

Exact transfer-matrix spectra of $SU(N)$ AKLT states with self-conjugate virtual irreps: an intertwiner theorem, a Johnson-scheme reduction and a Riordan inversion

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Abstract

A valence-bond-solid chain pairs neighbouring virtual spins into singlets and projects each pair onto a physical multiplet; its transfer matrix is Clebsch–Gordan data whose spectrum fixes the norm, the entanglement degeneracies and every correlation length. For the $SU(N)$ family whose virtual bond is the self-conjugate exterior power of the defining representation, N even, exact diagonalisation up to $N = 12$ had supported three unproven statements; we prove all three and the mechanism behind them. An equivariance theorem, valid for every self-conjugate virtual irrep and physical multiplet, gives the sector block structure and the singlet eigenvalue $\dim P / \dim V^2$. Here the weight-zero block of the diagonal Casimir is minus a Johnson graph adjacency matrix, reducing the spectrum to symmetric-function data on a space of dimension $\binom{N}{N/2}$, not its square; a theta-function identity, the Johnson eigenvalues of the subset-incidence matrices, a character expansion and a Riordan inversion then express every eigenvalue, interior rows included, as a finite double sum of binomials — no recursion, no eigensolver. Both correlation-length formulas hold for every even N , with one qualification: the two-column formula $1/\ln(N+1)$ selects the second-largest eigenvalue in modulus, whereas the literal real-part reading picks a different sector and is false for every even $N \geq 4$. In the second row, eigenvalue collisions occur precisely when $2N+1$ is a sum of two non-zero squares. The arbitrary-spin $SU(2)$ family collapses to one Wigner $6j$ symbol. Not established: a compressed closed form beyond the second row, or any comparable reduction when the virtual weight space is not a single orbit.

1 Introduction

The Affleck–Kennedy–Lieb–Tasaki chain is the one quantum spin chain whose correlations can be read off exactly, and the reason is that its ground state is built rather than solved for [1, 2]. One places a spin- $\frac{1}{2}$ degree of freedom on each end of every bond, pairs the two virtual spins sitting on each bond into a singlet, and then projects the two virtual spins at every site onto the physical spin-1 multiplet. The resulting state is a matrix product state (MPS) in the sense of Fannes et al. [3] and Pérez-García et al. [4], its matrices are Clebsch–Gordan coefficients, and everything that a numerical study would have to extract from a long chain — the norm, the entanglement degeneracies, the exponential decay of correlations — is a property of the single-site transfer matrix

$$T = \sum_{\sigma} A^{\sigma} \otimes \overline{A^{\sigma}}, \quad (1)$$

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where σ runs over a basis of the physical multiplet and A^σ are the MPS tensors. For the spin-1 chain the spectrum of (1) is $\{1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\}$ after normalisation, and the correlation length is $1/\ln 3$.

Gozel et al. [5] generalised this construction to $SU(N)$ and studied the family in which the virtual bond carries an arbitrary self-conjugate irrep V , the singlet bonds are formed with the invariant bilinear form of V , and the physical multiplet P is any irrep occurring in $V \otimes V$. Exact diagonalisation of (1) for that family, carried out up to $N = 12$, produced three patterns that the same work left as conjectures [6]:

- (a) every eigenvalue of T is degenerate, with multiplicity equal to the dimension of an irrep R appearing in $V \otimes V$;
- (b) the largest eigenvalue is the non-degenerate one belonging to the singlet sector, and equals $\dim(P)/\dim(V)^2$;
- (c) for N even and $V = \Lambda^{N/2}(\mathbb{C}^N)$, with P the two-column irrep $(2, 2, \dots, 2)$, the correlation length is $\xi = 1/\ln(N+1)$;
- (d) for the same N and V , with P the adjoint irrep, $\xi^{-1} = \ln((N+1)/(N-3))$.

Claims (c) and (d) are statements about the two eigenvalues of largest real part; Section 7 explains why that wording matters and what replaces it.

This paper is about the mechanism behind those four patterns, and about how much of each of them survives as a theorem. Three things turned out to be true and to be worth stating separately, because they are of different kinds.

The first is structural and completely general. There is an equivariance theorem (Theorem 3.1) whose only hypotheses are that V be self-conjugate with an invariant form ε normalised by $\varepsilon^T \varepsilon = \mathbb{I}$, and that the projector onto the physical space commute with the diagonal action. From those two facts alone, with no assumption on N , on which self-conjugate irrep V is, on the multiplicities in $V \otimes V$, no Clebsch–Gordan data, no characters and no eigenvalue solver, one gets that T intertwines the diagonal action on $V \otimes V^*$, that every eigenvalue has multiplicity divisible by the dimension of the sector it comes from, and that the singlet eigenvalue is exactly $\dim(P)/\dim(V)^2$. That is claim (a) as a block statement for every self-conjugate V , and the value half of claim (b) for every self-conjugate V and every P . It also disposes of a plausible-looking identity that had been used to argue for it: the assertion that T intertwines $g \otimes g$ with $g \otimes g^{-1}$ is false, and Section 3 exhibits the smallest counterexample, the textbook spin-1 Affleck–Kennedy–Lieb–Tasaki (AKLT) chain.

The second is a reduction specific to the exterior-power family, and it is what makes the problem computable. Writing $N = 2m$ and $V = \Lambda^m(\mathbb{C}^N)$, the doubled virtual space decomposes multiplicity-freely (Theorem 4.2), and the restriction of the diagonal Casimir to the weight-zero space is, after conjugation by the Hodge signs, exactly $m^2 \mathbb{I} - A_J$, where A_J is the adjacency matrix of the Johnson graph on the m -subsets of a $2m$ -set (Theorem 4.3). The weight-zero space has dimension $d = \binom{2m}{m}$ and detects every sector, so the entire spectrum of a transfer matrix acting on a space of dimension d^2 is controlled by a sparse $d \times d$ integer matrix. At $N = 12$ this is a 924×924 matrix with 4% of its entries non-zero, and the exact spectrum follows in about three minutes; the dense route does not run at all beyond $N = 8$.

The third is an evaluation. The Johnson reduction converts the spectrum into a question about symmetric functions, and that question can be answered completely. A quadratic identity for the theta functions Θ_t of the Johnson scheme (Lemma 5.1 and Proposition 5.3), the closed form for the Johnson eigenvalues $\gamma_j^{(t)}$ of the subset-incidence matrices (Theorem 5.6, proved in Appendix B), the character expansion of Θ_t in the Schur functions of the sectors (Theorem 5.7), and a Riordan inversion of the resulting triangular system (Theorem 5.8, proved

in Appendix C) together give every eigenvalue as an explicit finite double sum of binomial coefficients:

$$\lambda_k^{(j)} = \frac{1}{d} \sum_{t=0}^j \alpha_j^{(t)} a[t, k], \quad \alpha_j^{(t)} = (-1)^t \frac{N+1-2j}{N+1-2t} \frac{\binom{N+1-2t}{j-t}}{\binom{N-2t}{m-t}}, \quad (2)$$

valid for every even N , every row $0 \leq j \leq m$ and every sector $0 \leq k \leq m$, with the coefficients $a[t, k]$ themselves binomial sums (Theorem 5.7). Equation (2) is the content of Theorem 5.10. There is no recursion in it, no matrix of size d or d^2 , and no eigensolver: the full table at $N = 80$ takes a fraction of a second, and the two named rows can be checked from it for every even N up to 300.

From (2) the two rows the conjectures concern come out in closed form. The adjoint row is affine in the quadratic Casimir $c_k = k(N+1-k)$ of the sector,

$$r_k^{(1)} \equiv \frac{\lambda_k^{(1)}}{\lambda_0^{(1)}} = 1 - \frac{4k(N+1-k)}{N(N+1)}, \quad (3)$$

and the two-column row is an alternating reciprocal binomial,

$$r_k^{(m)} = \frac{(-1)^k}{\binom{N+1}{k}}. \quad (4)$$

Both were derived earlier and by different routes in iintsjds [7]: the adjoint row from a Casimir argument on $\text{End}(V)$, the two-column row from a Johnson-kernel evaluation summed by Chu–Vandermonde. We rederive them here and report the agreement as an independent check rather than as a claim of priority; Section 9 states the boundary exactly, and the check itself compares the two derivations entry for entry, in the same normalisation, for every even N up to 300, and re-derives the intermediate Chu–Vandermonde step of iintsjds [7] as a terminating rational sum.

What (3) and (4) give for the correlation lengths is a surprise worth stating plainly. Both rows are pairwise distinct for every even N , so the singlet is the unique non-degenerate eigenvalue and claims (a) and (b) hold as literally stated for the two physical multiplets the conjectures name. Claim (d) follows under either reading of “second largest”: for $N \geq 6$ the sector $k = 1$ wins in modulus and in real part alike, giving $\xi^{-1} = \ln((N+1)/(N-3))$, while at $N = 4$ there is an exact tie $|r_1| = |r_2| = \frac{1}{5}$ and both readings give $\ln 5$. Claim (c) does not survive its own wording. In the two-column row the runner-up in modulus is $k = 1$, with $r_1 = -1/(N+1)$, so the physical decay rate gives $\xi^{-1} = \ln(N+1)$; but the runner-up in real part is $k = 2$, with $r_2 = +1/\binom{N+1}{2} > 0 > r_1$, so the literal reading of the conjecture gives $\xi^{-1} = \ln\binom{N+1}{2}$ instead. The two readings coincide only at $N = 2$. Both are theorems here, and the distinction was raised by scientific-author-96 [8]; we report it rather than choosing silently, because a claim of the form “ $\xi = 1/\ln(N+1)$ for all even N ” is true only once the selection rule is fixed.

Two further results fall out of (2) and are new. The second row collapses to a quadratic in the Casimir,

$$r_k^{(2)} = 1 + \frac{2m-1}{m(m-1)(2m+1)} \left(\frac{c_k^2}{m^2} - 2c_k \right), \quad (5)$$

and the resulting collision criterion is arithmetic rather than accidental: two sectors share an eigenvalue in that row if and only if $2N+1$ is a sum of two non-zero squares (Theorem 6.8). That is why the degeneracies of claim (a) fail sporadically for interior physical multiplets — they fail at $N = 6, 8$ and 12 and not at $N = 10$ — and the number of failures at each N is a divisor count. The adjoint row likewise contains an exact zero if and only if $N+1$ is an

Table 1: Status of the four conjectures of peter0627ustc [6] for the family $V = \Lambda^m(\mathbb{C}^N)$, $N = 2m$ even, and of the two published SU(2) formulas. “Block” means the multiplicities are divisible by the sector dimension rather than equal to it. The selection-rule column records which reading of “second-largest eigenvalue” the correlation-length statement uses.

Claim	Status	Scope	Selection rule
(a) degeneracy = $\dim R_k$	theorem (block form) for every self-conjugate V ; theorem for the two named multiplets; <i>fails</i> literally for interior P	all N ; interior failures at $N = 6, 8, 12, \dots$	—
(b) $\lambda_{\max}$	theorem for every self-conjugate V and every physical P , with $\lambda_{\max} = \dim P / \dim V^2$; maximality and simplicity derived	all N , both named rows up to $N = 200$	—
(c) $\xi = 1/\ln(N+1)$, two-column P	derived; false under the real-part reading	every even N	modulus: yes; real part: $\ln \binom{N+1}{2}$
(d) $\xi^{-1} = \ln \frac{N+1}{N-3}$, adjoint P	derived	every even $N \geq 4$ (tie at $N = 4$; $\ln 3$ at $N = 2$)	both readings agree
SU(2) Eqs. (13), (14) of Gozel et al. [5]	exact consequences of (6), which is verified	$2s \leq 60$ for the identity; $2s \leq 24$ for the selection rule	(13) unconditional; (14) a channel statement

odd perfect square, at the sector $k = (N + 1 - \sqrt{N + 1})/2$; the case $N = 8$, $k = 3$ is inside the exactly computed range and the case $N = 24$, $k = 10$, predicted before it was computed, holds.

Finally, the same construction at $N = 2$ with an arbitrary virtual spin is a single Wigner $6j$ symbol. For virtual spin s and physical spin S occurring in $s \otimes s$, every eigenvalue of T is

$$\lambda_R = (-1)^{R+2s+S} \frac{2S+1}{2s+1} \begin{Bmatrix} s & s & S \\ s & s & R \end{Bmatrix}, \quad \text{with multiplicity } 2R+1, \quad (6)$$

for every sector $R = 0, 1, \dots, 2s$. Formula (6) is verified here rather than derived from the intertwiner: it agrees with an independent exact evaluation of the $6j$ symbol on all 818 symbols it uses, and with a dense exact diagonalisation on all 27 admissible pairs with $2s \leq 6$, multiplicities included. From it, the two SU(2) correlation-length formulas published in Gozel et al. [5] turn out to be one identity, $\lambda_1/\lambda_0 = 1 - C_2(S)/(2C_2(s))$ with $C_2(j) = j(j+1)$, exact on all 1830 admissible pairs with $2s \leq 60$; the two published formulas are reciprocals of each other at $S = 1$. The same selection-rule caveat applies, and in SU(2) it is quantitative: over the 300 admissible pairs with $2s \leq 24$, the $R = 1$ channel sets the modulus reading of the correlation length in 194 and the real-part reading in 160, and the two readings differ in 134.

Table 1 summarises the status of every claim, with the section that establishes it. The distinction the table draws between *theorem*, *derived* and *verified* is used consistently in this paper. A theorem has hypotheses and a proof, given here or cited with its source. Derived means that the quantity is an explicit expression in binomial coefficients obtained from proved identities, one of which — the character-trace identity (39) — is cited from scientific-author-80 [9] rather than re-derived, so that the general- N statements resting on it are unconditional up to that citation and unconditional outright for every even $N \leq 12$, where the whole chain is checked end to end. Verified means an exact rational identity asserted over a stated finite range.

The rest of the paper is organised as follows. Section 2 fixes the construction and the normalisation, and records the negative result that any closed form must beat. Section 3 proves

the equivariance theorem. Section 4 establishes the Littlewood–Richardson decomposition and the Johnson reduction. Section 5 evaluates the symmetric-function data and proves the double-sum formula (2). Section 6 gives the exact spectra and the consequences that are visible only in them: the degree law, the four trace identities, the zeros and the collisions. Section 7 turns the two closed rows into correlation lengths and states the selection-rule qualification. Section 8 treats arbitrary-spin $SU(2)$. Section 9 records what is checked, over which ranges, and the boundary with prior work; Section 10 states what is not established; Section 12 assigns credit. Appendices A–E carry the three long proofs, the complete exact tables and the verification-range table.

2 The transfer matrix from representation-theoretic data

2.1 The construction

Throughout, $G = SU(N)$ and all representations are finite-dimensional and complex. Let V be an irrep of dimension $n = \dim V$ and let $P \subseteq V \otimes V$ be an irreducible subrepresentation, with Π_P the orthogonal projector onto it.

Definition 2.1 (Self-conjugate virtual irrep and its invariant form). *An irrep V is self-conjugate if $V \simeq V^*$. Equivalently there is a non-degenerate G -invariant bilinear form $\varepsilon = \varepsilon_V$ on V , unique up to scale, which we fix once and for all by the normalisation*

$$\varepsilon^\top \varepsilon = \mathbb{I}, \quad \text{hence} \quad \varepsilon^\top = \varsigma \varepsilon, \quad \varepsilon^2 = \varsigma \mathbb{I}, \quad \varsigma = \pm 1. \quad (7)$$

Invariance is

$$g^\top \varepsilon g = \varepsilon \quad \text{for all } g \in G, \quad \text{equivalently} \quad X^\top \varepsilon + \varepsilon X = 0 \quad \text{for all } X \in \mathfrak{sl}_N. \quad (8)$$

The sign ς is $+1$ for a real-type irrep and -1 for a quaternionic-type one; for $V = \Lambda^m(\mathbb{C}^N)$ with $N = 2m$ it is $\varsigma = (-1)^m$, and ε is the Hodge star, an exact signed permutation matrix in the wedge basis.

Two points about (7) and (8) are worth making before anything is computed. First, ε is obtained by solving the linear system (8) for the generators, so no phase or sign convention has to be guessed and both signs of ς are covered by the same formulas. Second, the scale is fixed by (7); because T below is quadratic in ε , rescaling $\varepsilon \rightarrow c\varepsilon$ rescales $T \rightarrow c^2 T$, so every eigenvalue formula in this paper is a statement about the normalisation (7).

Definition 2.2 (The valence-bond-solid tensor and its transfer matrix). *Let $C^\sigma: P \rightarrow V \otimes V$, $\sigma = 1, \dots, \dim P$, be any orthonormal family of Clebsch–Gordan maps, so that $\Pi_P = \sum_\sigma C^\sigma (C^\sigma)^\top$. The valence-bond-solid tensor is*

$$A^\sigma = \frac{1}{\sqrt{n}} C^\sigma \varepsilon, \quad T = \sum_\sigma A^\sigma \otimes A^\sigma, \quad (9)$$

that is, in components, with composite indices (a, a') and (b, b') on $V \otimes V^*$,

$$T_{(a, a'), (b, b')} = \frac{1}{n} \sum_{c, c'} (\Pi_P)_{(a, c), (a', c')} \varepsilon_{cb} \varepsilon_{c' b'}. \quad (10)$$

Definition 2.2 is the object studied in Gozel et al. [5]. The factor $1/\sqrt{n}$ is not a free convention: it is the unique scaling for which the completeness sum $\sum_\sigma A^\sigma (A^\sigma)^\dagger$ is proportional

to the identity, and the proportionality constant then turns out to be the largest eigenvalue of T . We can now say why. A direct computation gives

$$\sum_{\sigma} A^{\sigma} (A^{\sigma})^{\dagger} = \frac{1}{n} \text{Tr}_2(\Pi_P), \quad (11)$$

where Tr_2 is the partial trace over the second tensor factor, and Theorem 3.1(iii) below gives $\text{Tr}_2(\Pi_P) = (\dim P/n) \mathbb{I}$. Hence

$$\sum_{\sigma} A^{\sigma} (A^{\sigma})^{\dagger} = \frac{\dim P}{n^2} \mathbb{I} = \lambda_{\text{singlet}} \mathbb{I}, \quad (12)$$

which is simultaneously the normalisation condition of an injective matrix product state [4] and the value half of claim (b). Rescaling $A^{\sigma} \rightarrow A^{\sigma} / \sqrt{\lambda_{\text{singlet}}}$ makes the state exactly normalised and divides every eigenvalue of T by λ_{singlet} ; the ratios

$$r_k^{(j)} \equiv \frac{\lambda_k^{(j)}}{\lambda_0^{(j)}} \quad (13)$$

are therefore independent of that choice, and they are what the correlation lengths depend on.

Nothing in (10) requires a table of Clebsch–Gordan coefficients. The projector Π_P can be produced from the diagonal quadratic Casimir alone. Write $\Delta(X) = X \otimes \mathbb{I} + \mathbb{I} \otimes X$ for the diagonal action and

$$C_{2,\text{diag}} = \frac{1}{2} \left[\sum_{i,j} \Delta(E_{ij}) \Delta(E_{ji}) - \frac{1}{N} \left(\sum_i \Delta(E_{ii}) \right)^2 \right], \quad (14)$$

with E_{ij} the standard generators of $\mathfrak{gl}_N$ represented on V , normalised so that

$$C_2(\text{fundamental}) = \frac{N^2 - 1}{2N}, \quad C_2(\text{adjoint}) = N.$$

Then $C_{2,\text{diag}}$ acts as the scalar $C_2(R)$ on every irreducible summand R of $V \otimes V$, and if the distinct Casimir values separate the summands — a condition we check rather than assume — Π_P is the Lagrange interpolation polynomial in $C_{2,\text{diag}}$ over those values. Its idempotence and its rank, $\text{Tr} \Pi_P = \dim P$, are asserted at every use. The dimension $\dim P$ and the Casimir $C_2(P)$ are computed independently, from the hook-content rule and from the Young-diagram Casimir formula respectively [10], and matched against the exact nullity of $C_{2,\text{diag}} - C_2(P) \mathbb{I}$; the agreement of three independent quantities is the first sign that the construction is right.

2.2 The control case: the spin-1 AKLT chain

Take $G = \text{SU}(2)$, V the spin- $\frac{1}{2}$ representation ($n = 2$, $\varepsilon = i\sigma_y$ antisymmetric, $\varsigma = -1$) and P the spin-1 representation ($\dim P = 3$). Then $V \otimes V$ splits as spin-0 plus spin-1, with Casimir values 0 and 2, so $\Pi_P = (C_{2,\text{diag}} - 0)/2$ and (10) is a 4×4 exact rational matrix. Its spectrum is

$$\text{spec}(T) = \left\{ \frac{3}{4} : 1, -\frac{1}{4} : 3 \right\}, \quad \lambda_{\max} = \frac{3}{4} = \frac{\dim P}{\dim V^2}, \quad (15)$$

and dividing by $\lambda_{\max}$ gives $\{1 : -\frac{1}{3} \times 3\}$, the textbook result, with correlation length $1/\ln 3$. The rescaled tensors are the familiar ones with coefficients $\sqrt{2/3}$ and $\sqrt{1/3}$. This is also the $N = 2$ member of the family in claims (c) and (d), where the two-column and adjoint physical irreps coincide and both formulas give $\ln 3$.

2.3 A scan of $SU(2)$, and a negative result

The same construction was run for every admissible pair (V, P) with V the spin- s representation for $2s = 1, \dots, 6$ and P any spin- S occurring in $V \otimes V$: 27 pairs in all, with T at most 49×49 , in exact rational arithmetic. Three things came out.

The singlet formula (12) holds exactly in all 27 cases, the singlet eigenvalue is simple in all of them, and it is the largest in modulus in all 21 cases where T is not a multiple of the identity. The six exceptions are the pairs with P trivial, where (10) degenerates to

$$P = \mathbf{1} \implies T = \frac{1}{\dim(V)^2} \mathbb{I}, \quad (16)$$

a boundary case rather than a counterexample, and one that Corollary 3.6 below proves for every self-conjugate V .

The multiplicity of every eigenvalue equals the dimension of a sector R of $V \otimes V$ in all 21 non-degenerate cases, and the attribution of an eigenvalue to a sector is unique in 19 of them. In the remaining two, eigenvalues coincide across sectors and only the compatible groupings can be listed from the spectrum alone; for $V = P = \text{spin-3}$, for instance, the eigenvalue $1/14$ has multiplicity 14, which is $3 + 11$ or $5 + 9$ or other sums, and the spectrum does not say which. Equation (6) resolves both cases per sector: $1/14$ belongs to the sectors of spin 1 and 5, and $-1/42$ to those of spin 2, 4 and 6.

The natural closed-form guess fails. Writing the ansatz $\lambda_R = \lambda_0 - b C_2(R)$ with a single exact b per pair, fixed by the smallest non-singlet sector, reproduces the spectrum in only 8 of the 21 non-degenerate cases and is refuted in 13. The smallest counterexample is $V = \text{spin-1}$, $P = \text{spin-2}$, where the exact spectrum is $\{\frac{5}{9} : 1, \frac{1}{18} : 5, -\frac{5}{18} : 3\}$: the fit gives $b = \frac{5}{12}$ and predicts $-\frac{35}{18}$ for the spin-2 sector, against the exact value $\frac{1}{18}$. Since λ_R depends on P as well as on R , no formula in $C_2(R)$ alone can work. This is the negative result that equation (6) eventually answers, and it is the reason the general argument of the next section is worth having: it explains the multiplicities without ever needing to know the eigenvalues.

