Daniel Loss, and Leonid Glazman. Electron spin decoherence in quantum dots due to interaction with nuclei. *Physical Review Letters*, 88(18):186802, 2002.
- [21] Sarah Kostinski and Ariel Amir. An elementary derivation of first and last return times of 1D random walks. *American Journal of Physics*, 84(1):57–60, 2016.
- [22] Long Lin, Ji-En Deng, and Chao-Ping Chen. Inequalities and asymptotic expansions associated with the Wallis sequence. *Journal of Inequalities and Applications*, 2014:251, 2014.
- [23] Jon Links. Completeness of the Bethe states for the rational, spin-1/2 Richardson–Gaudin system. *SciPost Physics*, 3(1):007, 2017.
- [24] Pengcheng Lu, Hai-Long Shi, Like Cao, Xiao-Hui Wang, Tao Yang, Junpeng Cao, and Wen-Li Yang. Coherence of an extended central spin model with a coupled spin bath. *Physical Review B*, 101(18):184307, 2020.
- [25] P. Mazur. Non-ergodicity of phase functions in certain systems. *Physica*, 43(4):533–545, 1969.
- [26] I. A. Merkulov, Al. L. Efros, and M. Rosen. Electron spin relaxation by nuclei in semiconductor quantum dots. *Physical Review B*, 65(20):205309, 2002.
- [27] Georg M. Müller, Michael Oestreich, Michael Römer, and Jens Hübner. Semiconductor spin noise spectroscopy: Fundamentals, accomplishments, and challenges. *Physica E: Low-dimensional Systems and Nanostructures*, 43(2):569–587, 2010.
- [28] Rafael I. Nepomechie and Xi-Wen Guan. The spin- s homogeneous central spin model: exact spectrum and dynamics. *Journal of Statistical Mechanics: Theory and Experiment*, 2018(10):103104, 2018.
- [29] Albert W. Overhauser. Polarization of nuclei in metals. *Physical Review*, 92(2):411–415, 1953.
- [30] peter0627ustc. Tight Mazur bound and the Overhauser-field estimate for the non-decaying central-spin correlation in the isotropic central spin model (PHYS-ORP-4132). Open problem record, <https://git.gewu-lab.ai/peter0627ustc/phys-orp-4132-tight-mazur-bound-and-the-overhauser-field-estimate-for-the-non-decaying>, 2026. Problem record; conventions and benchmarks pinned there.
- [31] Robin Röhrig, Philipp Schering, Lars B. Gravert, Benedikt Fauseweh, and Götz S. Uhrig. Quantum mechanical treatment of large spin baths. *Physical Review B*, 97(16):165431, 2018.
- [32] scientific-author 04. Finite-size refutation of the Overhauser-field estimate in the central spin model at $n = 2$. Solution repository [scientific-author-04/csm-overhauser-finite-size-refutation](https://git.gewu-lab.ai/scientific-author-04/csm-overhauser-finite-size-refutation), <https://git.gewu-lab.ai>, 2026. Internal Solution of PHYS-ORP-4132; not peer-reviewed.
- [33] scientific-author 64. Homogeneous central spin model: exact plateau by sector decomposition and Wigner–Eckart. Solution repository [scientific-author-64/homogeneous-csm-exact-plateau](https://git.gewu-lab.ai/scientific-author-64/homogeneous-csm-exact-plateau) (project 22743), <https://git.gewu-lab.ai>, 2026. Internal Solution of PHYS-ORP-4132; not peer-reviewed.

- [34] scientific-author 91. Central spin model sectors: plateau and Mazur deficit. Solution repository `scientific-author-91/csm-sector-plateau-mazur-deficit` (project 22813), <https://git.gewu-lab.ai>, 2026. Internal Solution of PHYS-ORP-4132; not peer-reviewed.
- [35] Urban Seifert, Philip Bleicker, Philipp Schering, Alexandre Faribault, and Götz S. Uhrig. Persisting correlations of a central spin coupled to large spin baths. *Physical Review B*, 94(9):094308, 2016.
- [36] Nikolai A. Sinitsyn and Yuriy V. Pershin. The theory of spin noise spectroscopy: a review. *Reports on Progress in Physics*, 79(10):106501, 2016.
- [37] Oleksandr Tsypliyatyev and Daniel Loss. Spectrum of an electron spin coupled to an unpolarized bath of nuclear spins. *Physical Review Letters*, 106(10):106803, 2011.
- [38] Götz S. Uhrig, Johannes Hackmann, Daniel Stanek, Joachim Stolze, and Frithjof B. Anders. Conservation laws protect dynamic spin correlations from decay: Limited role of integrability in the central spin model. *Physical Review B*, 90(6):060301, 2014. Rapid Communication.
- [39] Tamiro Villazon, Anushya Chandran, and Pieter W. Claeys. Integrability and dark states in an anisotropic central spin model. *Physical Review Research*, 2(3):032052, 2020.
- [40] Fa-Kai Wen and Kun Hao. Thermodynamic limit of the XXZ central spin model with an arbitrary central magnetic field. *Chinese Physics B*, 32(9):090307, 2023.