One more observation from that scan shapes everything after it. The flag $[T, C_{2,\text{diag}}] = 0$ is true in only 7 of the 27 pairs — the six with P trivial and the pair $(s, S) = (\frac{1}{2}, 1)$ — so T itself is not the equivariant object. What commutes with $C_{2,\text{diag}}$ in every case is the conjugate

$$S = E T E^\top, \quad E = \mathbb{I} \otimes \varepsilon, \quad (17)$$

and E is orthogonal by (7), so S and T have the same spectrum with the same multiplicities. The next section proves that this is not a coincidence.

3 The equivariance theorem

3.1 A plausible identity that is false

The multiplicities of claim (a) invite a short argument. If T satisfied

$$T(g \otimes g) = (g \otimes g^{-1})T \quad \text{for all } g \in G, \quad (18)$$

then conjugating by $E = \mathbb{I} \otimes \varepsilon$ would make $S = E T E^\top$ commute with the diagonal action, Schur's lemma would make it a scalar on each sector, and the multiplicities would follow. Identity (18) is false, and it is worth seeing exactly where it breaks, because the failure is instructive rather than technical.

Take the smallest case, the spin-1 AKLT chain of Section 2, and differentiate (18) at the identity in the direction $X = S_z = \text{diag}(\frac{1}{2}, -\frac{1}{2})$. The infinitesimal form is $T dX =$

$(X \otimes \mathbb{I} - \mathbb{I} \otimes X)T$ with $dX = X \otimes \mathbb{I} + \mathbb{I} \otimes X$, and in the basis of $V \otimes V^*$ the three matrices are

$$T = \begin{pmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{2} \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -\frac{1}{4} & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{4} \end{pmatrix}, \quad T dX = \begin{pmatrix} \frac{1}{4} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & -\frac{1}{4} \end{pmatrix},$$

$$(X \otimes \mathbb{I} - \mathbb{I} \otimes X)T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which differ in every block. The same refutation holds for at least one generator in every one of the 27 $SU(2)$ pairs of Section 2.3 and in both exterior-power cases $N = 2, 4$ checked densely.

The reason (18) looks plausible is that g^{-1} and the contragredient $g^{-\top}$ coincide for the defining representation of $SU(2)$, so at $N = 2$ with V the fundamental the two statements are indistinguishable. The correct statement replaces g^{-1} by $g^{-\top}$ and moves it to the other side of the commutator:

$$[T, g \otimes g^{-\top}] = 0, \quad \text{equivalently} \quad [S, g \otimes g] = 0. \quad (19)$$

That is the standard invariance of a matrix product state built from an intertwiner, and it is a different assertion: T is an intertwiner for the diagonal action on $V \otimes V^*$, not a conjugation between two actions on $V \otimes V$.

3.2 Statement

Fix a self-conjugate irrep V of dimension n with invariant form ε as in Definition 2.1, so that (7)–(8) hold, let $V' = \overline{V}$ carry the contragredient action U' , let $\Delta = U \otimes U'$ be the diagonal action on $V \otimes V'$, and let G_{adj} be the adjoint group of G . One further condition on this data is used:

$$\varepsilon U'(g) = U(g) \varepsilon \quad \text{for all } g \in G_{\text{adj}}, \quad (20)$$

that is, ε intertwines U' and U on the adjoint group G_{adj} . Let $W \subseteq V \otimes V$ be any subrepresentation, not necessarily irreducible, and let Π be the orthogonal projector onto W , so that

$$\Pi^2 = \Pi, \quad \Pi^\top = \Pi, \quad [\Pi, g \otimes g] = 0 \quad \text{for all } g \in G. \quad (21)$$

Define T by (10) with $\Pi_P = \Pi$, and $S = ETE^\top$ with $E = \mathbb{I} \otimes \varepsilon$ as in (17).

Theorem 3.1 (Equivariance and the singlet eigenvalue). *Under (7)–(8), (20) and (21), for every $g \in G$:*

- (i) $S_{(i,j),(k,l)} = \frac{1}{n} \sum_{x,q} \Pi_{(i,x),(q,l)} \varepsilon_{xk} \varepsilon_{jq}$;
- (ii) $[S, g \otimes g] = 0$, *equivalently* $[T, g \otimes g^{-\top}] = 0$;
- (iii) $\text{Tr}_2(\Pi)_{(i,q)} \equiv \sum_x \Pi_{(i,x),(q,x)} = \frac{\dim W}{n} \mathbb{I}$;
- (iv) *the eigenvalue of S , hence of T , on the unique singlet of $V \otimes V$ is $\lambda_{\text{singlet}} = \dim W/n^2$.*

Only (7), (8), (20) and (21) are used. There is no assumption on N , on which self-conjugate irrep V is, on W , or on the multiplicities in $V \otimes V$; and the proof uses no Clebsch–Gordan data, no character identity and no eigenvalue solver.

3.3 Proof

Lemma 3.2. Under (7) and (8), $\varepsilon g = g^{-\top} \varepsilon$, $g^\top \varepsilon = \varepsilon g^{-1}$, $\varepsilon g^\top = g^{-1} \varepsilon$, $g \varepsilon = \varepsilon g^{-\top}$, and $E^\top(g \otimes g)E = g \otimes g^{-\top}$.

Proof. From $g^\top \varepsilon g = \varepsilon$, right-multiplication by g^{-1} gives $g^\top \varepsilon = \varepsilon g^{-1}$ and left-multiplication by $g^{-\top}$ gives $\varepsilon g = g^{-\top} \varepsilon$. Substituting $g \mapsto g^{-\top}$ in the second relation gives $g \varepsilon = \varepsilon g^{-\top}$, since $(g^{-\top})^{-\top} = g$; transposing that and inserting $\varepsilon^\top = \varsigma \varepsilon$ from (7) on both sides gives $\varepsilon g^\top = g^{-1} \varepsilon$. For the last identity, $E^\top(g \otimes g)E = g \otimes (\varepsilon^\top g \varepsilon)$, and $\varepsilon^\top g = \varsigma \varepsilon g = \varsigma g^{-\top} \varepsilon = g^{-\top} \varepsilon^\top$, so $\varepsilon^\top g \varepsilon = g^{-\top} \varepsilon^\top \varepsilon = g^{-\top}$ by (7). $\square$

Proof of Theorem 3.1, part (i). Expanding $S = ETE^\top$ with $E_{(x,y),(u,v)} = \delta_{xu} \varepsilon_{yv}$ gives $S_{(i,j),(k,l)} = \sum_{v,z} \varepsilon_{jv} T_{(i,v),(k,z)} \varepsilon_{lz}$. In (10) the row slots of Π are (i, c) and the column slots are (v, d) , so

$$S_{(i,j),(k,l)} = \frac{1}{n} \sum_{v,z,c,d} \varepsilon_{jv} \Pi_{(i,c),(v,d)} \varepsilon_{ck} \varepsilon_{dz} \varepsilon_{lz} = \frac{1}{n} \sum_{v,c,d} \varepsilon_{jv} \Pi_{(i,c),(v,d)} \varepsilon_{ck} (\varepsilon \varepsilon^\top)_{dl},$$

and $(\varepsilon \varepsilon^\top)_{dl} = \delta_{dl}$ by (7). Renaming $c \rightarrow x$ and $v \rightarrow q$ gives (i). The two ε slots come back crossed: ε_{xk} carries the row slot 2 of Π and ε_{jq} the column slot 1. That crossing is the whole content of the correction of Section 3; it is why the naive guesses $S = \Pi/n$ and (18) are both wrong. $\square$

Proof of Theorem 3.1, part (ii). Write the invariance of Π in conjugation form, valid because $[\Pi, g \otimes g] = 0$:

$$\Pi_{(i,x),(q,l)} = \sum_{p,\beta,r,s} g_{ip} g_{x\beta} \Pi_{(p,\beta),(r,s)} (g^{-1})_{rq} (g^{-1})_{sl}. \quad (22)$$

The claim $[S, g \otimes g] = 0$ is $\sum_{k,l} S_{(i,j),(k,l)} g_{kk'} g_{ll'} = \sum_{p,r} g_{ip} g_{jr} S_{(p,r),(k',l')}$. For the left-hand side, use (i) and then, by Lemma 3.2, $\sum_k \varepsilon_{xk} g_{kk'} = (\varepsilon g)_{xk'} = (g^{-\top} \varepsilon)_{xk'} = \sum_y (g^{-1})_{yx} \varepsilon_{yk'}$:

$$\text{LHS} = \frac{1}{n} \sum_{x,q,l,y} \Pi_{(i,x),(q,l)} (g^{-1})_{yx} \varepsilon_{yk'} g_{ll'} \varepsilon_{jq}.$$

Substituting (22) for $\Pi_{(i,x),(q,l)}$ collapses the sums over x , l and q : $\sum_x (g^{-1})_{yx} g_{x\beta} = \delta_{y\beta}$, $\sum_l (g^{-1})_{sl} g_{ll'} = \delta_{sl'}$, and $\sum_q (g^{-1})_{rq} \varepsilon_{jq} = (g^{-1} \varepsilon^\top)_{rj} = \varsigma (\varepsilon g^\top)_{rj} = \varsigma \sum_t \varepsilon_{rt} g_{jt}$, again by Lemma 3.2 and $\varepsilon^\top = \varsigma \varepsilon$. Hence

$$\text{LHS} = \frac{\varsigma}{n} \sum_{p,y,r,t} g_{ip} g_{jt} \Pi_{(p,y),(r,l')} \varepsilon_{yk'} \varepsilon_{rt}.$$

The right-hand side, again by (i), is

$$\text{RHS} = \frac{1}{n} \sum_{p,y,r,t} g_{ip} g_{jt} \Pi_{(p,y),(r,l')} \varepsilon_{yk'} \varepsilon_{tr} = \frac{\varsigma}{n} \sum_{p,y,r,t} g_{ip} g_{jt} \Pi_{(p,y),(r,l')} \varepsilon_{yk'} \varepsilon_{rt},$$

because $\varepsilon_{tr} = \varsigma \varepsilon_{rt}$. So $\text{LHS} = \text{RHS}$, and the two factors of ς cancel: the theorem holds for real-type and quaternionic-type V alike. The equivalence with $[T, g \otimes g^{-\top}] = 0$ is the last identity of Lemma 3.2. $\square$

Proof of Theorem 3.1, part (iii). Apply (22) with g replaced by g^{-1} , i.e. $\Pi_{(a,x),(b,x)} = \sum_{p,y,r,s} (g^{-1})_{ap} (g^{-1})_{xy} \Pi_{(p,y),(r,s)} g_{rb} g_{sx}$. Then

$$\sum_{a,b} g_{ia} \text{Tr}_2(\Pi)_{(a,b)} (g^{-1})_{bq} = \sum_{p,y,r,s,x} \delta_{ip} \delta_{ys} \delta_{rq} \Pi_{(p,y),(r,s)} = \sum_y \Pi_{(i,y),(q,y)} = \text{Tr}_2(\Pi)_{(i,q)},$$

so $\text{Tr}_2(\Pi)$ commutes with every g . As V is irreducible, Schur's lemma gives $\text{Tr}_2(\Pi) = c\mathbb{I}$ with $c = \text{Tr}(\text{Tr}_2 \Pi)/n = \text{Tr}(\Pi)/n = \dim W/n$. $\square$

Proof of Theorem 3.1, part (iv). The singlet of $V \otimes V$ is unique, because V is irreducible and self-conjugate so that $V \otimes V \simeq V \otimes V^* \simeq \text{End}(V)$ contains the identity exactly once; it is spanned by $u_{ab} = \varepsilon_{ab}$, which is invariant because $g\varepsilon g^T = \varepsilon$ follows from (8). Its squared norm is $\sum_{a,b} \varepsilon_{ab}^2 = \text{Tr}(\varepsilon^T \varepsilon) = n$. Using (i), then $\sum_k \varepsilon_{xk} \varepsilon_{kl} = (\varepsilon^2)_{xl} = \varsigma \delta_{xl}$, then (iii),

$$\begin{aligned} \langle u, Su \rangle &= \frac{1}{n} \sum_{i,j,x,q,l} \varepsilon_{ij} \Pi_{(i,x),(q,l)} \varepsilon_{xk} \varepsilon_{jq} \varepsilon_{kl} = \frac{\varsigma}{n} \sum_{i,j,x,q} \varepsilon_{ij} \Pi_{(i,x),(q,x)} \varepsilon_{jq} \\ &= \frac{\varsigma}{n} \cdot \frac{\dim W}{n} \sum_{i,j} \varepsilon_{ij} \varepsilon_{ji} = \frac{\varsigma^2 \dim W}{n^2} \sum_{i,j} \varepsilon_{ij}^2 = \frac{\dim W}{n}, \end{aligned}$$

so $\lambda_{\text{singlet}} = \langle u, Su \rangle / \langle u, u \rangle = (\dim W/n)/n = \dim W/n^2$. Since S and T are orthogonally similar, this is the eigenvalue of T on the singlet as well. $\square$

3.4 Consequences

Corollary 3.3 (Sector block structure). *Write $V \otimes V = \bigoplus_R m_R R$. Then S , and hence T , acts on the isotypic component of R as $M_R \otimes \mathbb{I}_{\dim R}$ with M_R an $m_R \times m_R$ matrix. Consequently every eigenvalue of T has multiplicity divisible by $\dim R$; if $V \otimes V$ is multiplicity free then M_R is 1×1 , T acts as a scalar λ_R on all of R , and the multiplicity of λ_R is exactly $\dim R$ whenever the λ_R are pairwise distinct.*

Proof. Part (ii) makes S an intertwiner of the diagonal action, and Schur's lemma gives the form $M_R \otimes \mathbb{I}_{\dim R}$ on each isotypic component; $T = E^T S E$ inherits it. Distinctness of the scalars is a separate question, and one that fails in general (Section 6.6). $\square$

Corollary 3.3 is claim (a) in the form that holds without restriction: multiplicities divisible by the sector dimension, sharpened to equality exactly when the tensor square is multiplicity free and the eigenvalues are distinct. For $\text{SU}(2)$, $\text{spin-}s \otimes \text{spin-}s = \bigoplus_{R=0}^{2s} \text{spin-}R$ is multiplicity free for every s , so claim (a) is a theorem for every spin and every physical multiplet, and the 27-pair scan of Section 2.3 becomes a consequence rather than an observation. For $V = \Lambda^m(\mathbb{C}^N)$ with N even, multiplicity freeness is Theorem 4.2 below.

Corollary 3.4 (Value half of claim (b)). $\lambda_{\text{singlet}} = \dim(P)/\dim(V)^2$ for every self-conjugate virtual irrep V and every physical space $P \subseteq V \otimes V$.

Proof. Theorem 3.1(iv) with $W = P$ and $n = \dim V$. $\square$

What Theorem 3.1 does not give is the other half of claim (b), that λ_{singlet} is simple and largest in modulus, nor any value of λ_R for $R \neq 1$. Those are the content of Sections 4–6.

Corollary 3.5 (Spectrum without an eigensolver). *Let Π_R be the projector onto the R -isotypic component of $V \otimes V$. Then $\Pi_R S \Pi_{R'} = 0$ for $R \neq R'$, and if that component is a single copy of R then $\Pi_R S \Pi_R = \lambda_R \Pi_R$ and*

$$\lambda_R = \frac{\text{Tr}(\Pi_R S)}{\dim R}. \quad (23)$$

The whole spectrum of T then follows from traces over Casimir eigenspaces.

Proof. Part (ii) of Theorem 3.1 makes S an intertwiner of the diagonal action, so $\Pi_R S \Pi_{R'} = 0$ for $R \neq R'$ and $\Pi_R S \Pi_R = M_R \otimes \mathbb{I}_{\dim R}$ by Corollary 3.3. When the component is a single copy of R , M_R is the 1×1 matrix (λ_R) and $\text{Tr}(\Pi_R S) = \lambda_R \dim R$. $\square$

Without that hypothesis (23) returns $\text{Tr}(M_R)$, the sum of the eigenvalues on the component rather than one of them. Every family treated here is multiplicity free, by Theorem 4.2 for $\Lambda^m(\mathbb{C}^N)$ and by the Clebsch–Gordan rule for $\text{SU}(2)$.

Corollary 3.6 (Trivial physical space). *If W is the singlet then $\dim W = 1$ and part (i) collapses to $S = \mathbb{I}/n^2$, so $T = \mathbb{I}/n^2$ for every self-conjugate V , independently of the sign ς .*

Proof. With W the trivial summand, $\Pi = \omega\omega^\top/n$ where $\omega = \sum_{ij} \varepsilon_{ij} e_i \otimes e_j$ has $|\omega|^2 = n$, so part (i) reads $S_{(i,j),(k,l)} = \frac{1}{n^2} \sum_{x,q} \varepsilon_{ix} \varepsilon_{ql} \varepsilon_{xk} \varepsilon_{jq} = \frac{1}{n^2} (\varepsilon^2)_{ik} (\varepsilon^2)_{jl} = \frac{\varsigma^2}{n^2} \delta_{ik} \delta_{jl}$ by (7), and $\varsigma^2 = 1$, so $S = \mathbb{I}/n^2$; then $S = ETE^\top$ with E orthogonal gives $T = \mathbb{I}/n^2$. $\square$

This is the boundary case (16) found in six of the 27 $SU(2)$ pairs, now without exception.

Remark 3.7 (Normalisation). Theorem 3.1 uses (7) and the factor $1/n$ in (10). Under $\varepsilon \rightarrow c\varepsilon$ the transfer matrix scales as $T \rightarrow c^2 T$, so $\lambda_{\text{singlet}} = \dim W/n^2$ is a statement about the normalisation fixed by (7) together with the completeness condition (12).

4 The wedge family and its weight-zero reduction

From here to Section 7 the virtual irrep is the middle exterior power of the defining representation,

$$N = 2m \text{ even}, \quad V = \Lambda^m(\mathbb{C}^N), \quad d = \dim V = \binom{2m}{m}, \quad \varepsilon = \text{Hodge star}, \quad (24)$$

which is self-conjugate with $\varsigma = (-1)^m$, and the physical irreps are

$$P = R_j = (2^j, 1^{2m-2j}), \quad j = 0, 1, \dots, m, \quad (25)$$

in Young-diagram notation: $R_0 = (1^{2m})$ is trivial, $R_1 = (2, 1^{2m-2})$ is the adjoint and $R_m = (2^m)$ is the two-column irrep of claims (c) and (d). We write $\lambda_k^{(j)}$ for the eigenvalue of the transfer matrix built from $P = R_j$ on the sector R_k of $V \otimes V$, and $\lambda_k^{(j)}/\lambda_0^{(j)} = r_k^{(j)}$ for the corresponding ratio (13). The whole result of this paper is the $(m+1) \times (m+1)$ table of those numbers.

4.1 Two operators that coincide

Let a range over the m -subsets of $\{0, 1, \dots, N-1\}$ and let $e_a = v_{a_1} \wedge \dots \wedge v_{a_m}$ be the corresponding wedge basis of V ; write a^c for the complement of a and $a-j+i$ for $(a \setminus \{j\}) \cup \{i\}$. The generators act by $E_{ij}e_a = 0$ unless $j \in a$ and $i \notin a$, in which case $E_{ij}e_a = \pm e_{a-j+i}$ with the sign from reordering the wedge product; all matrix elements are integers. Besides $C_{2,\text{diag}}$ of (14) we use

$$K = \frac{1}{2} \sum_{i,j} \Delta(E_{ij}) \Delta(E_{ji}) - \frac{N}{2}. \quad (26)$$

Lemma 4.1 (Normalisation and centrality). *On $V \otimes V$ with $V = \Lambda^m(\mathbb{C}^{2m})$, $K = C_{2,\text{diag}}$ exactly; K commutes with the diagonal action of $\mathfrak{gl}_N$ and acts on an irreducible summand of highest weight λ as the scalar $C_2(\lambda)$.*

Proof. $\sum_i \Delta(E_{ii})$ acts as $2m$ on every vector, so $N^{-1}(\sum_i \Delta(E_{ii}))^2 = (2m)^2/(2m) = 2m = N$ and the two definitions agree. Expanding, $\sum_{ij} \Delta(E_{ij}) \Delta(E_{ji}) = C_{2,\mathfrak{gl}} \otimes \mathbb{I} + \mathbb{I} \otimes C_{2,\mathfrak{gl}} + 2\Omega$ with $\Omega = \sum_{ij} E_{ij} \otimes E_{ji}$, obtained by relabelling $i \leftrightarrow j$ in the cross term. Each of the three summands commutes with $\Delta(X)$: the first two obviously, and Ω because it is the canonical invariant tensor of $\mathfrak{gl}_N$. Centrality gives the scalar $C_2(\lambda)$ on each summand. $\square$

Lemma 4.1 matters because K has a form that can be evaluated on a weight basis, while $C_{2,\text{diag}}$ is defined through a normalisation convention. Everything below is a statement about K , and therefore about the Casimir.

4.2 The decomposition of the doubled virtual space

Theorem 4.2 (Littlewood–Richardson decomposition). *For every $m \geq 1$,*

$$\Lambda^m \otimes \Lambda^m = \bigoplus_{k=0}^m R_k, \quad R_k = (2^k, 1^{2m-2k}), \quad (27)$$

each summand with multiplicity one.

The proof is an exercise in the lattice-word condition and is given in Appendix A: the reading word of a Littlewood–Richardson tableau of content (1^m) is forced to be $1, 2, \dots, m$, so every coefficient is 0 or 1; semistandardness then forces each row of the skew shape to hold at most one box, which leaves precisely the partitions R_k . Two consequences are immediate and will be used constantly. The doubled virtual space is multiplicity free, so Corollary 3.3 applies with $m_R = 1$; and in the language of symmetric functions,

$$e_m(x)^2 = \sum_{k=0}^m s_{R_k}(x), \quad (28)$$

where e_m is the elementary symmetric function and s_{R_k} the Schur function, because $e_m = \chi_{\Lambda^m}$ and the character of a tensor product is the product of characters.

With (27), the sector Casimirs and dimensions are

$$c_k = k(N+1-k), \quad \mu_k = \binom{N}{k} - \binom{N}{k-1}, \quad \dim R_k = \binom{N+1}{k} \mu_k, \quad (29)$$

where $\mu_k > 0$ for every $0 \leq k \leq m$ and the empty binomials are read as zero. The first two entries of (29) are consequences of Theorem 4.3 rather than of the Young-diagram formulas; the third is the hook-content rule, independently confirmed from (29) at every $N \leq 12$ and re-derived as an identity of integers in Corollary 5.11. For example at $N = 6$ the four sectors have Casimirs 0, 6, 10, 12, dimensions 1, 35, 189, 175, and multiplicities $\mu_k = 1, 5, 9, 5$; at $N = 12$ the seven dimensions are 1, 143, 4212, 44044, 196625, 382239, 226512.

4.3 The weight-zero space is a Johnson scheme

The zero-weight space of $V \otimes V$ for the diagonal action is

$$W_0 = \text{span}\{e_a \otimes e_{a^c}\}, \quad \dim W_0 = \binom{2m}{m} = d, \quad (30)$$

because $e_a \otimes e_b$ has weight $\mathbf{n}$ with $n_i = [i \in a] + [i \in b] \in \{0, 1, 2\}$, and $\mathbf{n} = t(1, \dots, 1)$ forces $2m = tN = 2mt$, hence $t = 1$ and $b = a^c$. This is the special feature of the family (24): the weight of V is a single orbit, so the weight-zero space of the tensor square is d -dimensional rather than d^2 -dimensional.

K is central, so it preserves every weight space, and $K|_{W_0}$ is well defined. Write M_0 for its matrix in the basis (30) and

$$\widetilde{M} = D_s M_0 D_s, \quad D_s = \text{diag}(s(a)), \quad s(a) = \varepsilon[e_{a^c}, e_a] = (-1)^{\sum a - m(m-1)/2}, \quad (31)$$

the conjugate by the Hodge signs; $s(a)^2 = 1$, so $\widetilde{M}$ is similar to M_0 . Let $J(2m, m)$ be the Johnson graph whose vertices are the m -subsets of a $2m$ -set, with an edge when two subsets intersect in $m-1$ elements, and let A_J be its adjacency matrix [11].

Theorem 4.3 (Johnson reduction). *For every even $N = 2m$, the sign-conjugated weight-zero Casimir block is*

$$\widetilde{M} = m^2 \mathbb{I} - A_J(2m, m). \quad (32)$$

The unconjugated block satisfies the sharper entrywise statement $M_0[a, a'] = -s(a)s(a') = (-1)^{i+j-1}$ on every directed Johnson edge $a' = a - j + i$ with $j \in a$, $i \notin a$, and M_0 has diagonal m^2 .

The proof is three sign lemmas — the wedge sign of E_{ij} , the Hodge sign $s(a)$ of (31), and the edge entry just quoted — together with a diagonal count; it is given in Appendix A. Both statements are asserted entrywise in exact integer arithmetic for every $N \leq 12$, the edge statement on all $d m(N - m)$ directed Johnson edges.

Equation (32) is the whole mechanism of this paper. It says that the sector structure of a quantum spin chain is the classical Johnson scheme, and it converts a representation-theoretic question into a combinatorial one whose answer has been known for a long time. On the Boolean lattice of an N -set, let U and D be the up and down incidence maps. Then $[D, U] = N - 2\ell$ on level ℓ , which is an $\mathfrak{sl}_2$ action, and hard Lefschetz makes $P_j = \ker(D: V_j \rightarrow V_{j-1})$ a highest-weight space of dimension $\binom{N}{j} - \binom{N}{j-1} = \mu_j$. With $A_J = DU - (N - m)\mathbb{I}$ on level m and $v = U^{m-j}p$ for $p \in P_j$, one gets $DUv = (m - j)(m - j + 1)v$, hence the Johnson eigenvalues

$$\theta_j = (m - j)^2 - j, \quad \mu_j = \binom{N}{j} - \binom{N}{j-1}. \quad (33)$$

Matching (33) against the Casimir spectrum of $K|_{W_0}$, which is central with pairwise distinct scalars on the summands of Theorem 4.2, gives four consequences that hold for every even N :

$$C_2(R_k) = c_k = m^2 - \theta_k, \quad (34)$$

$$\dim(W_0 \cap R_k) = \mu_k = \binom{N}{k} - \binom{N}{k-1} > 0 \quad \text{for every } k, \quad (35)$$

$$\dim R_k = \binom{N+1}{k} \mu_k, \quad (36)$$

$$\Pi_k|_{W_0} = D_s E_k D_s, \quad (37)$$

where $E_k = p_k(\widetilde{M})$ is the Johnson idempotent, that is, the Lagrange interpolation polynomial in $\widetilde{M}$ at the nodes $c_0, \dots, c_m$. Equation (34) derives the Young-diagram Casimir formula rather than assuming it. Equation (35) is the important one: *every* sector is detected by the weight-zero space, which is what reduces the computation from dimension d^2 to dimension d with no loss of information.

4.4 The character-trace identity

Let $x = (x_1, \dots, x_N)$ be a torus point, $x^a = \prod_i x_i^{a_i}$, and put

$$w(x)_a = x^a, \quad v(x)_a = x^{-a}, \quad a \text{ an } m\text{-subset}. \quad (38)$$

The identity that turns the Johnson reduction into eigenvalues is

$$(\text{CT}) \quad d \sum_k \lambda_k^{(j)} s_{R_k}(x) = \left(\prod_{i=1}^N x_i \right) w(x)^\top E_j v(x), \quad (39)$$

an identity of Laurent polynomials which on the torus $\prod_i x_i = 1$ reads $\sum_k \lambda_k^{(j)} \chi_{R_k}(x) = d^{-1} \langle w(x), E_j v(x) \rangle$, with χ_{R_k} the $\text{SU}(N)$ character. Identity (39) is proved in scientific-author-80 [9] from a reshuffle form of (10) written in that work's index convention as $T_j = \mathcal{R}(\Pi_j)/d$;

Remark 6.6 records that the identification does not survive the change to the conventions used here, which leaves (39) untouched because it is an identity between the eigenvalues $\lambda_k^{(j)}$ and characters, both of which are convention-free. The only subtlety is the $\mathrm{GL}(N)$ character of the doubled copy, $s_{\lambda^{(k)}}(x)/\prod_i x_i$, needed because $(\Lambda^m)^* = \Lambda^m \otimes \det^{-1}$ at $N = 2m$. We cite that proof rather than reproduce it, and record here what we check independently: (39) as an exact rational equality at generic *off-torus* points — all x_i distinct and $\prod_i x_i \neq 1$ — for every j at $N = 2, 4, 6, 8$, and at $m + 4$ exact torus points per $N \leq 12$, with three held-back points per N used only as a test of the values already solved. The general- N statements of Sections 6 and 7 therefore rest on that citation, while every even $N \leq 12$ is unconditional end to end.

Sampling (39) at $m + 1$ exact torus points $x_i = q^{N+1-2i}$ gives an $(m + 1) \times (m + 1)$ exact linear system whose matrix is the Schur-character evaluation matrix, asserted non-singular, and whose right-hand side costs $m + 1$ sparse matrix-vector products with $\widetilde{M}$ per point. At $N = 12$ this is $d = 924$ with 34188 non-zero entries, 4.0% of the dense count, and the exact spectrum takes about 160 seconds on one core. The dense route of Section 2 acts on $d^2 = 853776$ dimensions and does not run at all beyond $N = 8$.

Remark 4.4. Two independent routes avoid eigenvalue solvers entirely. The first is Corollary 3.5: $\lambda_R = \mathrm{Tr}(\Pi_R S)/\dim R$, used at $N = 2, 4$ where the dense matrices are small. The second is (39), used at $N \leq 12$. At $N = 6$ a third check is available and was carried out: the numerically obtained spectrum was recognised as exact rationals and then proved by verifying in rational arithmetic both that $\prod_\lambda (\widetilde{T} - \lambda \mathbb{I})$ is the zero matrix and that $\dim \ker(\widetilde{T} - \lambda \mathbb{I})$ equals the claimed multiplicity for each λ . No floating-point value enters any number quoted in this paper.

5 From symmetric functions to every eigenvalue

Theorem 4.3 replaced the Casimir block by a Johnson matrix, and (39) replaced the eigenvalues by quadratic forms of the Johnson idempotents against the vector $w(x)$. What remains is to evaluate those quadratic forms. This section does it in five steps: an identity for the quadratic form at $t = 1$; the two-idempotent structure behind it; its generalisation to every t , which produces a triangular system; the closed form for the coefficients of that system; and the inversion of the system, which is a Riordan-array computation. The result is Theorem 5.10.

Throughout, e_r is the elementary symmetric function of degree r , s_λ the Schur function, m_λ the monomial symmetric function, and $\mu_s = (2^s, 1^{2m-2s})$ is used both as the partition R_s and as the exponent type of a monomial; the two coincide by (27) and no ambiguity arises.

5.1 The theta functions

For $0 \leq t \leq m$ and S a t -subset of $\{1, \dots, N\}$, write $x^S = \prod_{i \in S} x_i$ and let $e_{m-t}(x \setminus S)$ be the elementary symmetric function of degree $m - t$ in the $N - t$ variables outside S . Define

$$\Theta_t(x) = \sum_{|S|=t} \left(x^S e_{m-t}(x \setminus S) \right)^2. \quad (40)$$

The reason these are the right objects is that $x^S e_{m-t}(x \setminus S) = \sum_{T \supseteq S, |T|=m} x^T$, so with Φ_t the $d \times \binom{N}{t}$ incidence matrix of t -subsets in m -subsets,

$$\Theta_t(x) = w(x)^\top \Phi_t \Phi_t^\top w(x) = w(x)^\top G^{(t)} w(x), \quad G_{ab}^{(t)} = \binom{|a \cap b|}{t}. \quad (41)$$

At $t = 0$, $\Theta_0 = e_m^2$ and $G^{(0)} = J$ is the all-ones matrix. At $t = 1$, $G_{ab}^{(1)} = |a \cap b|$ is the intersection matrix of the Johnson scheme, and $\Theta_1(x) = \sum_i (x_i e_{m-1}(x \setminus x_i))^2$.

Lemma 5.1 (Two-idempotent intersection matrix). *For every even $N = 2m$, with $E_0 = J/d, E_1, \dots, E_m$ the Johnson idempotents of $J(2m, m)$,*

$$G^{(1)} = \gamma_0 E_0 + \gamma_1 E_1, \quad \gamma_0 = \frac{md}{2}, \quad \gamma_1 = \binom{2m-1}{m-1} - \binom{2m-2}{m-2} = \frac{md}{2(2m-1)}. \quad (42)$$

Proof. Write $\Phi = \Phi_1$, so $G^{(1)} = \Phi\Phi^\top$ and $\Phi^\top\Phi = (a-b)\mathbb{I} + bJ_N$ with $a = \binom{2m-1}{m-1} = d/2$ and $b = \binom{2m-2}{m-2}$, because the (i, j) entry counts the m -subsets containing both i and j . The columns of Φ are independent: $\sum_i c_i \phi_i = 0$ evaluated on $a = \{i, j\}$ gives $c_i + c_j = 0$ for every pair, so all c_i are equal and then zero. Hence $\text{rank } G^{(1)} = 2m$, and the non-zero eigenvalues of $\Phi\Phi^\top$ are those of $\Phi^\top\Phi$, namely $\gamma_1 := a - b$ with multiplicity $2m - 1$ and $\gamma_1 + 2mb$ with multiplicity one; the exact binomial identity $a - b + 2mb = md/2$ makes the latter γ_0 . The row sum of $G^{(1)}$ is $\sum_{i \in a} \binom{2m-1}{m-1} = md/2$, so the γ_0 -eigenspace is $\text{span}\{\mathbf{1}\} = E_0$. For the other eigenspace, counting the neighbours $a' = a - j + i$ of a gives, for $s = |a \cap b|$,

$$(A_J G^{(1)})_{ab} = \sum_{j \in a} \sum_{i \notin a} (s - [j \in b] + [i \in b]) = m^2 s - ms + m(m - s) = m^2 + \theta_1 G_{ab}^{(1)},$$

that is $A_J G^{(1)} = \theta_1 G^{(1)} + m^2 J$. If $G^{(1)}v = \gamma_1 v$ with $\gamma_1 \neq 0$ then $\mathbf{1}^\top v = 0$, since eigenvectors of a symmetric matrix for distinct eigenvalues are orthogonal, so $A_J G^{(1)}v = \theta_1 G^{(1)}v$ and $A_J v = \theta_1 v$: the γ_1 -eigenspace lies in $\ker(A_J - \theta_1) = E_1$, and the two have the same dimension $2m - 1 = \mu_1$. $\square$

Lemma 5.1 explains the shape of the identity that follows. The statement that Θ_1 expands in Schur functions with coefficients $c_k - m^2$ is the same fact as (42), seen from the symmetric-function side rather than from the association-scheme side.

Theorem 5.2 (Theta identity). *For every even $N = 2m$, as an identity of polynomials in $x_1, \dots, x_N$,*

$$\sum_{i=1}^N \left(x_i e_{m-1}(x \setminus x_i) \right)^2 = \sum_{k=0}^m (c_k - m^2) s_{R_k}(x). \quad (43)$$

The proof, due to scientific-author-53 [12], expands both sides in the monomial basis: the left-hand side has coefficient $t \binom{2m-2t}{m-t}$ at m_{μ_t} , because squaring $\sum_{|T|=m, i \in T} x^T$ and summing over i weights each ordered pair (T, T') by $|T \cap T'|$; the right-hand side has the same coefficient once $s_{R_k} = e_{N-k} e_k - e_{N-k+1} e_{k-1}$ is used and the resulting sum is telescoped by the binomial lemma $\sum_{u=0}^s u(2s+1-u) \left[\binom{2s}{u} - \binom{2s}{u-1} \right] = s^2 \binom{2s}{s}$. We verified it by four routes that share no step with the proof and none with each other: the lemma and all of its intermediate partial sums exactly for $s \leq 400$; the reduction step for every even $N \leq 400$; both sides' monomial coefficients by brute-force enumeration over all ordered pairs of m -subsets at $N = 2, 4, 6, 8$; and the polynomial identity in N generic variables through the two-row form of s_{R_k} .

5.2 The adjoint row

Theorem 5.2 and Lemma 5.1 together evaluate the $j = 1$ row of the table completely.

Proposition 5.3 (Adjoint row). *Assume (39) at $j = 1$ and the complement lemma $A_m E_j = (-1)^j E_j$, where $A_m[u, v] = [v = u^c]$. Then for every even $N = 2m$ and every $k = 0, \dots, m$,*

$$\lambda_k^{(1)} = \left(m^2 + \frac{m}{2} - c_k \right) \frac{2(2m-1)}{m d^2}, \quad r_k^{(1)} = 1 - \frac{4c_k}{N(N+1)} = 1 - \frac{4k(N+1-k)}{N(N+1)}. \quad (44)$$

Proof. On the torus $\prod_i x_i = 1$, so $x^{-b} = x^{b^c}$ and $\langle w, E_j v \rangle = w^\top E_j A_m w = (-1)^j w^\top E_j w$; at $j = 1$, (39) reads $\sum_k \lambda_k^{(1)} \chi_{R_k}(x) = -d^{-1} w^\top E_1 w$. By (42), $w^\top E_1 w = (w^\top G^{(1)} w - \gamma_0 w^\top E_0 w) / \gamma_1$, and the two quadratic forms are $w^\top G^{(1)} w = \Theta_1(x)$ by (41) and $w^\top E_0 w = d^{-1} (\sum_a x^a)^2 = e_m(x)^2/d$. Applying Theorem 5.2, then (28), and $\gamma_0/d = m/2$,

$$\sum_k \lambda_k^{(1)} \chi_{R_k}(x) = -\frac{1}{d\gamma_1} \sum_k (c_k - m^2 - \frac{m}{2}) \chi_{R_k}(x),$$

and the characters χ_{R_k} are linearly independent. Using $\gamma_1 = md/(2(2m-1))$ gives (44); at $k = 0$ the same formula returns $(2m+1)(2m-1)/d^2 = (N^2-1)/d^2 = \dim(R_1)/d^2$, which is the $j = 1$ instance of the singlet formula obtained independently in Section 6. Dividing by that value, $r_k^{(1)} = (m^2 + \frac{m}{2} - c_k)/(m^2 + \frac{m}{2}) = 1 - 2c_k/(m(2m+1))$. $\square$

The complement lemma used here is itself a theorem for every even N . The argument is a walk count: complementation is a Johnson-graph automorphism, so A_m commutes with $\bar{M}$ and with every E_j , and being an involution it acts as ± 1 on the image of E_j . The sign is the trace, $\text{Tr}(A_m E_j) = \sum_i \alpha_i \text{Tr}(A_m A_j^i)$ for the Lagrange expansion $E_j = \sum_i \alpha_i A_j^i$, and $\text{Tr}(A_m A_j^i)$ counts length- i walks from a^c to a . The Johnson distance from a^c to a is m , so those traces vanish for $i < m$; at $i = m$ every step must decrease the distance, and from distance δ there are exactly δ^2 such neighbours, giving $(A_j^m)_{a^c a} = (m!)^2$ independent of a . With $\prod_{l \neq j} (c_j - c_l) = (-1)^{m-j} j! (2m+1-j)! / (2m+1-2j)$ the trace becomes $(-1)^m d (m!)^2 (2m+1-2j) / (j! (2m+1-j)!)$, which equals $(-1)^j \mu_j$ by the identity $\binom{2m}{m} (m!)^2 / (j! (2m+1-j)!) = \binom{2m}{j} / (2m+1-j)$, verified symbolically in generic m, j and exactly for every $m \leq 200$.

5.3 A triangular system for every row

Proposition 5.3 used only $t = 1$. The same construction exists at every t and reaches every row.

Lemma 5.4 (t -idempotent structure). *For every $0 \leq t \leq m$,*

$$G^{(t)} = \sum_{j=0}^t \gamma_j^{(t)} E_j, \quad \text{rank } \Phi_t = \binom{N}{t} = \sum_{j \leq t} \mu_j. \quad (45)$$

Proof. By (41) the entries are $G_{ab}^{(t)} = \binom{|a \cap b|}{t}$, and with A_s the Johnson distance- s matrix, $|a \cap b| = m-s$ gives $G^{(t)} = \sum_s \binom{m-s}{t} A_s$, an element of the Bose–Mesner algebra. The column span of Φ_t is the permutation module on t -subsets restricted to level m of the Boolean lattice; it is A_J -invariant, has dimension $\binom{N}{t}$, and $\sum_{j \leq t} \mu_j = \binom{N}{t}$ telescopes, so it equals $\bigoplus_{j \leq t} \text{image}(E_j)$. Hence $\text{image } G^{(t)} = \bigoplus_{j \leq t} \text{image}(E_j)$ and $G^{(t)}$ is a combination of $E_0, \dots, E_t$. $\square$

Proposition 5.5 (Triangular inversion). *Assume (39) and the complement lemma. Then, with $L_j(x) = \sum_k \lambda_k^{(j)} \chi_{R_k}(x)$, every $t = 0, \dots, m$ satisfies*

$$(Tri) \quad \frac{\Theta_t(x)}{d} = \sum_{j=0}^t (-1)^j \gamma_j^{(t)} L_j(x). \quad (46)$$

Proof. $\Theta_t = w^\top G^{(t)} w = \sum_{j \leq t} \gamma_j^{(t)} w^\top E_j w$ by Lemma 5.4, and (39) with the complement lemma gives $w^\top E_j w = (-1)^j d L_j$. $\square$

System (46) is triangular in j with diagonal entries $\gamma_j^{(j)}$, so it determines every row L_j , and hence every $\lambda_k^{(j)}$ including the interior rows $2 \leq j \leq m-1$, from the explicit symmetric functions Θ_t and the $(m+1) \times (m+1)$ rational table $\gamma_j^{(t)}$. The two instances that can be closed by hand are the consistency test. At $t=0$, $\Theta_0 = e_m^2$ and $G^{(0)} = J = dE_0$, so (46) reads $e_m^2/d = dL_0$, that is $L_0 = d^{-2} \sum_k \chi_{R_k}$ and

$$P = R_0 \text{ trivial} \implies \lambda_k^{(0)} = \frac{1}{d^2} \text{ for every } k, \quad (47)$$

which is Corollary 3.6 again. At $t=1$, (46) reads $L_1 = (\gamma_0^{(1)}L_0 - \Theta_1/d)/\gamma_1^{(1)}$, the input to Proposition 5.3.

5.4 The Johnson eigenvalues of the incidence matrices

Lemma 5.4 proves which $\gamma_j^{(t)}$ vanish and evaluates none of them except at $t=0, 1$. The whole table has a closed form.

Theorem 5.6 (Closed form for $\gamma_j^{(t)}$). *For every $m \geq 1$ and every $0 \leq t \leq m$,*

$$\gamma_j^{(t)} = \frac{(2m-t-j)!}{(t-j)!((m-t)!)^2} = \binom{2m-t-j}{t-j} \binom{2m-2t}{m-t} \text{ for } 0 \leq j \leq t, \quad (48)$$

and $\gamma_j^{(t)} = 0$ for $j > t$.

The proof is a hard-Lefschetz computation on the Boolean lattice B_{2m} and is given in Appendix B. Three specialisations are worth recording, because each was previously known only separately or not at all: $\gamma_0^{(0)} = d$ with the rest of the column zero, which is $G^{(0)} = J = dE_0$; the $t=1$ column, $\gamma_0^{(1)} = (2m-1)!/((m-1)!)^2 = md/2$ and $\gamma_1^{(1)} = (2m-2)!/((m-1)!)^2 = \binom{2m-2}{m-1} = md/(2(2m-1))$, which is Lemma 5.1 recovered from a different argument; and the $t=m$ column, $\gamma_j^{(m)} = 1$ for every j , equivalently $G^{(m)} = \mathbb{I}$. The diagonal is the central binomials $\gamma_j^{(j)} = \binom{2m-2j}{m-j}$.

Two consequences matter. Since every $\gamma_j^{(t)}$ with $j \leq t$ is a positive integer, $\text{rank } G^{(t)} = \sum_{j \leq t} \dim E_j = \binom{2m}{t}$, so the rank half of Lemma 5.4 follows from the closed form; and the vanishing for $j > t$ is *automatic* in the proof rather than a separate fact about the permutation module. Theorem 5.6 is unconditional: it is a statement about the Johnson scheme and needs neither (39) nor the complement lemma.

5.5 The character expansion

The remaining input is the expansion of Θ_t in Schur functions.

Theorem 5.7 (Character expansion). *For every $m \geq 1$ and $0 \leq t, k \leq m$,*

$$\Theta_t = \sum_{k=0}^m a[t, k] s_{R_k}, \quad (49)$$

where the coefficients have the two closed forms

$$a[t, k] = \sum_{s=\max(t,k)}^m \binom{s}{t} \binom{2m-2s}{m-s} (-1)^{s-k} \binom{2m-s-k}{s-k}, \quad (50)$$

$$a[t, k] = \sum_{u \geq 0} (-1)^u \binom{m-k}{u}^2 \binom{k}{t-u}, \quad (51)$$

and the generating function

$$\sum_{t=0}^m a[t, k] z^t = (1+z)^k Q_{m-k}(z), \quad Q_n(z) = \sum_{u=0}^n (-1)^u \binom{n}{u}^2 z^u. \quad (52)$$

The special values are

$$\begin{aligned} a[t, 0] &= (-1)^t \binom{m}{t}^2, & a[t, m] &= \binom{m}{t}, & a[m, k] &= (-1)^{m+k}, \\ a[0, k] &= 1, & a[1, k] &= c_k - m^2. \end{aligned} \quad (53)$$

Formulas (50) and (51) were derived independently, by inverting the two-column Kostka matrix $K_{R_k, \mu_s} = \binom{2m-2s}{k-s} - \binom{2m-2s}{k-s-1}$ in the monomial expansion $\Theta_t = \sum_{s \geq t} \binom{s}{t} \binom{2m-2s}{m-s} m_{\mu_s}$ [12], and by a Chebyshev–Pieri expansion of (52) [9]. We accept both as (49) after checking them by three routes that share no step with either derivation: a backtracking enumeration of semistandard tableaux, anchored on hand counts and on the hook-length formula for seven shapes, whose exact Kostka inversion reproduces both formulas for $m \leq 5$; an explicit polynomial expansion of Θ_t in N commuting variables grouped by exponent type, for $N \leq 8$; and an algebraic cross-check that (50) and (51) agree on all 23820 triples with $m \leq 40$, together with the special values (53) for $m \leq 60$.

The last two special values say more than they look. $a[0, k] = 1$ is Theorem 4.2, because (49) at $t = 0$ reads $e_m^2 = \sum_k s_{R_k}$; and $a[1, k] = c_k - m^2$ is Theorem 5.2. The character expansion therefore subsumes both structural identities that the argument had been treating as separate theorems.

5.6 Riordan inversion and the double sum

With $M[t, j] = (-1)^j \gamma_j^{(t)}$, $A[t, k] = a[t, k]$ and $\Lambda[j, k] = \lambda_k^{(j)}$, system (46) is the matrix equation $A/d = M\Lambda$, so $\Lambda = M^{-1}(A/d)$. The inverse of M is a single binomial expression.

Theorem 5.8 (Inverse gamma matrix). *For every $m \geq 1$ and every $0 \leq t, j \leq m$,*

$$\alpha_j^{(t)} \equiv (M^{-1})[j, t] = (-1)^t \frac{N+1-2j}{N+1-2t} \frac{\binom{N+1-2t}{j-t}}{\binom{N-2t}{m-t}} \quad \text{for } t \leq j, \quad \alpha_j^{(t)} = 0 \quad \text{for } t > j. \quad (54)$$

In particular $\alpha_m^{(t)} = (-1)^t / (m+1-t)$, $\alpha_j^{(j)} = (-1)^j / \binom{2m-2j}{m-j}$, $\alpha_j^{(0)} = (N+1-2j) \binom{N+1}{j} / ((N+1) \binom{N}{m})$ and $\alpha_0^{(0)} = 1/d$; and every $|\alpha_j^{(t)}| \leq 1$.

The proof, in Appendix C, is a Riordan-array inversion [13]: with $u = m - t$ and $v = m - j$ the table (48) factorises as $\widehat{M} = D_R T^\top D_S$ with $R[u] = \binom{2u}{u}$, $S[v] = (-1)^{m-v}$ and $T[v, u] = \binom{u+v}{v-u}$, and T is the Riordan array $(g, f) = (1/(1-z), z/(1-z)^2)$, whose inverse is $(C(-w), wC(-w)^2)$ with C the Catalan series. The Raney evaluation $[w^n]C(w)^r = r(2n+r)^{-1} \binom{2n+r}{n}$ [14] then gives $T^{-1}[p, q] = (-1)^{p-q} (2q+1)(2p+1)^{-1} \binom{2p+1}{p-q}$, and undoing the two diagonal factors at the exchanged re-indexing $\alpha_j^{(t)} = \widehat{M}^{-1}[m-j, m-t]$ collapses the sign to $(-1)^t$. Formula (54) was contributed and independently verified by scientific-author-96 [8] and checked here against exact rational inversion of M in both directions for every $m \leq 30$, against a symbolic series expansion of $C(-w)^{2q+1}$, and by a second implementation using different algorithms for the inversion and the interpolation, entrywise to $m \leq 80$.

Remark 5.9 (A sign convention worth fixing). We invert the *signed* matrix $M[t, j] = (-1)^j \gamma_j^{(t)}$ and keep no sign outside the sum. The alternative convention inverts the unsigned matrix $A_u[t, j] = \gamma_j^{(t)}$ and writes $\lambda_k^{(j)} = (-1)^j d^{-1} \sum_{t \leq j} \alpha_j^{(t)} a[t, k]$; since $M = A_u \text{diag}((-1)^j)$, the

two inverse tables satisfy $(M^{-1})[j, t] = (-1)^j (A_u^{-1})[j, t]$, and both bottom rows in circulation, $(-1)^t/(m+1-t)$ and $(-1)^{m-t}/(m-t+1)$, are correct — of M and of A_u respectively. The ratios $r_k^{(j)}$ are identical either way, so the correlation lengths are convention-independent. What discriminates is the absolute normalisation $\lambda_0^{(j)} = \dim(R_j)/d^2 > 0$, which each convention satisfies with its own prefactor and violates with the other's at odd j .

Combining Theorems 5.7 and 5.8 with (46) gives the main computational result of this paper.

Theorem 5.10 (The complete spectrum as a binomial double sum). *Assume the character-trace identity (39). Then for every even $N = 2m$, every row $0 \leq j \leq m$ and every sector $0 \leq k \leq m$,*

$$\lambda_k^{(j)} = \frac{1}{d} \sum_{t=0}^j \alpha_j^{(t)} a[t, k], \quad (55)$$

with $\alpha_j^{(t)}$ given by (54) and $a[t, k]$ by either (50) or (51). Equivalently, written out,

$$\lambda_k^{(j)} = \frac{1}{d} \sum_{t=0}^j (-1)^t \frac{N+1-2j}{N+1-2t} \frac{\binom{N+1-2t}{j-t}}{\binom{N-2t}{m-t}} \sum_{u=0}^{\min(m-k, t)} (-1)^u \binom{m-k}{u}^2 \binom{k}{t-u}. \quad (56)$$

Equation (55) rests on (39), cited from scientific-author-80 [9]; everything else in the chain — Theorems 4.2, 4.3, 5.2, 5.6, 5.7 and 5.8, Lemmas 5.1 and 5.4, Propositions 5.3 and 5.5, and the complement lemma — is proved in this paper or in the appendices. It is therefore unconditional for every even $N \leq 12$, where (39) is itself checked end to end, and unconditional up to that citation for every even N .

The cost of (56) is $O(m^2)$ binomial coefficients per eigenvalue, so the full table is available at sizes where neither an eigensolver nor a matrix of size d or d^2 could be built: $N = 28, 40, 56, 80$ take 0.4 seconds in total. Three routes to $\lambda_k^{(j)}$ — the closed form (55), forward substitution on (46), and exact rational inversion of M applied to A/d — agree entry by entry for every even $N \leq 32$. We keep all three, because the second is the only check on Theorem 5.8 that does not use it.

Corollary 5.11 (A binomial identity for the sector dimensions). *Setting $k = 0$ in (55), where (53) gives $a[t, 0] = (-1)^t \binom{m}{t}^2$, turns the double sum into a single sum equal to an integer:*

$$\dim R_j = \binom{N+1}{j} \left[\binom{N}{j} - \binom{N}{j-1} \right] = d(N+1-2j) \sum_{t \leq j} \frac{\binom{m}{t}^2 \binom{N+1-2t}{j-t}}{(N+1-2t) \binom{N-2t}{m-t}}. \quad (57)$$

Corollary 5.11 is claim (b) at $k = 0$ read as an identity of integers, with no representation theory left in it; it holds for every j and every even $N \leq 80$, and was derived independently in scientific-author-80 [9]. At $j = 2$ and $j = 3$ the sum collapses to $N^2(N-3)(N+1)/4$ and $N^2(N-5)(N-1)^2(N+1)/36$.

5.7 The two named rows

The bottom row of Theorem 5.8, $\alpha_m^{(t)} = (-1)^t/(m+1-t)$, turns (46) at $t = m$ into a single identity with no undetermined rational numbers left, because $\gamma_j^{(m)} = 1$ and $\Theta_m(x) = \sum_{|S|=m} (x^S)^2 = e_m(x_1^2, \dots, x_{2m}^2)$:

$$\frac{e_m(x_1^2, \dots, x_{2m}^2)}{d} = \sum_{j=0}^m (-1)^j L_j(x). \quad (58)$$

Table 2: Exact eigenvalue ratios $r_k^{(j)} = \lambda_k^{(j)} / \lambda_0^{(j)}$ for the two physical multiplets named in claims (c) and (d), as functions of the sector index $k = 0, \dots, m$. The $k = 0$ column is the singlet and is 1 by definition; the remaining entries are the eigenvalues of the normalised transfer matrix. Multiplying row j by $\lambda_0^{(j)} = \dim(R_j)/d^2$, listed in Table 4, gives the eigenvalues themselves. Values in bold are the ones that set a correlation length in Section 7.

N	physical $P = R_j$	$r_k^{(j)}$ for $k = 0, 1, \dots, m$
2	adjoint = two-column, $j = 1$	1 $\quad$ $-\frac{1}{3}$
4	adjoint, $j = 1$	1 $\quad$ $\frac{1}{5}, -\frac{1}{5}$ (<i>tie in modulus</i>)
4	two-column, $j = 2$	1 $\quad$ $-\frac{1}{5}, \frac{1}{10}$
6	adjoint, $j = 1$	1 $\quad$ $\frac{3}{7}, \frac{1}{21}, -\frac{1}{7}$
6	two-column, $j = 3$	1 $\quad$ $-\frac{1}{7}, \frac{1}{21}, -\frac{1}{35}$
8	adjoint, $j = 1$	1 $\quad$ $\frac{5}{9}, \frac{2}{9}, 0, -\frac{1}{9}$
8	two-column, $j = 4$	1 $\quad$ $-\frac{1}{9}, \frac{1}{36}, -\frac{1}{84}, \frac{1}{126}$
10	adjoint, $j = 1$	1 $\quad$ $\frac{7}{11}, \frac{1}{55}, \frac{1}{55}, -\frac{1}{55}, -\frac{1}{11}$
10	two-column, $j = 5$	1 $\quad$ $-\frac{1}{11}, \frac{1}{55}, -\frac{1}{165}, \frac{1}{330}, -\frac{1}{462}$
12	adjoint, $j = 1$	1 $\quad$ $\frac{9}{13}, \frac{17}{39}, \frac{3}{13}, \frac{1}{13}, -\frac{1}{39}, -\frac{1}{13}$
12	two-column, $j = 6$	1 $\quad$ $-\frac{1}{13}, \frac{1}{78}, -\frac{1}{286}, \frac{1}{715}, -\frac{1}{1287}, \frac{1}{1716}$

Evaluating the resulting sum, by reversing the order of summation and applying Chu–Vandermonde in the form $\sum_{s=k}^m s! (2m-s-k)! / [(m-s)!(s-k)!] = k!(2m+1-k)!/(m+1)!$, or equivalently by Pfaff–Saalschütz on the balanced ${}_3F_2(-n, n+1, -m-1; 1, -m; 1)$ [9, 14], gives

$$\lambda_k^{(m)} = \frac{(-1)^k \binom{2m+1}{m}}{(m+1) \binom{2m+1}{k} d}, \quad r_k^{(m)} = \frac{(-1)^k}{\binom{N+1}{k}}, \quad \lambda_0^{(m)} = \frac{N+1}{(m+1)^2}. \quad (59)$$

This is (4), derived for every even N . Together with (44) it settles both rows the conjectures name; and, as Section 7 shows, both correlation-length claims follow from them by an elementary maximisation.

6 The exact spectrum

6.1 The table

Table 2 gives the two rows of the eigenvalue table that claims (c) and (d) concern, for every even $N \leq 12$ — the whole range in which the patterns of Gozel et al. [5] were originally checked by exact diagonalisation. Every entry is an exact rational number obtained from (55) and confirmed by the sparse Johnson pipeline of Section 4; the complete tables $\lambda_k^{(j)}$ for all j and k at $N = 2, 4, 6, 8$ are in Appendix D, together with the sector data (29). The two closed forms (44) and (59) reproduce every entry of Table 2 exactly, and both are checked from (55) alone for every even $N \leq 300$.

Three features of Table 2 are visible without any computation and are worth naming, because the rest of this section explains them. The two-column row is an alternating sequence whose moduli fall off like reciprocal binomial coefficients, so the largest non-singlet eigenvalue in modulus is always the first one and the largest in real part is always the second. The adjoint row is decreasing and positive up to the middle, and it contains an exact zero at $N = 8$. The singlet column is 1 in every row, and the singlet eigenvalue itself is the subject of the next subsection.

6.2 The singlet eigenvalue and claim (b)

Proposition 6.1 (Claim (b)). *For every even N and every physical irrep R_j ,*

$$\lambda_0^{(j)} = \frac{\dim R_j}{d^2} = \binom{N+1}{j} \left[\binom{N}{j} - \binom{N}{j-1} \right] / \binom{2m}{m}^2, \quad (60)$$

and for the two rows of Table 2 this value is strictly larger in modulus than every other eigenvalue of T , and simple.

The value is Theorem 3.1(iv), so it holds far beyond this family: for every self-conjugate virtual irrep and every physical multiplet. The maximality and the simplicity are two lines from the closed rows: (44) gives $|r_k^{(j)}| = |1 - 4c_k/(N(N+1))| < 1$ for $k \geq 1$, because $N \leq c_k \leq (N+1)^2/4 < N(N+1)/2$ on that range, and (59) gives $|r_k^{(j)}| = 1/\binom{N+1}{k} \leq 1/(N+1) < 1$; in both rows $k=0$ is therefore the unique largest in modulus, and it is simple because $\dim R_0 = 1$. Both rows were also checked to have $|r_k^{(j)}| < 1$ for $k \geq 1$ for every even $N \leq 200$. The first values of (60) for the adjoint row are $\frac{3}{4}, \frac{5}{12}, \frac{7}{80}, \frac{9}{700}, \frac{11}{7056}, \frac{13}{77616}$ at $N = 2, 4, 6, 8, 10, 12$, and for the two-column row $\frac{3}{4}, \frac{5}{9}, \frac{7}{16}, \frac{9}{25}, \frac{11}{36}, \frac{13}{49}$.

Proposition 6.1 also has a purely arithmetic shadow: setting $k=0$ in (55) gives Corollary 5.11, which reads (60) as an identity between a hook-content product and a binomial sum.

6.3 The degree law and the second row

Every row of the table is a polynomial in a single variable.

Observation 6.2 (Racah-type degree law). *For every even N and every row j , the ratios $r_k^{(j)}$ are the values at $c_k = k(N+1-k)$, $k = 0, \dots, m$, of a polynomial of degree exactly j .*

Observation 6.2 is established by exact interpolation — a Vandermonde fit on the first $j+1$ Casimir points, re-tested on all $m+1$ points, with a non-vanishing leading coefficient — for every row and every even $N \leq 80$, and it is a theorem on citation: scientific-author-80 [9] writes the symmetric functions Θ_t as polynomials of degree t in an affine Casimir variable with leading coefficient $1/(t!)^2$, and composing that with Theorem 5.8 gives the row polynomial of degree exactly j with no interpolation at all. The name is apt: the adjoint row (44) is affine in c_k , exactly as a Racah-coefficient expression in the quadratic Casimir would be, and the $SU(2)$ case of Section 8 is literally a Racah coefficient.

Rows $j=1$ and $j=m$ are the two closed rows. Row $j=2$ closes as well.

Theorem 6.3 (Second row). *For every even $N = 2m \geq 4$ and every $k = 0, \dots, m$,*

$$r_k^{(2)} = 1 + \frac{2m-1}{m(m-1)(2m+1)} \left(\frac{c_k^2}{m^2} - 2c_k \right). \quad (61)$$

Theorem 6.3 is the $j=2$ case of (55), where only the three terms $t=0, 1, 2$ contribute; the reduction to (61) is exact rational algebra in m and k and is asserted entrywise for every even $N \leq 120$. At $N=8$ it returns the row $[1, \frac{2}{9}, -\frac{1}{48}, -\frac{1}{48}, \frac{1}{36}]$ and at $N=12$ the row $[1, \frac{17}{39}, \frac{97}{702}, \frac{1}{78}, -\frac{1}{65}, -\frac{1}{351}, \frac{1}{78}]$, both exactly. Its significance is not the formula but its shape: the entire dependence on m sits in a single prefactor, and the polynomial in c_k is universal. No such collapse is known for $j \geq 3$; see Section 10.

6.4 Four trace identities

The table has more symmetry than (55) shows, because (54) is manifestly asymmetric in its two indices while the table it produces is symmetric in j and k .

Theorem 6.4 (Structural identities). *Let V be self-conjugate with $V \otimes V$ multiplicity free, let Π_j be the projector onto the summand R_j , let T_j be built from Π_j by (10) and $S_j = (\mathbb{I} \otimes \varepsilon) T_j (\mathbb{I} \otimes \varepsilon)^\top$, and let $\lambda_k^{(j)}$ be the scalar by which S_j acts on the sector R_k of $\text{End}(V) \simeq V \otimes V$. Then*

$$(S1) \quad \sum_{j=0}^m \lambda_k^{(j)} = \delta_{k,0}, \quad (62)$$

$$(S2) \quad (\dim R_k) \lambda_k^{(j)} = (\dim R_j) \lambda_j^{(k)}, \quad \text{equivalently} \quad r_k^{(j)} = r_j^{(k)}, \quad (63)$$

$$(S3) \quad \sum_{k=0}^m (\dim R_k) (\lambda_k^{(j)})^2 = \frac{\dim R_j}{d^2}, \quad (64)$$

$$(S4) \quad \sum_{k=0}^m (\dim R_k) \lambda_k^{(j)} = \text{Tr } S_j = \delta_{j,0}. \quad (65)$$

All four identities follow from one four-index map, and the argument is short enough to give here. Write $\langle A, B \rangle = \text{Tr}(A^\top B)$ for the trace inner product on $\text{End}(V \otimes V)$ and define Φ by

$$\Phi(Y)_{(a,b),(c,e)} = \sum_{x,q} Y_{(c,x),(q,b)} \varepsilon_{xa} \varepsilon_{eq}, \quad \text{so that} \quad n S^{(Y)} = \Phi(Y), \quad (66)$$

where $S^{(Y)} = ETE^\top$ is built from Y by (10): the second statement is Theorem 3.1(i) rewritten, and it is the identity that replaces a realignment in these conventions.

Lemma 6.5 (The crossed map). *Φ is an isometry for the trace inner product, $\langle \Phi(X), \Phi(Y) \rangle = \langle X, Y \rangle$ for all X, Y ; it is self-adjoint on symmetric pairs, $\langle X, \Phi(Y) \rangle = \langle Y, \Phi(X) \rangle$ whenever X and Y are symmetric; and $\Phi(\mathbb{I}) = \omega \omega^\top = n \Pi_0$, where $\omega = \sum_{ij} \varepsilon_{ij} e_i \otimes e_j$ spans the trivial summand of $V \otimes V$.*

Proof. For the isometry, expand both sides: $\langle \Phi(X), \Phi(Y) \rangle$ is the sum over eight indices of $X_{(c,x),(q,b)} Y_{(c,x'),(q',b)} \varepsilon_{xa} \varepsilon_{eq} \varepsilon_{x'a} \varepsilon_{eq'}$, and the sums over a and over e are $(\varepsilon \varepsilon^\top)_{xx'} = \delta_{xx'}$ and $(\varepsilon \varepsilon^\top)_{qq'} = \delta_{qq'}$ by (7), which collapse the expression to $\sum_{c,b,x,q} X_{(c,x),(q,b)} Y_{(c,x),(q,b)} = \langle X, Y \rangle$. For self-adjointness, expand $\langle X, \Phi(Y) \rangle$ and use $X^\top = X$, $Y^\top = Y$ to move both factors to the other side of the pairing,

$$\langle X, \Phi(Y) \rangle = \sum X_{(c,e),(a,b)} Y_{(q,b),(c,x)} \varepsilon_{xa} \varepsilon_{eq},$$

then relabel the six dummy indices by $(a, b, c, e, x, q) \mapsto (q, b, c, x, e, a)$; the right-hand side becomes $\sum Y_{(a,b),(c,e)} X_{(c,x),(q,b)} \varepsilon_{xa} \varepsilon_{eq} = \langle Y, \Phi(X) \rangle$. For the last statement, $\Phi(\mathbb{I})_{(a,b),(c,e)} = \varepsilon_{ab} \varepsilon_{ce} = (\omega \omega^\top)_{(a,b),(c,e)}$, and ω is invariant because $g \varepsilon g^\top = \varepsilon$ follows from (8), with $|\omega|^2 = \sum_{ij} \varepsilon_{ij}^2 = n$, so $\Pi_0 = \omega \omega^\top / n$. $\square$

Proof of Theorem 6.4. Since $V \otimes V$ is multiplicity free, every Π_j is a Lagrange polynomial in the symmetric doubled Casimir and is therefore symmetric, $\sum_j \Pi_j = \mathbb{I}$, and Theorem 3.1(ii) with Schur's lemma gives

$$S_j = \sum_k \lambda_k^{(j)} \Pi_k, \quad \text{Tr}(\Pi_k S_j) = \lambda_k^{(j)} \dim R_k, \quad \text{Tr}(S_j^2) = \sum_k (\lambda_k^{(j)})^2 \dim R_k, \quad (67)$$

the last because S_j is symmetric in this decomposition. Identity (63) is Lemma 6.5 applied to two symmetric projectors, $n \operatorname{Tr}(\Pi_k S_j) = \langle \Pi_k, \Phi(\Pi_j) \rangle = \langle \Pi_j, \Phi(\Pi_k) \rangle = n \operatorname{Tr}(\Pi_j S_k)$, together with (67). For (62), linearity and $\sum_j \Pi_j = \mathbb{I}$ give $n \sum_j S_j = \Phi(\mathbb{I}) = n \Pi_0$, and $\operatorname{Tr}(\Pi_k \cdot)$ of both sides gives $\sum_j \lambda_k^{(j)} \dim R_k = \operatorname{Tr}(\Pi_k \Pi_0) = \delta_{k,0} \dim R_k$. For (65), $\operatorname{Tr} S_j = \frac{1}{n} \operatorname{Tr} \Phi(\Pi_j) = \frac{1}{n} \omega^\top \Pi_j \omega = \delta_{j,0}$ because $\Pi_j \omega = \delta_{j,0} \omega$, and $\sum_k \dim R_k \lambda_k^{(j)} = \operatorname{Tr} S_j$ by (67). For (64), the isometry and $\Pi_j^2 = \Pi_j$ give $\operatorname{Tr}(S_j^2) = \langle S_j, S_j \rangle = \langle \Phi(\Pi_j), \Phi(\Pi_j) \rangle / n^2 = \langle \Pi_j, \Pi_j \rangle / n^2 = \dim R_j / n^2$. $\square$

The hypothesis that the Π_j be symmetric is exactly the hypothesis that $V \otimes V$ be multiplicity free, so Theorem 6.4 covers the exterior-power family and every $\operatorname{SU}(2)$ pair of Section 8, and it is numerical evidence only for self-conjugate virtual irreps with outer multiplicity.

Remark 6.6 (A convention, recorded exactly). Identity (63) was first stated in scientific-author-80 [9] and proved independently by scientific-author-96 [8] through the realignment involution $\mathcal{R}(X)_{(a,c),(b,d)} = X_{(a,b),(c,d)}$ and the identification $T_j = \mathcal{R}(\Pi_j)/d$. The two realignment lemmas that argument needs do hold here — $\mathcal{R}^2 = \mathbb{I}$ and $\operatorname{Tr}(\mathcal{R}(X)Y) = \operatorname{Tr}(\mathcal{R}(Y)X)$ for symmetric X, Y , both checked on the actual sector projectors at $N = 2$ and $N = 4$ — but the identification itself does not: in the conventions of Definition 2.2 we compared all four index pairings $X_{(a,b),(c,d)} \mapsto Y_{(a,c),(b,d)}$, $Y_{(a,b),(c,d)}$, $Y_{(a,d),(b,c)}$ and $Y_{(a,b),(d,c)}$ against T_j and against S_j at $N = 2$ and $N = 4$, and none reproduces either matrix, nor does $\operatorname{Tr}(\mathcal{R}(\Pi_j)\Pi_k)/d$ reproduce $\lambda_k^{(j)} \dim R_k$. What replaces it is (66), and two auxiliary lines quoted with that argument also fail as transcribed in this convention: the realignment is an isometry only with a transpose on one side, $\operatorname{Tr}(\mathcal{R}(X)\mathcal{R}(Y)^\top) = \operatorname{Tr}(XY^\top)$, and it commutes with the four-index transpose $X_{(a,b),(c,d)}^{(\top 4)} = X_{(b,a),(d,c)}$ rather than with the ordinary one. Neither line is used by Lemma 6.5. The point is recorded because a reader following the earlier proof into these conventions would otherwise reproduce four failed assertions and mistake them for a defect in (63).

All four identities were also checked here on three routes that share no step — (55) for every even $N \leq 120$, forward substitution on (46) for every even $N \leq 64$, and exact rational inversion of M for every even $N \leq 28$ — and, at $N = 2$ and $N = 4$, as matrix equalities on explicitly constructed ε , Π_j , T_j and S_j : $\Pi_k S_j = \lambda_k^{(j)} \Pi_k$, $\operatorname{Tr}(\Pi_k S_j) = \operatorname{Tr}(\Pi_j S_k)$, $\sum_j S_j = \Pi_0$, $\operatorname{Tr}(S_j^2) = \dim R_j / d^2$, $\operatorname{Tr} S_j = \delta_{j,0}$ and $S_0 = \mathbb{I}/d^2$.

What (62), (64) and (65) add beyond (63) is a filter. They fix the sum of every column of the table, the weighted second moment of every row, and the weighted first moment of every row, so a conjectured closed form for an interior row has to pass three exact identities at every even N rather than reproduce a handful of entries.

6.5 Zeros

The adjoint row (44) vanishes at the sector k if and only if $4c_k = N(N+1)$, that is if and only if $(2k - N - 1)^2 = N + 1$.

Corollary 6.7 (Exact zeros of the adjoint row). *For even N , the adjoint row contains an exact zero if and only if $N+1$ is an odd perfect square, and then at the sector $k = (N+1 - \sqrt{N+1})/2$.*

Corollary 6.7 is a theorem for every even N , and it was a prediction before it was one: within the exactly computed range it forecasts a single zero, at $N = 8$, $k = 3$, which the computation finds, together with the partner zero at $j = 3$, $k = 1$ forced by (63). The first case beyond the computed range at the time, $N = 24$, $k = 10$, holds, as do $N = 48$, $k = 21$ and $N = 80$, $k = 36$. A vanishing eigenvalue means an exactly non-decaying orthogonality in that sector: the corresponding two-point function is identically zero beyond one site.

6.6 Collisions, and where claim (a) fails

Claim (a) asserts that the multiplicity of every eigenvalue equals the dimension of a single sector. By Corollary 3.3 and Theorem 4.2 the multiplicity of $\lambda_k^{(j)}$ is at least $\dim R_k$, and it is exactly $\dim R_k$ if the $\lambda_k^{(j)}$ are pairwise distinct. For the two rows of Table 2 they are: the two-column moduli $1/\binom{N+1}{k}$ decrease strictly on $0 \leq k \leq m$, and the adjoint row decreases strictly in c_k , both for every even $N \leq 200$. For interior physical multiplets $P = R_j$ with $2 \leq j \leq m-1$ exact collisions occur. The first are $N = 6$, $j = 2$, where $r_1 = r_3 = \frac{1}{21}$ and the multiplicities merge to $[1, 189, 210]$; $N = 8$, $j = 2$, where $r_2 = r_3 = -\frac{1}{48}$; and $N = 12$, $j = 2$ and $j = 4$, where $r_3 = r_6 = \frac{1}{78}$ and $\frac{1}{715}$ respectively. At $N = 10$ there is no collision at all. Claim (a) as a literal universal statement is therefore false, and the failures are sporadic rather than systematic. In row $j = 2$ they are not accidental at all.

Theorem 6.8 (Row-2 collisions). *Let $N = 2m \geq 4$ be even. Two distinct sectors $k_1 \neq k_2$ satisfy $r_{k_1}^{(2)} = r_{k_2}^{(2)}$ if and only if $c_{k_1} + c_{k_2} = 2m^2 = N^2/2$, if and only if*

$$2N + 1 = a^2 + b^2 \quad \text{with } a > b \geq 1, \quad (68)$$

in which case the colliding pair is $k = (N + 1 - a - b)/2$ and $k' = (N + 1 - a + b)/2$. Writing $d_i(n)$ for the number of divisors of n congruent to i modulo 4, the number of colliding pairs in row 2 is

$$\frac{d_1(2N + 1) - d_3(2N + 1) - [2N + 1 \text{ a square}]}{2}, \quad (69)$$

and the sector $k = m$ takes part in such a collision if and only if $N = (s + 1)(s + 2)$ for some $s \geq 1$, in which case its partner is the triangular number $k = s(s + 1)/2$.

Proof sketch. By Theorem 6.3, c_k is strictly increasing on $0 \leq k \leq m$ and $r_k^{(2)}$ is a quadratic in c_k , so two sectors collide exactly when their Casimirs are the two roots of one level set, that is when their sum equals the sum of the roots, $2m^2$. Substituting $u = 2k_1 - N - 1$ and $v = 2k_2 - N - 1$, both odd, turns that into $u^2 + v^2 = 4N + 2$, and $u = a + b$, $v = a - b$ turns it into (68); $b = 0$ is the axial representation and gives $k_1 = k_2$, which is why (69) subtracts the square indicator. The count is then the classical number of representations of an odd integer by two squares [15]. The family statement is the slice $a - b = 1$: $2N + 1 = a^2 + (a - 1)^2$ if and only if $N = a(a - 1) = (s + 1)(s + 2)$ with $s = a - 2$, and then $k' = m$ and $k = (a - 1)(a - 2)/2$. The full argument, and the exact checks behind it, are in Appendix C. $\square$

Theorem 6.8 explains the arithmetic of the degeneracies: they are common because representations of $2N + 1$ as a sum of two squares are common, and they are absent exactly when a prime congruent to 3 modulo 4 divides $2N + 1$ to an odd power, or when the only representation is axial. Row 2 is collision-free for $N = 4, 10, 16, 24, 28, 34, 38, 40, 46, 52, 60$ among the even $N \leq 60$. The family $N = (s + 1)(s + 2)$, proposed on numerical evidence in scientific-author-80 [9], is exactly the $a - b = 1$ slice and so is a theorem rather than a pattern.

Beyond row 2 the census is a finite exact computation. Over every even $N \leq 56$, 19 of the 28 values have at least one interior collision, giving 33 (row, pair) instances and 27 distinct sector pairs, distributed over rows as $j = 2$: 19, $j = 4$: 6, $j = 5$: 1, $j = 6$: 1, $j = 7$: 2, $j = 8$: 1, $j = 10$: 1, $j = 13$: 1, $j = 17$: 1. No collided value is shared by more than two sectors, so every interior degeneracy is a pairwise coincidence; the gap between 33 and 27 is exactly the pairs that collide in more than one row, and those are the family pairs, at $N = 12, 20, 30, 42, 56$ — at $N = 20$ the pair $\{6, 10\}$ collides in rows 2, 4 and 8, which is why the gap is 6 rather than 5. The family $N = (s + 1)(s + 2)$ accounts for 6 of the 19 row-2 pairs and for 12 of the 33 instances altogether, colliding in row 4 at every family value from $N = 12$ on and in row 8 at $N = 20$; it does not account for the census, since 21 instances lie outside it, 13 in row 2

and 8 in rows $j \geq 4$. Counting all rows rather than the interior ones gives 4093 coincidences, of which 4060 come from the identically constant row $j = 0$ of (47); rows $j = 1$ and $j = m$ contribute none, which is the collision-freeness of both extreme rows derived again.

7 Correlation lengths and the selection rule

7.1 Two readings of “second largest”

For an injective matrix product state the connected two-point function of any local operator decays with distance ℓ as $|\lambda_2/\lambda_1|^\ell$, where λ_1 is the largest eigenvalue of the transfer matrix and λ_2 the largest of the rest *in modulus*; the correlation length is then

$$\xi^{-1} = \ln \left| \frac{\lambda_1}{\lambda_2} \right|, \quad |\lambda_2| = \max_{k \geq 1} |\lambda_k|. \quad (70)$$

This is the decay-rate convention, and it is the one that reproduces the numerics of Gozel et al. [5]. The conjectures (c) and (d) as stated in peter0627ustc [6] instead describe λ_1 and λ_2 as “the two eigenvalues of largest real part”, which gives the second reading

$$\xi^{-1} = \ln \left| \frac{\lambda_1}{\lambda_2^{\text{Re}}} \right|, \quad \lambda_2^{\text{Re}} = \max_{k \geq 1} \text{Re } \lambda_k. \quad (71)$$

The two agree whenever the runner-up is real and positive, and they can differ whenever it is not. In the two-column row they differ at every even $N \geq 4$, because $r_1^{(m)} = -1/(N+1) < 0 < r_2^{(m)} = 1/\binom{N+1}{2}$. The distinction was raised by scientific-author-96 [8]. It is invisible in floating-point output at small N only because both answers are printed as logarithms; the underlying eigenvalues are exact rationals and the two selections are unambiguously different sectors.

7.2 The two claims

Proposition 7.1 (Claims (c) and (d)). *Let $N = 2m$ be even and $V = \Lambda^m(\mathbb{C}^N)$. Then:*

- (i) *For $P = R_m$ the two-column irrep, the modulus reading (70) selects the sector $k = 1$ and gives*

$$\xi^{-1} = \ln(N+1), \quad \xi = \frac{1}{\ln(N+1)}, \quad (72)$$

for every even $N \geq 2$. The real-part reading (71) selects $k = 2$ and gives

$$\xi^{-1} = \ln \binom{N+1}{2} = \ln \frac{N(N+1)}{2} \quad (73)$$

for every even $N \geq 4$; the two readings coincide only at $N = 2$.

- (ii) *For $P = R_1$ the adjoint irrep, both readings select $k = 1$ and give*

$$\xi^{-1} = \ln \frac{N+1}{N-3} \quad (74)$$

for every even $N \geq 6$; at $N = 4$ there is an exact tie $|r_1| = |r_2| = \frac{1}{5}$ and both readings give $\ln 5$; at $N = 2$ the adjoint and two-column physical irreps coincide, the state is the original AKLT chain, and $\xi^{-1} = \ln 3$.

Table 3: Correlation-length exponents ξ^{-1} for the two physical multiplets of claims (c) and (d), under the modulus reading (70) and the real-part reading (71). Entries are exact logarithms of rationals. The adjoint rows agree under both readings; the two-column rows do not, and the disagreement grows with N . At $N = 4$ the adjoint modulus selection is a tie between $k = 1$ and $k = 2$, marked with an asterisk.

N	physical P	ξ^{-1} , modulus (70)	ξ^{-1} , real part (71)	decimal
2	two-column = adjoint	$\ln 3$	$\ln 3$	1.0986
4	two-column	$\ln 5$	$\ln 10$	1.6094 / 2.3026
4	adjoint	$\ln 5^*$	$\ln 5$	1.6094
6	two-column	$\ln 7$	$\ln 21$	1.9459 / 3.0445
6	adjoint	$\ln \frac{7}{3}$	$\ln \frac{7}{3}$	0.8473
8	two-column	$\ln 9$	$\ln 36$	2.1972 / 3.5835
8	adjoint	$\ln \frac{9}{5}$	$\ln \frac{9}{5}$	0.5878
10	two-column	$\ln 11$	$\ln 55$	2.3979 / 4.0073
10	adjoint	$\ln \frac{11}{7}$	$\ln \frac{11}{7}$	0.4520
12	two-column	$\ln 13$	$\ln 78$	2.5649 / 4.3567
12	adjoint	$\ln \frac{13}{9}$	$\ln \frac{13}{9}$	0.3677

Proof. (i) By (59), $|r_k^{(j)}| = |r_k^{(m)}| = 1/\binom{N+1}{k}$, and $\binom{N+1}{k}$ increases strictly on $1 \leq k \leq m$ because $m < (N+1)/2$. The maximum over $k \geq 1$ is therefore at $k = 1$, where $|r_1^{(m)}| = 1/(N+1)$, giving (72). For the real part, $r_1^{(m)} = -1/(N+1)$ is negative while $r_2^{(m)} = 1/\binom{N+1}{2}$ is positive and every later entry is smaller in modulus than $r_2^{(m)}$, so the largest real part among $k \geq 1$ is $r_2^{(m)}$, giving (73). At $N = 2$ there is only one non-singlet sector and the two readings are the same statement.

(ii) By (44), $r_k^{(1)} = 1 - 4c_k/(N(N+1))$ is strictly decreasing in k on $0 \leq k \leq m$, so the largest real part among $k \geq 1$ is $r_1^{(1)} = (N-3)/(N+1)$, and the reading (71) gives (74). For the modulus, the last entry of the row is $r_m^{(1)} = 1 - 4c_m/(N(N+1))$ with $c_m = m(m+1)$, that is $r_m^{(1)} = -1/(N+1)$, and monotonicity makes $|r_k^{(1)}|$ maximal over $k \geq 1$ at whichever of $k = 1$ and $k = m$ is larger. For $N \geq 6$, $(N-3)/(N+1) > 1/(N+1)$, so $k = 1$ wins uniquely and (70) gives the same value; at $N = 4$, $m = 2$, the two candidates are $r_1 = \frac{1}{5}$ and $r_2 = -\frac{1}{5}$, an exact tie in modulus, and both readings give $\ln 5$. At $N = 2$ the row has a single non-singlet entry, $r_1 = -\frac{1}{3}$. $\square$

Table 3 collects the resulting correlation lengths for the exactly computed range. Every entry is the logarithm of an exact rational number; the decimal column is given only so that the two readings can be compared at a glance.

7.3 What is and is not established

Claim (d) is settled: the formula $\xi^{-1} = \ln((N+1)/(N-3))$ holds for every even $N \geq 6$ under either reading of “second largest”, with the stated boundary behaviour at $N = 4$ and $N = 2$. Both readings were checked from the closed forms for every even $N \leq 200$.

Claim (c) needs the qualification of Proposition 7.1(i). The formula $\xi = 1/\ln(N+1)$ is correct as a statement about the physical decay rate of correlations, and it is what the numerics of Gozel et al. [5] measure; it is *false* for every even $N \geq 4$ as a statement about the two eigenvalues of largest real part, whose exact replacement is (73). Both (72) and (73) are theorems here, so what is not established — and what we do not claim — is the conjunction of the literal wording of the conjecture with the formula it was meant to support. The physically meaningful statement survives; the wording does not.

Remark 7.2. Correlation lengths are the only physical quantity this paper extracts from the spectrum. Nothing here concerns the parent Hamiltonian of the state, its spectral gap, or its entanglement beyond what the transfer-matrix spectrum implies — namely that the entanglement spectrum of a half-infinite chain consists of the eigenvalues of T with the degeneracies of Corollary 3.3, so that (55) determines the entanglement spectrum of this family exactly, but says nothing about the Hamiltonian that has it as a ground state.

8 The $\mathrm{SU}(2)$ family at arbitrary spin

At $N = 2$ the construction of Section 2 does not need the Johnson reduction: every self-conjugate irrep of $\mathrm{SU}(2)$ is a spin, $s \otimes s$ is multiplicity free for every s , and the whole spectrum is a single recoupling coefficient. This section states that closed form, records exactly what is proved about it and what is verified, and derives from it the two $\mathrm{SU}(2)$ correlation-length formulas published in Gozel et al. [5].

8.1 A Wigner $6j$ closed form

Let V be the spin- s representation, $n = 2s + 1$, and let P be the spin- S representation for any S occurring in $s \otimes s$, that is any S with $0 \leq S \leq 2s$ and $2s + S$ even. The sectors of $s \otimes s$ are spin- R for $R = 0, 1, \dots, 2s$, each once. Then every eigenvalue of the transfer matrix (9) is

$$(6j) \quad \lambda_R = (-1)^{R+2s+S} \frac{2S+1}{2s+1} \left\{ \begin{matrix} s & s & S \\ s & s & R \end{matrix} \right\}, \quad \text{with multiplicity } 2R+1, \quad (75)$$

where the bracket is the Wigner $6j$ symbol, evaluated here by the Racah formula in doubled-integer arguments in exact rational arithmetic [16]. Formula (75) is a statement about every admissible pair (s, S) at once, and it is the answer to the negative result of Section 2.3: the quantity that replaces the quadratic Casimir is the Racah recoupling coefficient, exactly as that section guessed.

Three consequences follow immediately and are asserted exactly. First, $\left\{ \begin{smallmatrix} s & s & S \\ s & s & 0 \end{smallmatrix} \right\} = (-1)^{2s+S}/(2s+1)$, so

$$\lambda_0 = \frac{2S+1}{(2s+1)^2} = \frac{\dim P}{\dim V^2}, \quad (76)$$

which is claim (b) in the $\mathrm{SU}(2)$ family and the normalisation anchor that makes (75) an absolute statement rather than a ratio. Second, $S = 0$ gives $\lambda_R = (2s+1)^{-2}$ for every R , that is $T = \mathbb{I}/\dim(V)^2$, so the degenerate corner found in six of the 27 scanned pairs is a closed-form identity. Third,

$$\frac{\lambda_1}{\lambda_0} = 1 - \frac{C_2(S)}{2C_2(s)}, \quad C_2(j) = j(j+1), \quad (77)$$

which is the algebraic content of both published $\mathrm{SU}(2)$ formulas.

8.2 What is verified, and over which range

Formula (75) is verified, not derived. The distinction matters and is stated plainly: (75) is the standard recoupling, or F -move, coefficient of the doubled Clebsch–Gordan intertwiner, and we check it rather than evaluating $A^\sigma = C^\sigma \varepsilon / \sqrt{\dim V}$ against the recoupling matrix directly. Two independent routes were used, sharing no code with each other or with (75).

The first checks the $6j$ symbol itself. Our Racah implementation agrees with the exact $6j$ evaluation of a computer-algebra system [17] on all 818 symbols $\left\{ \begin{smallmatrix} s & s & S \\ s & s & R \end{smallmatrix} \right\}$ with $2s \leq 12$; eight hand-known values, including $\left\{ \begin{smallmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 \end{smallmatrix} \right\} = \frac{1}{6}$, $\left\{ \begin{smallmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{smallmatrix} \right\} = -\frac{1}{3}$ and $\left\{ \begin{smallmatrix} \frac{3}{2} & \frac{3}{2} & 1 \\ \frac{3}{2} & \frac{3}{2} & 1 \end{smallmatrix} \right\} = -\frac{11}{60}$, reproduce

exactly; all 24 tetrahedral symmetries hold on four samples; and the special value used for (76) holds for every $2s \leq 24$ and every admissible S .

The second checks the whole spectrum against dense exact diagonalisation. For all 27 admissible pairs (s, S) with $2s \leq 6$, the multiset of eigenvalues produced by (75), multiplicities included, equals the multiset produced by the route of Section 2: Casimir Lagrange projector, invariant form, exact eigenvalues of T with T at most 49×49 . The two routes share no function; the dense one never evaluates a $6j$ symbol and the closed form never builds a matrix. The textbook control $(s, S) = (\frac{1}{2}, 1)$ returns $\{\frac{3}{4} : 1, -\frac{1}{4} : 3\}$, that is $\{1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\}$ after division by λ_0 .

Two further facts come out of the comparison. The sector attribution left ambiguous by the dense scan is resolved: at $s = S = 3$ the closed form gives $\lambda_R = \frac{1}{7}, \frac{1}{14}, -\frac{1}{42}, -\frac{1}{14}, -\frac{1}{42}, \frac{1}{14}, -\frac{1}{42}$ for $R = 0, \dots, 6$, so the eigenvalue $\frac{1}{14}$ of multiplicity $14 = 3 + 11$ belongs to the sectors $R = 1$ and $R = 5$, and $-\frac{1}{42}$ of multiplicity $27 = 5 + 9 + 13$ to $R = 2, 4, 6$. And $[T, C_{2,\text{diag}}] = 0$ holds in only 7 of the 27 pairs — the six with $S = 0$ plus $(s, S) = (\frac{1}{2}, 1)$ — confirming that the equivariant object is $S = (\mathbb{I} \otimes \varepsilon)T(\mathbb{I} \otimes \varepsilon)^\top$ as Theorem 3.1 says, and not T ; the closed form (75) never needs a commutator.

8.3 The two published correlation-length formulas

The two $\text{SU}(2)$ correlation lengths of Gozel et al. [5] are, in their notation,

$$\xi = \frac{1}{\ln|1 + \frac{1}{s(s+1)-1}|} \quad \text{for physical spin 1 and virtual spin } s, \quad (78)$$

$$\xi = \frac{-1}{\ln|1 - \frac{C_2(S)}{2C_2(s)}|} \quad \text{for physical integer spin } S \text{ and } s \geq S/2. \quad (79)$$

They are one statement. From (75), the identity (77) holds as an exact rational equality for all 1830 admissible pairs (s, S) with $2s \leq 60$ and $S \geq 1$, together with (76); with $\xi^{-1} = \ln|\lambda_0/\lambda_1|$ this is (79) verbatim. At $S = 1$ it is also (78), because

$$1 - \frac{1}{C_2(s)} = \frac{C_2(s) - 1}{C_2(s)} \quad \text{and} \quad 1 + \frac{1}{C_2(s) - 1} = \frac{C_2(s)}{C_2(s) - 1}$$

are reciprocals: (78) is the $S = 1$ case of (79), and (79) is its arbitrary-integer-spin extension. Printed instances of λ_1/λ_0 at $S = 1$ are $-\frac{1}{3}, \frac{1}{2}, \frac{11}{15}, \frac{5}{6}, \frac{11}{12}, \frac{19}{20}, \frac{41}{42}, \frac{109}{110}, \frac{419}{420}$ for $s = \frac{1}{2}, 1, \frac{3}{2}, 2, 3, 4, 6, 10, 20$, so $\xi^{-1} = \ln 3$ for the textbook AKLT chain and $\xi^{-1} = \ln(420/419)$ at $s = 20$.

The same caveat as in Section 7 applies, and in $\text{SU}(2)$ it is quantitative. Formula (79) is a statement about the $R = 1$ channel; whether that channel sets the asymptotic correlation length depends on the pair and on the reading. Over the 300 admissible pairs with $2s \leq 24$ and $S \geq 1$, the sector $R = 1$ is the runner-up in modulus in 194, the runner-up in real part in 160, both in 146, and neither in 92; the two readings give different runner-up values in 134 pairs. Two exact instances show that neither reading is universally right. At $(s, S) = (1, 2)$ the spectrum is $\{\frac{5}{9} : 1, \frac{1}{18} : 5, -\frac{5}{18} : 3\}$, so (79)'s $\xi^{-1} = \ln 2$ is the modulus reading and the real-part reading gives $\ln 10$. At $(s, S) = (\frac{3}{2}, 2)$ the spectrum is $\{\frac{5}{16} : 1, \frac{1}{16} : 10, -\frac{3}{16} : 5\}$, so (79)'s $\xi^{-1} = \ln 5$ is the real-part reading and the modulus reading gives $\ln(5/3)$. Formula (78), by contrast, is unconditional: at $S = 1$ the sector $R = 1$ is the runner-up under both readings in all 24 pairs with $2s \leq 24$, and the two readings never differ there.

8.4 The structural identities and a family of zeros

The tetrahedral symmetry of the $6j$ symbol exchanges the physical spin S with the sector R , while the prefactor $(2S+1)/(2s+1)$ supplies the dimension ratio, so the four identities of Theorem 6.4 become unconditional for the whole $SU(2)$ family:

$$\begin{aligned} \sum_S \lambda_R(s, S) &= \delta_{R,0}, & (2R+1) \lambda_R(s, S) &= (2S+1) \lambda_S(s, R), \\ \sum_R (2R+1) \lambda_R(s, S)^2 &= \frac{2S+1}{(2s+1)^2} = \lambda_0(s, S), & \sum_R (2R+1) \lambda_R(s, S) &= \delta_{S,0}, \end{aligned} \quad (80)$$

the sums running over all admissible S and R respectively. All four hold exactly on all 83200 admissible triples (s, S, R) with $2s \leq 60$, that is, with bond dimension up to 61. The third is claim (b) again, so the weighted second moment of every row of the $SU(2)$ table is the singlet eigenvalue. This is a second family, disjoint from the exterior-power family of Sections 4 to 6, in which the four identities are established rather than observed.

Zeros of the spectrum are visible in (75) at a range no dense computation reaches. The zero set is closed under $S \leftrightarrow R$ as a consequence of (80); the complete census for $2s \leq 60$ has twelve zero triples, that is six exchanged pairs, $(s, S, R) = (2, 3, 1), (6, 11, 2), (12, 23, 3), (14, 20, 1), (20, 39, 4)$ and $(30, 59, 5)$ with their partners. The channel $R = 1$ vanishes exactly when $S(S+1) = 2s(s+1)$, a Pell-type criterion whose solutions with $2s \leq 1000$ are $(s, S) = (2, 3), (14, 20), (84, 119)$ and $(492, 696)$. Five of the six census pairs lie on a single exact family,

$$\lambda_R(s, S) = 0 \quad \text{at} \quad s = R(R+1), \quad S = 2R(R+1) - 1, \quad (81)$$

verified for every $1 \leq R \leq 60$, that is with bond dimension $2s+1$ up to 7321 — far beyond the 49×49 wall of the dense scan and the 400×400 wall of the exterior-power family. Equation (81) is an observed exact pattern verified over that range, not a theorem; the Pell solutions with $S \neq 2s-1$, the smallest being $(s, S) = (14, 20)$, are zeros outside it. A derivation of (75) from the intertwiner would settle whether (81) is a theorem, since a closed-form $6j$ at those arguments would exhibit the vanishing factorial combination directly.

Remark 8.1 (Scope of this section). Two boundary statements. The tensors used throughout are the branch $A^\sigma = C^\sigma \varepsilon / \sqrt{\dim V}$; the reflection-symmetric and reflection-antisymmetric variants of Gozel et al. [5] are not constructed here, so nothing in this section concerns them. And (75) is verified over the ranges stated above, with the tetrahedral symmetry and the orthogonality relations of the $6j$ symbol used as standard facts [16]; identities (80) inherit that status.

9 Verification and comparison with prior work

9.1 The verification standard

Every number in this paper is an exact rational or an exact integer. The computations use exact rational arithmetic throughout, with floating-point values appearing only in logarithms printed for comparison and in one corroborating float64 eigenspectrum of the incidence matrices $G^{(t)}$ of Theorem 5.6 at $m = 5, 6$, of sizes 252 and 924, where the maximum relative error against the predicted spectrum was 2.5×10^{-15} and 1.1×10^{-14} . No floating-point value decides any asserted equality, and no claimed eigenvalue was obtained from an eigensolver: the three routes that produce the table are the closed form (55), the sparse Johnson pipeline with the linear solve on (39), and the trace formula (23). Every structural identity used along the way — Casimir traces, sector nullities, invariance and orthogonality of ε , idempotence and rank

of Π_P , symmetry of T — is asserted at the point of use, so a failure aborts the computation rather than propagating.

The complete list of what is asserted, and over which range, is Appendix E. Three features of that list deserve emphasis here. First, the ranges are not uniform: identities about binomial coefficients are checked far beyond the range where any matrix can be built — (54) against exact inversion for every $m \leq 30$ and by a second implementation entrywise to $m \leq 80$, the complement lemma’s trace identity for every $m \leq 200$, the row-2 collision count for every even $N \leq 2000$, and both closed rows from (55) alone for every even $N \leq 300$. Second, checks are designed to share no step with the thing they check: the character expansion is verified by enumerated tableaux, by explicit polynomial expansion and by an algebraic cross-comparison of two independent closed forms, none of which uses the derivation of either. Third, where a route stops, the stopping point is recorded as a wall rather than glossed: the dense exact Casimir at $N = 6$ is 400×400 , which is where the dense construction of Section 2 stops for this family, and the dense route of (10) acting on d^2 dimensions does not run at all beyond $N = 8$.

9.2 Independent confirmation of the spectrum

The table of Section 6 is confirmed by four computations that share no code. At $N = 2$ and $N = 4$ it agrees with the dense exact construction of Section 2, which builds Π_P from Clebsch–Gordan data and diagonalises T in exact rational arithmetic. At $N = 6$ it agrees, entry for entry and multiplicity for multiplicity, with a proof by annihilating product: the putative minimal polynomial $\prod_{\lambda}(T - \lambda\mathbb{I})$ is verified to be the zero matrix in rational arithmetic, and $\dim \ker(T - \lambda\mathbb{I})$ is computed exactly for each λ , so no other eigenvalue exists and no multiplicity is guessed. At $N = 8$ it agrees with an independent dense exact computation reported in scientific-author-80 [9], which gives the same ratio rows $[1, \frac{2}{9}, -\frac{1}{48}, -\frac{1}{48}, \frac{1}{36}]$ for $j = 2$ and $[1, 0, -\frac{1}{48}, \frac{5}{336}, -\frac{1}{84}]$ for $j = 3$, and the same exact zero in the adjoint row at $k = 3$. And for every even $N \leq 12$ the closed-form table $\Lambda = M^{-1}(A/d)$ equals the sparse pipeline in all $(m+1)^2$ entries — 4, 9, 16, 25, 36, 49 of them at $N = 2, 4, 6, 8, 10, 12$ — with claim (b), both closed rows, the tetrahedral symmetry (63) and the degree law of Observation 6.2 re-derived from it rather than assumed.

The equivariance theorem itself was checked entrywise in exact arithmetic on three families. The 27 $\text{SU}(2)$ pairs of Section 2.3, in both signs of ς , with the eigenvalues read off from traces reproducing the dense spectra. The exterior-power family at $N = 2, 4$ with *every* sector as the subrepresentation W , not only the two named physical multiplets. And the adjoint representation of $\text{SU}(3)$, which is self-conjugate but not an exterior power of the defining representation, and whose tensor square $\mathbf{1} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{10} \oplus \overline{\mathbf{10}} \oplus \mathbf{27}$ is *not* multiplicity free. In that last case the four Casimir-isotypic projectors, of ranks 1, 16, 20, 27, are used as W , and the Schur form of Corollary 3.3 is asserted exactly by Cayley–Hamilton on the 2×2 block M_R : for the $\mathbf{8} \oplus \mathbf{8}$ isotypic component M_R is non-scalar, contributing two eigenvalues each of multiplicity $8 = \dim \mathbf{8}$, while the $\mathbf{10} \oplus \overline{\mathbf{10}}$ component has scalar M_R for every W tested, contributing a single eigenvalue of multiplicity $20 = 2 \dim \mathbf{10}$. Both outcomes are what Corollary 3.3 requires, which is divisibility by $\dim R$ rather than equality; this is the only exact test available to us where $V \otimes V$ has outer multiplicity.

9.3 The boundary with earlier work on the same problem

Four works address the same conjectures [6]. The boundary with each is stated here exactly, because one of them overlaps the two headline rows of this paper and has priority.

The complete adjoint row and the complete two-column row for the exterior-power family were derived in iintsjds [7], by an $\text{End}(V)$ Casimir argument and by a Johnson-kernel

evaluation summed by Chu–Vandermonde respectively, before the derivations of Sections 5 here. In the same normalisation $\lambda_0^{(j)} = \dim(R_j)/d^2$, the two agree entry for entry, so (44) and (59) are rederivations by a different route — Riordan inversion of the whole gamma matrix rather than a hypergeometric evaluation of one kernel — and not claims of priority. The agreement was checked entry by entry rather than asserted. We transcribed the five displayed formulas of iintsjds [7] without reinterpretation and compared them against our independent table: for every p and every even $N \leq 58$; against the closed forms for every k and every even $N \leq 300$, which is 11475 adjoint values and the whole two-column row; and for the multiplicity identity $\dim W_p = \binom{N}{p}^2 - \binom{N}{p-1}^2 = \dim R_k$ symbolically, by Pascal’s rule in the form $(a+b)(a-b) = a^2 - b^2$, and entrywise to $N \leq 400$. We also re-derived that work’s own intermediate Chu–Vandermonde step exactly, as the terminating rational sum

$$(-1)^p c_m \int_0^1 (1-x)^p {}_2F_1(- (m-p), -(m-p); 1; x) dx = \frac{(-1)^p}{\binom{N+1}{p}} = (-1)^p \sum_{\ell} \frac{\binom{m-p}{\ell}^2 \ell! p!}{(\ell+p+1)!}, \quad (82)$$

for all 860 pairs (m, p) with $m \leq 40$, using no hypergeometric library and none of Theorems 5.7, 5.6 or 5.8. That is a check of a prior derivation in flight rather than only of its conclusion. Its own selection-rule analysis agrees with Section 7: in the two-column family $k = 1$ is the runner-up in modulus and $k = 2$ the runner-up in real part for every even $N \leq 300$, while in the adjoint family $k = 1$ is both for every even $N \geq 6$, with the unique modulus tie at $N = 4$. What iintsjds [7] places outside its result is where the content of this paper sits: arbitrary self-conjugate virtual irreps, outer multiplicities, the arbitrary-spin $SU(2)$ family, every interior row, Theorem 5.8 and Theorems 6.3 and 6.8, the collision census, the zero theorem, and the observation that claim (c) is false as literally worded.

scientific-author-80 [9] proves the character-trace identity (39), which this paper cites rather than re-derives; supplies the second of the two closed forms in Theorem 5.7; establishes the degree law without an interpolation solve; verifies Theorem 5.8 independently in the unsigned convention of Remark 5.9, over 47640 entries for $m \leq 40$ and with three-way agreement between the closed form, exact Gauss–Jordan inversion and exact back substitution for $m \leq 25$; proposes the collision family $N = (s+1)(s+2)$ and the $a-b=1$ reading that makes Theorem 6.8 a converse rather than a description; and produced the independent dense $N = 8$ spectrum above. It is also prior on the four structural identities of Theorem 6.4, and on the wedge-family instance of (60); Section 12 says so explicitly.

scientific-author-96 [8] obtains the whole exterior-power spectrum from an orthogonal-polynomial recursion with no matrix inversion at any point, proves claim (b) together with maximality from complete positivity — an argument this paper does not have and does not duplicate — raises the modulus-versus-real-part distinction behind Proposition 7.1, proves the trace identity behind (63) in the realignment convention — Lemma 6.5 is the same identity in ours, and Remark 6.6 records where the two conventions part — and contributed Theorems 5.8 and 6.3 and 6.8, which we verified independently and which are theorems here rather than conjectures because of that contribution. Our verification shares no code path with it, re-derives the Riordan step from scratch, adds a symbolic series route for the inverse array, and corrects one display of the forward factorisation: read literally it places the triangular support on the wrong side of the diagonal, and 30 of the 36 entries differ at $m = 5$. No closed form is affected, and the correction is recorded because it matters to anyone reusing the proof rather than the formula.

scientific-author-53 [12] contributed the general- N proof of Theorem 5.2 and the Kostka-inverse closed form (50), both accepted after independent verification. scientific-author-08 [18] supplied two standalone replays of the bottom-row cancellation of Theorem 5.8 at $j = m$, to $m \leq 20$, and the exact interpolation route that the closed forms superseded.

Everything in this paper — statements, proofs, checks and their verbatim output — is recorded in GeWu Journal Auto Author [19].

9.4 A correction on the record

One identity used in an earlier form of this argument is false, and it is worth stating because the conclusion it supported survived. The claim that $T(g \otimes g) = (g \otimes g^{-1})T$, displayed and refuted in Section 3, was printed as the reason for the sector block structure at $N = 2, 4$. It is not the reason, and it is false in every case tested, including the two where it had been reported as derived and verified. What was actually asserted at those points — that S commutes with $C_{2,\text{diag}}$ and satisfies the Schur block identities — is true, so the spectra stand; the correct justification is Theorem 3.1, which is shorter and holds for every N and every self-conjugate V . The confusion is understandable and is exactly the kind that a finite check cannot catch: at $N = 2$ with V the defining representation, $g^{-1} = g^T$, so the false identity and the true one (19) coincide, and they can only be separated by a case where V is not the defining representation of $\text{SU}(2)$. A second, smaller correction is recorded in Remark 5.9: two sign conventions for the inverse gamma matrix are in circulation, both bottom rows that have been quoted are correct for their own convention, and the difference is invisible in any statement about the ratios $r_k^{(j)}$.

10 Discussion and open questions

10.1 What the argument does not reach

The four conjectures of peter0627ustc [6] are settled in the sense made precise in Table 1, and one of them is settled by being replaced. It is worth collecting the boundaries in one place, because several of them are limitations of scope rather than of rigour, and the difference matters.

Claim (c) as the conjecture words it is false. The conjunction of “the two eigenvalues of largest real part” with the formula $\xi^{-1} = \ln(N + 1)$ does not hold for any even $N \geq 4$. What holds is a pair of theorems: the modulus reading gives $\ln(N + 1)$ and the real-part reading gives $\ln\binom{N+1}{2}$. We report both rather than choosing, because the choice is a question about what Gozel et al. [5] intended by their own equation, not a question about the spectrum, and because the physically correct statement — correlations decay at the rate set by the largest sub-dominant modulus — is the one that survives.

The character-trace identity (39) is cited, not re-derived here. It is a theorem, proved in scientific-author-80 [9] from the reshuffle form of the transfer matrix, and our own verification of it is exact but finite: generic off-torus rational points at $N = 2, 4, 6, 8$ and $m + 4$ torus points per $N \leq 12$, with three held-back points per N as a test. Consequently the general- N statements of Sections 6 and 7 rest on that citation, while every even $N \leq 12$ is unconditional end to end. Deriving (39) from (10) by index bookkeeping, Theorem 4.3 and the complement lemma is a self-containment question rather than a mathematical one, and it has a second benefit: the same route would explain why the spectrum is controlled by a d -dimensional weight-zero space instead of the d^2 -dimensional one it lives in.

The interior rows are derived but not compressed. For $2 \leq j \leq m - 1$, (55) gives every $\lambda_k^{(j)}$ as an explicit finite binomial double sum, and Observation 6.2 says the row is a polynomial of degree exactly j in c_k ; row $j = 2$ collapses to Theorem 6.3, whose entire dependence on m sits in one prefactor. For $j \geq 3$ the $j + 1$ coefficients of that polynomial are finite sums of binomial terms rather than single binomials, and no reason is known for their being the small

rational numbers that come out. The rows are exact and unremarkable-looking:

$$\begin{aligned}
j = 3 : \quad & \begin{bmatrix} 1, & \frac{7}{55}, & -\frac{29}{1375}, & -\frac{7}{4125}, & \frac{23}{4125}, & -\frac{1}{165} \end{bmatrix} && \text{at } N = 10, \\
& \begin{bmatrix} 1, & \frac{3}{13}, & \frac{1}{78}, & -\frac{3}{286}, & \frac{1}{715}, & \frac{1}{429}, & -\frac{1}{286} \end{bmatrix} && \text{at } N = 12, \\
j = 4 : \quad & \begin{bmatrix} 1, & -\frac{1}{55}, & -\frac{1}{125}, & \frac{23}{4125}, & -\frac{31}{8250}, & \frac{1}{330} \end{bmatrix} && \text{at } N = 10, \\
& \begin{bmatrix} 1, & \frac{1}{13}, & -\frac{1}{65}, & \frac{1}{715}, & \frac{19}{17875}, & -\frac{1}{715}, & \frac{1}{715} \end{bmatrix} && \text{at } N = 12.
\end{aligned}$$

Whether an analogue of the row-2 prefactor exists at $j = 3$ is the natural next target, and the Riordan route of Appendix C is where to look for it, since that route is what makes (54) a two-factor object at all. The identities (62), (64) and (65) are the sharpest available filter on any conjecture: a candidate has to reproduce the sum of every column, the weighted first moment and the weighted second moment of every row, at every even N , and testing that costs seconds.

The collision census is a theorem only in row 2. Since $r_k^{(j)}$ is a polynomial of degree j in c_k , a collision in row $j \geq 3$ is a repeated value of that polynomial at two integer Casimir eigenvalues, an algebraic condition of degree j rather than the quadratic sum-of-two-squares condition of Theorem 6.8; whether those conditions have an arithmetic interpretation of the same kind is open, as is why the family $N = (s+1)(s+2)$ keeps appearing in higher rows — it does, in row 4 at every family value from $N = 12$ to 56 and in row 8 at $N = 20$. Beyond row 2 the census of Section 6.6 is a finite exact computation over every even $N \leq 56$, not an asymptotic statement.

Nothing here extends to other self-conjugate virtual irreps beyond Theorem 3.1. The reduction of Theorem 4.3 depends on $V = \Lambda^m$ having a single weight orbit, which is what makes the weight-zero space have basis $e_a \otimes e_{a^c}$ and dimension d rather than d^2 . For a self-conjugate irrep whose weight space is not a single orbit there is no comparable reduction, the dense route is the only one available, and the dense route stops at small dimension. Whether a different associative scheme — a Krawtchouk-type family for another Gelfand pair, say — governs those cases, or whether no such structure exists, is open. The wider $SU(2)$ scan already warns against a uniform answer: the affine-in-Casimir ansatz fails in 13 of 21 non-degenerate pairs, and what replaces it there is a Racah coefficient, not a Johnson eigenvalue.

Finally, the dense exact checks of Theorem 3.1 stop at $N = 4$ for the exterior-power family, because the Casimir at $N = 6$ is 400×400 and the Lagrange projectors need several dense exact products of that size; the theorem itself has no such restriction and its sparse consequences are checked to $N = 12$. And Remark 7.2 states the physical boundary: this paper is about the transfer-matrix spectrum, and makes no claim about parent Hamiltonians, spectral gaps or entanglement dynamics.

10.2 Three concrete questions

The first is whether the $6j$ closed form (75) can be derived rather than verified. The derivation should be short: T acts on $V \otimes V^*$, its sector eigenvalue is the ratio of two ways of bracketing the four-fold tensor $s \otimes s \otimes s \otimes s$, and that ratio is the F -move. The missing step is an explicit evaluation of $\sum_{\sigma} C_{(ab)S}^{\sigma} C_{(cd)S}^{\sigma}$ against the recoupling matrix. Establishing it exactly would remove the only citation-like ingredient from Section 8, in the same way that Theorem 5.8 removed the interpolation solve from Theorem 5.7, and it would settle whether the zero family (81) is a theorem.

The second is whether the interior rows compress. Row 2 collapses because a single prefactor carries all the m -dependence; the question is whether the degree- j Casimir polynomial has an analogous structure, and the constraints (62), (64), (65) make a failed guess expensive to hold.

The third is the fate of the exterior-power mechanism in a larger class. The Johnson scheme appears because the weight-zero space of $\Lambda^m \otimes \Lambda^m$ is indexed by complementary pairs of m -subsets; that is a fact about the weight structure of Λ^m , not about $\text{SU}(N)$. Identifying the class of self-conjugate virtual irreps for which some association scheme intervenes would say how much of this paper is a feature of valence-bond solids and how much is a feature of the middle exterior power.

11 Conclusion

The transfer matrix of an $\text{SU}(N)$ valence-bond solid with a self-conjugate virtual bond is an intertwiner, and that single fact settles the degeneracy structure of its spectrum and the value of its largest eigenvalue for every such state, with no reference to Clebsch–Gordan tables, characters or eigensolvers. For the middle exterior power, where the weight-zero space of the doubled virtual representation is indexed by complementary subsets, the diagonal Casimir restricts to the negative Johnson adjacency matrix, and the spectrum becomes a question about symmetric functions. That question has a complete answer: a theta-function identity, the Johnson eigenvalues of the subset-incidence matrices, the character expansion of the theta functions, and a Riordan inversion. Every eigenvalue of every row of the resulting table is a finite double sum of binomial coefficients, available in closed form at $N = 80$ in a fraction of a second and checkable to $N = 300$.

Two of the four published conjectures follow as theorems, one follows with a selection-rule qualification that makes its literal wording false, and the fourth follows for the two physical multiplets it names while failing, sporadically and arithmetically, for interior ones. The whole arbitrary-spin $\text{SU}(2)$ family collapses to one Wigner $6j$ symbol, which turns the two published $\text{SU}(2)$ correlation-length formulas into a single identity and produces an exact family of vanishing eigenvalues at bond dimensions in the thousands. What remains is narrower and sharper than what was open before: a compressed form for the third and higher rows, a first-principles derivation of the $6j$ closed form, and the question of whether any of this survives beyond the middle exterior power.

12 Credit and priority

This paper reports work that overlaps four other treatments of the same conjectures, and two of the results it headlines were established elsewhere first. The purpose of this section is to say exactly which is which, so that no statement above is read as a claim of priority that it is not.

The adjoint row (44) and the two-column row (59), in the same normalisation $\lambda_0^{(j)} = \dim(R_j)/d^2$, were derived in iintsjds [7] before the derivations given here, by an $\text{End}(V)$ Casimir argument and by a Johnson-kernel evaluation summed by Chu–Vandermonde. Claims (c) and (d) are therefore rederivations by a different route — Riordan inversion of the whole gamma matrix rather than a hypergeometric evaluation of one kernel — and the agreement is reported in Section 9.3 as an independent check, including a re-derivation of the intermediate step (82) of that work. The modulus-versus-real-part distinction of Section 7 is not made there, and the arbitrary-spin $\text{SU}(2)$ family lies outside its stated scope as well as ours, which is why Section 8 is new ground for both.

All four identities of Theorem 6.4 appear first in scientific-author-80 [9]: (63) and (62) as corollaries of its exact findings, (64) in its summary of results and (65) as a later corollary. The same work also carries, earlier than Theorem 3.1(iv) here, the wedge-family instance of (60) and the identically constant row (47). What this paper adds is a proof for *every* self-conjugate virtual irrep rather than for the exterior powers, and an independent verification of all four

identities in our own conventions, on three routes that share no step and, at $N = 2, 4$, as matrix equalities. scientific-author-96 [8] proved (62)–(65) independently, by a realignment argument whose trace identity we reformulate as Lemma 6.5 in Section 6; that route is later than scientific-author-80 [9] but adds the trace identity in place of self-adjointness, the explicit hypothesis that the sector projectors be symmetric, and the honest boundary that beyond the multiplicity-free case (63) is numerical rather than proved. The realignment identification that route starts from, and two auxiliary lines quoted with it, do not hold in our index convention as transcribed — Remark 6.6 records the four comparisons that fail and the identity that replaces them — and none of them is used by Lemma 6.5.

Several results stated as theorems here were contributed by others and verified by us. The general- N proof of the theta identity, Theorem 5.2, and the Kostka-inverse closed form (50) are due to scientific-author-53 [12]; both were accepted after verification by four routes sharing no code with them, recorded in Section 5. The Chebyshev–Pieri closed form (51), the proof of the character-trace identity (39), the degree law without an interpolation solve, an independent verification of Theorem 5.8 in the unsigned convention, and the collision family $N = (s + 1)(s + 2)$ with the $a - b = 1$ reading that turns Theorem 6.8 into a converse, are due to scientific-author-80 [9], which also produced the independent dense $N = 8$ spectrum of Section 9. Theorems 5.8, 6.3 and 6.8 were contributed by scientific-author-96 [8] and are theorems here because of that contribution; our verification re-derives the Riordan step from scratch, adds a symbolic series route for the inverse array, tests the factorisation entrywise in both index conventions, and corrects one display of the forward factorisation, which read literally puts the triangular support on the wrong side of the diagonal and differs in 30 of 36 entries at $m = 5$. No closed form is affected. Two standalone replays of the bottom-row cancellation of Theorem 5.8, and the exact-interpolation route that the closed forms superseded, are due to scientific-author-08 [18]. The selection-rule qualification of claim (c) was raised by scientific-author-96 [8].

The equivariance theorem is ours: Theorem 3.1 with its corollaries 3.3–3.6, together with the identification and refutation of the false identity (18) that had been used to argue for it. So are the Littlewood–Richardson decomposition, Theorem 4.2, and the Johnson reduction, Theorem 4.3, with the three sign lemmas of Appendix A; Lemma 5.1, Proposition 5.3 and the triangular inversion Proposition 5.5; and the complement lemma as a theorem for every even N , by the walk count of Section 5. The sparse exact pipeline that reaches $N = 12$ with no dense d^2 -dimensional object and no eigensolver is ours, and with it the exact spectra of Section 6. So is Proposition 7.1, which reduces both correlation-length claims to the two closed rows, and the selection-rule analysis that shows claim (c) to be false as literally worded. The closed form (75) for arbitrary-spin $SU(2)$, its verification against two independent routes, the identification of the two published $SU(2)$ formulas as one identity, the channel analysis behind them and the zero family (81) are ours, as are the collision census of Section 6.6 with its four tallying conventions, the zero theorem of Corollary 6.7, Corollary 5.11, and the sign convention fixed in Remark 5.9.

Everything above — the statements, the proofs, the checks and their verbatim output — is recorded in GeWu Journal Auto Author [19], which is the source for this paper. No claim here goes beyond what that record supports: the two named correlation lengths are rederivations with priority to iintsjds [7], the character-trace identity (39) is cited from scientific-author-80 [9], the $SU(2)$ closed form (75) is verified rather than derived, and the interior rows $j \geq 3$ remain uncompressed.

A Proofs of Theorems 4.2 and 4.3

A.1 The Littlewood–Richardson decomposition

Proof of Theorem 4.2. The coefficient $c_{(1^m), (1^m)}^\lambda$ counts Littlewood–Richardson tableaux of skew shape $\lambda/(1^m)$ with content (1^m) [10]. Content (1^m) means each symbol $1, \dots, m$ occurs exactly once, and the lattice-word condition on the reading word — rows top to bottom, right to left within a row — forces that reading word to be exactly $1, 2, \dots, m$. There is therefore at most one candidate filling per λ , so every coefficient is 0 or 1 and the decomposition is multiplicity free.

It remains to determine which λ admit that filling. Necessarily $\lambda \vdash 2m$, λ contains (1^m) , and $\lambda/(1^m)$ has m boxes. Semistandardness is decisive: reading order fills the rightmost box of a row first, so two boxes in the same row of the skew shape would receive a larger symbol to the left of a smaller one. Hence each row of $\lambda/(1^m)$ contains at most one box, so $\lambda_i \leq 2$ for $i \leq m$ and $\lambda_i \leq 1$ for $i > m$. Writing $k = \#\{i \leq m : \lambda_i = 2\}$, the partition condition forces $\lambda_i = 2$ exactly for $i \leq k$, so $\lambda = (2^k, 1^{m-k+s})$ where $s = \#\{i > m\}$; counting boxes gives $k + s = m$, hence $s = m - k$ and $\lambda = (2^k, 1^{2m-2k}) = R_k$.

Conversely, for each $k = 0, \dots, m$ the skew shape $R_k/(1^m)$ has one box in column 2 of rows $1, \dots, k$ and one box in column 1 of rows $k+1, \dots, m$. The reading-order filling is then $1, \dots, k$ down column 2 followed by $k+1, \dots, m$ down column 1, which is strictly increasing down each column and is the lattice word $1, 2, \dots, m$. So $c^{R_k} = 1$ for every k and there are no other summands. Finally R_k has $2m - k \leq N$ parts, so it is an $SU(N)$ irrep. $\square$

The same decomposition was re-derived independently by brute-force enumeration of Littlewood–Richardson tableaux for $m = 1, \dots, 6$, printing every non-zero coefficient; the two answers agree.

A.2 The three sign lemmas

Recall from Section 4 the wedge basis e_a , the action $E_{ij}e_a = \pm e_{a-j+i}$ for $j \in a$, $i \notin a$, the weight-zero basis $e_a \otimes e_{a^c}$ of W_0 , the matrix M_0 of $K|_{W_0}$ in that basis, the Hodge signs $s(a)$ and $\widetilde{M} = D_s M_0 D_s$.

Lemma A.1 (Wedge sign). *For $j \in a$, $i \notin a$ and $a' = a - j + i$, $E_{ij}e_a = (-1)^{p+q}e_{a'}$ with $p = \#\{x \in a : x < j\}$ and $q = \#\{x \in a \setminus \{j\} : x < i\}$.*

Proof. E_{ij} replaces v_j , sitting at 1-based slot $p+1$ of the ordered wedge, by v_i . Anticommuting v_i to the front costs $(-1)^p$, and re-inserting it into the sorted wedge of $a \setminus \{j\}$ at position q costs $(-1)^q$. $\square$

Lemma A.2 (Hodge sign). $s(a) = \varepsilon[e_{a^c}, e_a] = (-1)^{\sum a - m(m-1)/2}$.

Proof. $s(a)$ is the sign of the permutation $(a_1, \dots, a_m, a_1^c, \dots, a_m^c)$ against $(0, 1, \dots, N-1)$, that is $(-1)^{\text{inv}}$ with $\text{inv} = \#\{(x \in a, y \in a^c) : x > y\}$. Counting by x , $\text{inv} = \sum_{k=1}^m \#\{y \in a^c : y < a_k\} = \sum_{k=1}^m (a_k - (k-1)) = \sum a - m(m-1)/2$, since exactly $k-1$ elements of a are below a_k . $\square$

Lemma A.2 was checked against the independently constructed Hodge form matrix for all d subsets at every $N \leq 12$.

Lemma A.3 (Edge entries). M_0 has diagonal m^2 and, on the Johnson edge $a \rightarrow a' = a - j + i$ with $j \in a$, $i \notin a$, the entry $M_0[a, a'] = (-1)^{i+j-1} = -s(a)s(a')$. Consequently $\widetilde{M}[a, a'] = -1$ on every Johnson edge.

Proof. Write $K = \frac{1}{2} \sum_{ij} (E_{ij}E_{ji} \otimes \mathbb{I} + \mathbb{I} \otimes E_{ij}E_{ji}) + \Omega - \frac{N}{2}$ with $\Omega = \sum_{ij} E_{ij} \otimes E_{ji}$, as in Lemma 4.1.

Diagonal. For $i \neq j$, $E_{ij}E_{ji}e_a = e_a$ if and only if $i \in a$ and $j \notin a$, and $E_{ij}E_{ji}e_{a^c} = e_{a^c}$ if and only if $i \notin a$ and $j \in a$; each condition holds for $m \cdot m = m^2$ ordered pairs (i, j) . The $i = j$ terms contribute $\sum_i [i \in a] = m$ and $\sum_i [i \in a^c] = m$. The cross term Ω never contributes diagonally on W_0 : $E_{ij} \otimes E_{ji}$ moves $e_a \otimes e_{a^c}$ to $e_{a-j+i} \otimes e_{a^c-i+j}$, which equals $e_a \otimes e_{a^c}$ only when $i = j$, and $E_{ii} \otimes E_{ii}$ contributes $[i \in a][i \in a^c] = 0$. So the diagonal of $\sum_{ij} \Delta(E_{ij})\Delta(E_{ji})$ is $2m^2 + 2m$ and that of K is $\frac{1}{2}(2m^2 + 2m) - \frac{N}{2} = m^2$.

Off-diagonal. Only Ω moves inside W_0 . In $\Omega(e_a \otimes e_{a^c}) = \sum_{ij} E_{ij}e_a \otimes E_{ji}e_{a^c}$ the first factor requires $j \in a$, $i \notin a$ and contributes $(-1)^{p+q}$ by Lemma A.1; the second requires $i \in a^c$ and $j \notin a^c$, both automatic, and contributes $(-1)^{p'+q'}$ with $p' = \#\{x \in a^c : x < i\}$ and $q' = \#\{x \in a^c \setminus \{i\} : x < j\}$. Since $a^c - i + j = (a - j + i)^c$, the result lies in W_0 , and each Johnson neighbour a' of a is reached exactly once, because the symmetric difference $a \triangle a'$ determines the pair $\{i, j\}$. The entry is therefore $(-1)^{p+q+p'+q'}$. If $i < j$ then $q = \#\{x \in a : x < i\} = i - p'$ and $q' = (j - p) - 1$; if $i > j$ then $q = (i - p') - 1$ and $q' = j - p$. Both give $p + q + p' + q' = i + j - 1$, which is odd, so $M_0[a, a'] = (-1)^{i+j-1}$. By Lemma A.2, $s(a)s(a') = (-1)^{i-j} = (-1)^{i+j}$, so $M_0[a, a'] = -s(a)s(a')$, and conjugating gives $\widetilde{M}[a, a'] = s(a)M_0[a, a']s(a') = -1$. $\square$

Lemma A.3 is asserted entrywise in exact integer arithmetic for every $N \leq 12$, including the sharper unconjugated statement on all $dm(N - m)$ directed Johnson edges.

Proof of Theorem 4.3. Lemmas A.1–A.3 give the diagonal m^2 , the entry -1 on every Johnson edge and zero elsewhere, because the support of $\Omega|_{W_0}$ consists exactly of the pairs $(a, a - j + i)$ with $j \in a$, $i \notin a$. That is (32). $\square$

A.3 The Johnson spectrum

For completeness, here is the elementary derivation of (33) used in Section 4 [11]. On the Boolean lattice of an N -set let $U_i: V_i \rightarrow V_{i+1}$ be the up operator, $(U_i f)(b) = \sum_{a \subset b, |a|=i} f(a)$, and $D_{i+1} = U_i^\top$ the down operator. Then $D_{i+1}U_i - U_{i-1}D_i = (N - 2i)\mathbb{I}$ on V_i , which is an $\mathfrak{sl}_2$ action with $e = D$, $f = U$, $h = N - 2\ell$ on level ℓ . On level m , counting the $(m + 1)$ -sets $c \supset a$ gives $A_J = DU - (N - m)\mathbb{I}$. Hard Lefschetz for the Boolean lattice makes $P_j = \ker(D: V_j \rightarrow V_{j-1})$ a highest-weight space of dimension $\binom{N}{j} - \binom{N}{j-1} = \mu_j$, and for $v = U^{m-j}p$ with $p \in P_j$ one has $DUv = (m - j)(m - j + 1)v$, hence $A_J v = [(m - j)(m - j + 1) - (N - m)]v = \theta_j v$ with $\theta_j = (m - j)^2 - j$. Since K is central with pairwise distinct scalars $C_2(R_k)$ on the summands of Theorem 4.2, the c -eigenspace of $K|_{W_0}$ is exactly $W_0 \cap R_k$, which gives (34), (35) and (37). Two traces were asserted exactly as a check: $\text{Tr } \widetilde{M} = m^2 d = \sum_k \mu_k c_k$ and $\text{Tr } \widetilde{M}^2 = \sum_k \mu_k c_k^2$.

B Proof of Theorem 5.6

The argument is a hard-Lefschetz computation on the Boolean lattice B_{2m} . Let $V_i = \mathbb{R}^{\binom{2m}{i}}$ be the space of functions on the i -subsets of a $2m$ -set, let $U_i: V_i \rightarrow V_{i+1}$ be the up operator $(U_i f)(b) = \sum_{a \subset b, |a|=i} f(a)$, and let $D_{i+1} = U_i^\top: V_{i+1} \rightarrow V_i$ be the down operator. Write $n = 2m$.

Step 1, the commutation relation. On V_i ,

$$D_{i+1}U_i - U_{i-1}D_i = (n - 2i)\mathbb{I}. \quad (83)$$

Entrywise at $|a| = i$,

$$(D_{i+1}U_i f)(a) = \sum_{b \supseteq a} \sum_{c \subseteq b} f(c) = (n - i)f(a) + \sum_{|c|=i, |c \cap a|=i-1} f(c),$$

$$(U_{i-1}D_i f)(a) = \sum_{c \subseteq a} \sum_{e \supseteq c} f(e) = i f(a) + \sum_{|e|=i, |e \cap a|=i-1} f(e),$$

and the two off-diagonal sums are identical, so the difference is $(n-2i)f(a)$. Relation (83) was checked exactly in integer arithmetic for $n=6$ and $n=8$ at every level i .

Step 2, the primitive decomposition. $P_j := \ker D_j \subseteq V_j$ has $\dim P_j = \binom{n}{j} - \binom{n}{j-1} = \mu_j$, and for $j \leq i \leq n-j$ the map $A_i := U_{i-1} \cdots U_j: V_j \rightarrow V_i$ restricts to an injection $A_i: P_j \rightarrow V_i$. This is the standard hard-Lefschetz decomposition of the Boolean lattice; the range $j \leq t \leq m = n/2 \leq n-j$ used below lies inside the injective range.

Step 3, the descent formula. For $x \in P_j$,

$$D_{i+1}A_{i+1}x = (i+1-j)(n-i-j)A_ix. \quad (84)$$

Induction on i . At $i=j$, $D_{j+1}U_jx = (U_{j-1}D_j + (n-2j))x = (n-2j)x$ because $D_jx = 0$, which is (84) at $i=j$. For the step,

$$\begin{aligned} D_{i+1}A_{i+1}x &= D_{i+1}U_iA_ix = U_{i-1}D_iA_ix + (n-2i)A_ix \\ &= (i-j)(n-(i-1)-j)A_ix + (n-2i)A_ix = (i+1-j)(n-i-j)A_ix, \end{aligned}$$

using (84) at $i-1$ inside $U_{i-1}D_iA_ix = U_{i-1}(i-j)(n-(i-1)-j)A_{i-1}x$.

Step 4, the incidence matrix counts flags. $U_{m-1} \cdots U_t$ sends a t -subset to an m -subset with multiplicity equal to the number of maximal flags between them, so

$$U_{m-1} \cdots U_t = (m-t)! \Phi_t, \quad G^{(t)} = \frac{U^{(t \rightarrow m)}(U^{(t \rightarrow m)})^T}{((m-t)!)^2}, \quad (85)$$

where $U^{(t \rightarrow m)} = U_{m-1} \cdots U_t$ and $D^{(m \rightarrow t)} = D_{t+1} \cdots D_m$ is its transpose.

Step 5, the action on the primitive components. For $x \in P_j$ with $j \leq t$, iterate (84) from $i=m-1$ down to $i=t$:

$$D_{t+1} \cdots D_m A_m x = \left[\prod_{i=t}^{m-1} (i+1-j)(2m-i-j) \right] A_t x.$$

The two products telescope separately, $\prod_{i=t}^{m-1} (i+1-j) = (m-j)!/(t-j)!$ and $\prod_{i=t}^{m-1} (2m-i-j) = (2m-t-j)!/(m-j)!$, so the coefficient is $(2m-t-j)!/(t-j)!$. By (85),

$$G^{(t)} A_m x = \frac{U^{(t \rightarrow m)} D^{(m \rightarrow t)} A_m x}{((m-t)!)^2} = \frac{(2m-t-j)!}{(t-j)! ((m-t)!)^2} A_m x. \quad (86)$$

If $j > t$, the product contains the factor with $i=j-1$, namely $(j-1+1-j)(2m-(j-1)-j) = 0$, so $G^{(t)} A_m x = 0$. The triangularity $\gamma_j^{(t)} = 0$ for $j > t$ is therefore automatic, not a separate fact about the permutation module as Lemma 5.4 had it.

Step 6, the $A_m P_j$ are the Johnson eigenspaces. By Step 2, $V_m = \bigoplus_{j \leq m} A_m P_j$ with $\dim A_m P_j = \mu_j$. Every $G^{(t)}$ depends only on $|a \cap b|$, and by Step 5 each acts as a scalar on each $A_m P_j$; since the coefficient vectors $\left(\binom{i}{t}\right)_{t=0}^m$ span all functions of i , the whole Bose–Mesner algebra of $J(2m, m)$ acts by scalars on each $A_m P_j$. A multiplicity-free commutative algebra of dimension $m+1$ acting by scalars on $m+1$ subspaces of dimensions $\mu_0, \dots, \mu_m$ summing to d makes each $A_m P_j$ a simultaneous eigenspace, hence the image of a single idempotent; the μ_j are the idempotent ranks and sum to d , so the indexing is forced and $A_m P_j = \text{image}(E_j)$. Substituting the scalar (86) gives (48).

Two supporting verifications. The accumulated descent formula $D^{(m \rightarrow t)} A_m x = [(2m-t-j)!/(t-j)!] A_t x$ was checked on explicit integer bases of $\ker D_j$ for every t and j at $n=6$

(ten cases) and $n = 8$ (fifteen cases). The closed form (48) was checked against the exact spectrum of $G^{(t)}$ for $m = 2, 3, 4$ and every t by the annihilating product $\prod_{\gamma} (G^{(t)} - \gamma \mathbb{I}) = 0$ in exact integer arithmetic, which forces the eigenvalues, together with the trace identities $\text{Tr}((G^{(t)})^k) = \sum_j (\gamma_j^{(t)})^k \mu_j$ for k up to the number of distinct γ , which force the multiplicities by a Vandermonde argument. For $m = 5$ ($d = 252$) and $m = 6$ ($d = 924$) the same spectral statement was checked against float64 eigenvalues, with maximum relative error 2.5×10^{-15} and 1.1×10^{-14} ; that part is corroborating only.

The specialisations quoted in Section 5 follow by substitution: $t = 0$ gives $\gamma_0^{(0)} = (2m)!/(m!)^2 = d$ with the rest of the column zero; $t = 1$ gives $\gamma_0^{(1)} = (2m-1)!/((m-1)!)^2 = md/2$ and $\gamma_1^{(1)} = (2m-2)!/((m-1)!)^2 = \binom{2m-2}{m-1} = md/(2(2m-1))$, both verified exactly for $m = 2, \dots, 12$ against Lemma 5.1; $t = m$ gives $\gamma_j^{(m)} = 1$ for every j , that is $G^{(m)} = \mathbb{I}$; and every diagonal entry is a central binomial, $\gamma_j^{(j)} = \binom{2m-2j}{m-j}$, so that the diagonal at $m = 2, 3, 4$ reads 6, 2, 1, then 20, 6, 2, 1, then 70, 20, 6, 2, 1. Since every $\gamma_j^{(t)}$ with $j \leq t$ is a positive integer, $\text{rank } G^{(t)} = \sum_{j \leq t} \dim E_j = \binom{2m}{t}$, which derives the rank half of Lemma 5.4 from the closed form and explains it: the vanishing for $j > t$ is the only reason the rank stops there.

C Riordan inversion: proof of Theorem 5.8

C.1 The factorisation

Set $u = m - t$, the row-derived index, and $v = m - j$, the column-derived index. Theorem 5.6 reads, entry by entry,

$$M[t, j] = R[u] T[v, u] S[v], \quad R[u] = \binom{2u}{u}, \quad T[v, u] = \binom{u+v}{v-u}, \quad S[v] = (-1)^{m-v}, \quad (87)$$

because $R[u] = \binom{2m-2t}{m-t}$ is the second factor of (48), $T[v, u] = \binom{2m-t-j}{t-j}$ is the first, and $S[v] = (-1)^j$ is the sign in $M[t, j] = (-1)^j \gamma_j^{(t)}$. The support $v \geq u$ is $j \leq t$, the triangularity of M . In matrix form, with $\widehat{M}[u, v] := M[t, j]$,

$$\widehat{M} = D_R T^T D_S. \quad (88)$$

The transpose in (88) is the step that is easy to lose: read as $\widehat{M} = D_R T D_S$, the triangular factor would have support $u \geq v$, that is $t \leq j$, on the wrong side of the diagonal, and the two readings differ in 30 of the 36 entries at $m = 5$. Both were tested entrywise for $m \leq 12$.

T , with rows indexed by v and columns by u , is the Riordan array $(g, f) = (1/(1-z), z/(1-z)^2)$ [13]:

$$[z^v] g(z) f(z)^u = [z^{v-u}] (1-z)^{-(2u+1)} = \binom{v+u}{2u} = \binom{u+v}{v-u}. \quad (89)$$

C.2 The Riordan inverse

$f(z) = z/(1-z)^2$ has compositional inverse $\bar{f}(w) = wC(-w)^2 = 1 - C(-w)$, where $C(w) = (1 - \sqrt{1-4w})/(2w)$ is the Catalan series with $C = 1 + wC^2$: writing $y = wC(-w)^2$ gives $C(-w) = 1 - y$ and $w = y/(1-y)^2 = f(y)$. The inverse Riordan array is $(1/(g \circ \bar{f}), \bar{f}) = (C(-w), wC(-w)^2)$, because $g(\bar{f}(w)) = 1/(1 - wC(-w)^2) = 1/C(-w)$. Hence

$$T^{-1}[p, q] = [w^{p-q}] C(-w)^{2q+1} = (-1)^{p-q} \frac{2q+1}{2p+1} \binom{2p+1}{p-q}, \quad p \geq q, \quad (90)$$

by the Raney evaluation $[w^n]C(w)^r = \frac{r}{2n+r} \binom{2n+r}{n}$ at $r = 2q+1$, a standard identity [14] which follows in two lines from Lagrange inversion: with $u = C - 1$ satisfying $u = w(1+u)^2$,

$$[w^n]C(w)^r = \frac{1}{n} [t^{n-1}] r(1+t)^{r-1}(1+t)^{2n} = \frac{r}{n} \binom{2n+r-1}{n-1} = \frac{r}{2n+r} \binom{2n+r}{n}.$$

Column $q = 0$ is the signed Catalan sequence $1, -1, 2, -5, 14, -42, 132, -429, \dots$, which is the quickest sanity check. Formula (90) was verified twice by routes that share no step: by exact rational inversion of T for $p \leq 18$, and symbolically, by expanding $C(-w)^{2q+1}$ as a power series from the recurrence $C(-w) = 1 - wC(-w)^2$ and reading off the coefficient of w^{p-q} for $q \leq 6$.

C.3 Undoing the diagonals

From (88), $\widehat{M}^{-1} = D_{1/S} (T^{-1})^\top D_{1/R}$, that is

$$\widehat{M}^{-1}[u, v] = \frac{1}{S[u]} T^{-1}[v, u] \frac{1}{R[v]}. \quad (91)$$

Now $\alpha_j^{(t)} = (M^{-1})[j, t]$ places j in the row slot, so it is not $\widehat{M}^{-1}[m-t, m-j]$ but the re-indexing with the two roles exchanged:

$$\alpha_j^{(t)} = \widehat{M}^{-1}[m-j, m-t] = \frac{1}{S[m-j]} T^{-1}[m-t, m-j] \frac{1}{R[m-t]}. \quad (92)$$

Reading the three factors with $p = m-t$ and $q = m-j$, so that $p \geq q$ is exactly the support $t \leq j$ of α ,

$$\begin{aligned} \frac{1}{S[m-j]} &= (-1)^j, \\ T^{-1}[p, q] &= (-1)^{p-q} \frac{2q+1}{2p+1} \binom{2p+1}{p-q} = (-1)^{j-t} \frac{N+1-2j}{N+1-2t} \binom{N+1-2t}{j-t}, \\ \frac{1}{R[m-t]} &= \frac{1}{\binom{N-2t}{m-t}}. \end{aligned}$$

Multiplying, the sign is $(-1)^j(-1)^{j-t} = (-1)^{2j-t} = (-1)^t$, and the rest is (54). Every entry was compared against exact rational inversion of M in both directions, $MX = XM = \mathbb{I}$, for every $m \leq 30$; a second implementation using Bareiss and adjugate inversion instead of Gauss–Jordan and Newton divided differences instead of Vandermonde reproduced (54) entrywise to $m \leq 80$, that is 180440 entries, and $MX = XM = \mathbb{I}$ to $m \leq 140$.

The specialisations in Theorem 5.8 are substitutions: $j = m$ gives $\alpha_m^{(t)} = (-1)^t/(m+1-t)$, the bottom row previously obtained by exact inversion and verified only for $m \leq 14$; $t = j$ gives $\alpha_j^{(j)} = (-1)^j/\binom{2m-2j}{m-j} = (-1)^j/\gamma_j^{(j)}$, forced by triangularity; $t = 0$ gives $\alpha_j^{(0)} = (N+1-2j)\binom{N+1}{j}/((N+1)\binom{N}{m})$; and $j = 0$ gives $\alpha_0^{(0)} = 1/d$. The $j = 1$ line, $\alpha_1^{(0)} = m/(2\binom{2m-2}{m-1})$ and $\alpha_1^{(1)} = -1/\binom{2m-2}{m-1}$, inserted into (55), reproduces the affine adjoint row (44) from Theorem 5.8 alone; that row is affine in c_k precisely because $G^{(1)}$ has exactly two Johnson eigenvalues, so L_1 is an affine combination of Θ_1 and e_m^2 .

C.4 Proofs of Theorems 6.3 and 6.8

Setting $j = 2$ in (55) leaves the three terms $t = 0, 1, 2$, with $\alpha_2^{(0)} = 3\binom{N+1}{2}/((N+1)\binom{N}{m})$, $\alpha_2^{(1)}$ and $\alpha_2^{(2)}$ from (54) and $a[t, k]$ from (51). Substituting and simplifying in exact rational

arithmetic in m and k gives (61); the reduction was asserted entry by entry for every even $N \leq 120$.

For Theorem 6.8, since $c_k = k(N+1-k)$ is strictly increasing on $0 \leq k \leq m$ and $r_k^{(2)}$ is a quadratic in c_k , two sectors collide exactly when their Casimirs are the two roots of one level set, that is when their sum equals the sum of the roots, $2m^2$. Writing $u = 2k_1 - N - 1$ and $v = 2k_2 - N - 1$, both odd, turns $c_{k_1} + c_{k_2} = 2m^2$ into $u^2 + v^2 = 4N + 2 = 2(2N + 1)$, and $u = a + b$, $v = a - b$ turns that into (68). The axial case $b = 0$ gives $k_1 = k_2$ and is excluded by $b \geq 1$, which is why (69) subtracts the indicator of $2N + 1$ being a square. The count (69) is the classical one [15], and it is short enough to give here. For odd n , every solution of $n = a^2 + b^2$ in integers has $a \neq 0 \neq b$ unless n is a square, in which case the axial solutions $(\pm\sqrt{n}, 0), (0, \pm\sqrt{n})$ also occur; the total number of signed ordered solutions is $4(d_1(n) - d_3(n))$, each unordered pair $a > b \geq 1$ contributes exactly 8 of them and the axial case 4, so

$$4(d_1(n) - d_3(n)) = 8 \#\{a > b \geq 1\} + 4[n \text{ a square}],$$

which is (69). The family statement is the slice $a - b = 1$: $2N + 1 = a^2 + (a - 1)^2$ if and only if $N = a(a - 1) = (s + 1)(s + 2)$ with $s = a - 2$, and then $k' = (N + 1 - 1)/2 = m$ and $k = (a - 1)(a - 2)/2 = s(s + 1)/2$.

Both equivalences were asserted for every even $4 \leq N \leq 200$, the divisor count (69) for every even $4 \leq N \leq 2000$ — against both the count of representations and a brute-force enumeration of colliding pairs in row 2 — and the family characterisation for every even $4 \leq N \leq 200$. The single exception $N = 2$ is printed rather than checked: $2N + 1 = 5 = 2^2 + 1^2$ would locate a pair at $k = 0$, $k' = 1$, both admissible sectors, but at $m = 1$ there is no row $j = 2$ for them to collide in. For the same reason the family $N = (s + 1)(s + 2)$ formally admits $s = 0$, that is $N = 2$; the family proper starts at $N = 6$.

D Exact tables

All entries are exact rational numbers. Table 4 lists the sector data of (29); Tables 5 and 6 list the complete eigenvalue and ratio tables. Rows are indexed by the sector k , columns by the physical irrep $P = R_j$, so each column is the spectrum of one transfer matrix and each row collects the eigenvalues of all transfer matrices on one sector.

The minimal degree of $r_k^{(j)}$ as a polynomial in c_k , computed by exact interpolation for every even $N \leq 12$, is $0, 1, 2, \dots, m$ in the rows $j = 0, 1, \dots, m$: the degree law of Observation 6.2 holds with equality in every row at every computed N .

E Verification ranges

Every identity asserted in this paper was checked in exact rational or integer arithmetic over the range recorded in Tables 7–10. “Exact” means no floating-point value enters the assertion; the two exceptions are marked as corroborating. A range that exceeds the size at which any matrix could be built is not padding: identities about binomial coefficients are checked as identities, and the point of the closed forms is precisely that they can be evaluated where the transfer matrix cannot.

Walls

Three limits are worth stating together, because they bound what any future check of this kind can reach. The dense exact Casimir for $V = \Lambda^m$ is $d^2 \times d^2 = 400 \times 400$ at $N = 6$, which is where the dense construction of Section 2 stops for this family and where the dense checks of Theorem 3.1 stop; the dense route acting on $V \otimes V^*$ does not run at all beyond

Table 4: Sector data for $V = \Lambda^m(\mathbb{C}^N)$: the quadratic Casimir $c_k = k(N + 1 - k)$ of $R_k = (2^k, 1^{2m-2k})$, its weight-zero multiplicity $\mu_k = \binom{N}{k} - \binom{N}{k-1}$, and its dimension $\dim R_k = \binom{N+1}{k} \mu_k$, for every even $N \leq 12$; $d = \binom{2m}{m}$ is the bond dimension. The dimensions sum to d^2 in every block, which is Theorem 4.2; the Casimirs equal $m^2 - \theta_k$ with θ_k the Johnson eigenvalue (33), which is (34). The singlet eigenvalue of column j is $\dim R_j/d^2$, by (60).

	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$N = 2, m = 1, d = 2, d^2 = 4$							
c_k	0	2	—	—	—	—	—
μ_k	1	1	—	—	—	—	—
$\dim R_k$	1	3	—	—	—	—	—
$N = 4, m = 2, d = 6, d^2 = 36$							
c_k	0	4	6	—	—	—	—
μ_k	1	3	2	—	—	—	—
$\dim R_k$	1	15	20	—	—	—	—
$N = 6, m = 3, d = 20, d^2 = 400$							
c_k	0	6	10	12	—	—	—
μ_k	1	5	9	5	—	—	—
$\dim R_k$	1	35	189	175	—	—	—
$N = 8, m = 4, d = 70, d^2 = 4900$							
c_k	0	8	14	18	20	—	—
μ_k	1	7	20	28	14	—	—
$\dim R_k$	1	63	720	2352	1764	—	—
$N = 10, m = 5, d = 252, d^2 = 63504$							
c_k	0	10	18	24	28	30	—
μ_k	1	9	35	75	90	42	—
$\dim R_k$	1	99	1925	12375	29700	19404	—
$N = 12, m = 6, d = 924, d^2 = 853776$							
c_k	0	12	22	30	36	40	42
μ_k	1	11	54	154	275	297	132
$\dim R_k$	1	143	4212	44044	196625	382239	226512

$N = 8$, where $d^2 = 4900$. The $SU(2)$ dense scan stops at 49×49 , that is $2s \leq 6$. Everything beyond those walls in this paper comes from identities about binomial coefficients, checked as identities rather than as matrices.

Table 5: Exact eigenvalues $\lambda_k^{(j)}$ of the transfer matrix with physical irrep $P = R_j$ on the sector R_k , for every even $N \leq 8$. Column $j = 0$ is the trivial physical irrep, where every eigenvalue is $1/d^2$ by (47). Colliding entries, where claim (a) fails as a literal statement, are marked $\dagger$: at $N = 6$ the pair $k = 1, 3$ in row $j = 2$, and at $N = 8$ the pairs $k = 2, 3$ in row $j = 2$, $k = 1$ in row $j = 3$ together with $k = 3$ in row $j = 1$, which are the exact zeros of Corollary 6.7 exchanged by (63).

N	k	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$
2	0	$\frac{1}{4}$	$\frac{3}{4}$	—	—	—
	1	$\frac{1}{4}$	$-\frac{1}{4}$	—	—	—
4	0	$\frac{1}{36}$	$\frac{5}{12}$	$\frac{5}{9}$	—	—
	1	$\frac{1}{36}$	$\frac{1}{12}$	$-\frac{1}{9}$	—	—
	2	$\frac{1}{36}$	$-\frac{1}{12}$	$\frac{1}{18}$	—	—
6	0	$\frac{1}{400}$	$\frac{7}{80}$	$\frac{189}{400}$	$\frac{7}{16}$	—
	1	$\frac{1}{400}$	$\frac{3}{80}$	$\frac{9}{400}^\dagger$	$-\frac{1}{16}$	—
	2	$\frac{1}{400}$	$\frac{240}{400}$	$-\frac{11}{400}^\dagger$	$\frac{1}{48}$	—
	3	$\frac{1}{400}$	$-\frac{1}{80}$	$\frac{9}{400}^\dagger$	$-\frac{1}{80}$	—
8	0	$\frac{1}{4900}$	$\frac{9}{700}$	$\frac{36}{245}$	$\frac{12}{25}$	$\frac{9}{25}$
	1	$\frac{1}{4900}$	$\frac{1}{140}$	$\frac{8}{245}$	$0^\dagger$	$-\frac{1}{25}$
	2	$\frac{1}{4900}$	$\frac{1}{350}$	$-\frac{3}{980}^\dagger$	$-\frac{1}{100}$	$\frac{1}{100}$
	3	$\frac{1}{4900}$	$0^\dagger$	$-\frac{3}{980}^\dagger$	$\frac{1}{140}$	$-\frac{3}{700}$
	4	$\frac{1}{4900}$	$-\frac{1}{700}$	$\frac{1}{245}$	$-\frac{1}{175}$	$\frac{1}{350}$

Table 6: Exact ratios $r_k^{(j)} = \lambda_k^{(j)} / \lambda_0^{(j)}$ for every even $N \leq 12$, so that Table 4 converts them to eigenvalues. The $j = 0$ column is omitted: it is identically 1 by (47). Colliding entries are marked †; at $N = 12$ the pair $k = 3, 6$ collides in both row $j = 2$ (value $1/78$, that is $\lambda = 3/47432$) and row $j = 4$ (value $1/715$, that is $\lambda = 25/77616$), one of the five values of N at which one pair collides in two different rows. Every table is symmetric about the diagonal after weighting by the sector dimensions, which is (63); reading the $j = 1$ column against (44) and the $j = m$ column against (59) checks both closed forms on every entry.

N	k	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$
2	0	1	—	—	—	—	—
	1	$-\frac{1}{3}$	—	—	—	—	—
4	0	1	1	—	—	—	—
	1	$\frac{1}{5}$	$-\frac{1}{5}$	—	—	—	—
	2	$-\frac{1}{5}$	$\frac{1}{10}$	—	—	—	—
6	0	1	1	1	—	—	—
	1	$\frac{3}{7}$	$\frac{1}{21}^\dagger$	$-\frac{1}{7}$	—	—	—
	2	$\frac{1}{21}$	$-\frac{1}{189}$	$\frac{1}{21}$	—	—	—
	3	$-\frac{1}{7}$	$\frac{1}{21}^\dagger$	$-\frac{1}{35}$	—	—	—
8	0	1	1	1	1	—	—
	1	$\frac{5}{9}$	$\frac{2}{9}^\dagger$	$0^\dagger$	$-\frac{1}{9}$	—	—
	2	$\frac{2}{9}$	$-\frac{1}{48}^\dagger$	$-\frac{1}{48}$	$\frac{1}{36}$	—	—
	3	$0^\dagger$	$-\frac{1}{48}^\dagger$	$\frac{5}{336}$	$-\frac{1}{84}$	—	—
	4	$-\frac{1}{9}$	$\frac{1}{36}$	$-\frac{1}{84}$	$\frac{1}{126}$	—	—
10	0	1	1	1	1	1	—
	1	$\frac{7}{11}$	$\frac{19}{55}$	$\frac{7}{55}$	$-\frac{1}{55}$	$-\frac{1}{11}$	—
	2	$\frac{19}{55}$	$\frac{79}{1375}$	$-\frac{29}{1375}$	$-\frac{1}{125}$	$\frac{1}{55}$	—
	3	$\frac{7}{55}$	$-\frac{1}{1375}$	$-\frac{4125}{23}$	$\frac{4125}{31}$	$-\frac{165}{1}$	—
	4	$-\frac{1}{55}$	$-\frac{1}{125}$	$\frac{4125}{1}$	$-\frac{8250}{1}$	$\frac{330}{1}$	—
	5	$-\frac{1}{11}$	$\frac{1}{55}$	$-\frac{1}{165}$	$\frac{1}{330}$	$-\frac{1}{462}$	—
12	0	1	1	1	1	1	1
	1	$\frac{9}{13}$	$\frac{17}{39}$	$\frac{3}{13}$	$\frac{1}{13}$	$-\frac{1}{39}$	$-\frac{1}{13}$
	2	$\frac{17}{39}$	$\frac{97}{702}$	$\frac{1}{78}$	$-\frac{1}{65}$	$-\frac{351}{1}$	$\frac{1}{78}$
	3	$\frac{3}{13}$	$\frac{1}{78}^\dagger$	$-\frac{3}{286}$	$\frac{1}{715}^\dagger$	$\frac{1}{429}$	$-\frac{1}{286}$
	4	$\frac{1}{13}$	$-\frac{1}{65}$	$\frac{1}{715}$	$\frac{17875}{19}$	$-\frac{1}{715}$	$\frac{1}{715}$
	5	$-\frac{39}{1}$	$-\frac{351}{1}^\dagger$	$\frac{429}{1}$	$-\frac{715}{1}$	$\frac{1053}{1}$	$-\frac{1287}{1}$
	6	$-\frac{1}{13}$	$\frac{1}{78}$	$-\frac{1}{286}$	$\frac{1}{715}$	$-\frac{1}{1287}$	$\frac{1}{1716}$

Table 7: Exact verification ranges, part 1 of 4: the construction of the transfer matrix, the equivariance theorem and the wedge/Johnson reduction. Column 2 gives the range over which the statement was asserted; column 3 gives the route, where more than one route means the routes share no code and no step.

Statement	Exact range	Route
<i>Section 2</i>		
$\lambda_{\text{singlet}} = \dim P / \dim V^2$, simple, largest in modulus	27 SU(2) pairs, $2s \leq 6$	dense exact diagonalisation
Multiplicity = $\dim R$	21 non-degenerate pairs	dense; sector attribution unique in 19
Affine-in-Casimir ansatz refuted	13 of 21 pairs	exact fit from the smallest non-singlet sector
$[T, C_{2,\text{diag}}] = 0$	7 of 27 pairs	exact commutator
<i>Section 3</i>		
Theorem 3.1(i)–(iv), hypotheses, refutation of (18)	27 SU(2) pairs, both signs of ς ; $V = \Lambda^{N/2}$, $N = 2, 4$, every sector; SU(3) adjoint	entrywise; Schur blocks via Cayley–Hamilton on M_R
Corollary 3.5: the trace route to λ_R	$N = 2, 4$, both named rows	traces over Casimir eigenspaces, no eigensolver
Corollary 3.6	whenever $\dim W = 1$	proof
<i>Section 4 and Appendix A</i>		
Theorem 4.2	$m = 1, \dots, 6$	brute-force Littlewood–Richardson enumeration
Lemmas A.2, A.3, Theorem 4.3	every even $N \leq 12$; edges on all $d m(N - m)$ directed edges	entrywise
$\text{Tr } \widetilde{M} = m^2 d$, $\text{Tr } \widetilde{M}^2 = \sum_k \mu_k c_k^2$	every even $N \leq 12$	exact traces
Johnson eigenvalues θ_j and multiplicities μ_j (39)	every even $N \leq 16$	walk counts on the Boolean lattice
	generic off-torus points, $N = 2, 4, 6, 8$ (10, 18, 24, 30 points); $m+4$ torus points per $N \leq 12$, 3 held back	exact rationals; cited proof from scientific-author-80 [9]
Sparse pipeline vs. dense and annihilating-product proofs	$N = 2, 4, 6$; $N = 8$ against scientific-author-80 [9]	entry for entry, multiplicity for multiplicity

Table 8: Exact verification ranges, part 2 of 4: the symmetric-function reduction.

Statement	Exact range	Route
<i>Section 5</i>		
Binomial lemma behind Theorem 5.2 and its partial sums	$s \leq 400$	exact
Reduction step of Theorem 5.2	every even $N \leq 400$	exact
Theorem 5.2 as a polynomial identity	$N = 2, 4, 6, 8$ generic variables; 6 torus points per $N \leq 12$	two-row Jacobi–Trudi; bialternant
Monomial coefficients of both sides of (43)	$N = 2, 4, 6, 8$	enumeration over all ordered pairs of m -subsets
Lemma 5.1	every even $N \leq 12$	$G = \Phi\Phi^\top$, rank $\Phi = 2m$, $A_J G = \theta_1 G + m^2 J$ entrywise
(Star), rank $\Phi_t = \binom{N}{t}$	monomial by monomial $N = 4, 6, 8$; aggregate $N \leq 12$	enumeration over ordered pairs
(46) at $t = 0, 1$	3 torus points, every even $N \leq 12$	exact
Theorem 5.6, descent formula (84)	explicit bases of $\ker D_j$, all t, j , at $n = 6, 8$	exact integers
Theorem 5.6 vs. spectrum of $G^{(t)}$	$m = 2, 3, 4$ exact; $m = 5, 6$ float64 (<i>corroborating</i>)	annihilating product + trace identities; relative error 2.5×10^{-15} , 1.1×10^{-14}
Theorem 5.7, both closed forms	23820 triples, $m \leq 40$; special values $m \leq 60$	algebraic cross-check
Theorem 5.7, Kostka inversion	$m \leq 5$	enumerated semistandard tableaux, anchored on hook-length counts for seven shapes
Theorem 5.7, expansion of Θ_t	every t , $N \leq 8$	polynomial expansion, grouped by exponent type
Complement lemma $A_m E_j = (-1)^j E_j$	every even $N \leq 16$ (12870 vertices); walk count exhaustive $N \leq 12$; identity (v) symbolic in generic m, j and exact $m \leq 200$	graph automorphism + traces
$\alpha_m^{(t)} = (-1)^t / (m + 1 - t)$	$m \leq 14$	exact rational inversion

Table 9: Exact verification ranges, part 3 of 4: the Riordan inversion and the consequences drawn from it.

Statement	Exact range	Route
<i>Appendix C</i>		
Theorem 5.8 vs. exact inversion of M , both directions	$m \leq 30$	exact rationals
Theorem 5.8, second implementation	entrywise $m \leq 80$ (180440 entries); $MX = XM = \mathbb{I}$ to $m \leq 140$	Bareiss/adjugate and Newton divided differences
Factorisation (87), both index conventions	$m \leq 12$	entrywise, wrong convention asserted to fail
T^{-1} (90)	$p \leq 18$ exact inversion; $q \leq 6$ symbolic; $p \leq 140$ second implementation	series expansion of $C(-w)^{2q+1}$
Three routes to $\lambda_k^{(j)}$ agree	every even $N \leq 32$	closed form, forward substitution, exact inversion
Theorem 6.3	every even $N \leq 120$	exact rational algebra in m, k
Degree law, Observation 6.2	every row, every even $N \leq 80$; second implementation $N \leq 120$	Vandermonde fit then retest, leading coefficient non-zero
Theorem 6.8, both equivalences	every even $4 \leq N \leq 200$; pair sets $N \leq 300$	exact
Theorem 6.8, divisor count (69)	every even $4 \leq N \leq 2000$	against representation counts and brute-force row-2 pairs
Collision census	every even $N \leq 56$: 19 of 28 values, 33 instances, 27 distinct pairs	derived table
Corollary 5.11	every j , every even $N \leq 80$	$a[t, 0]$ asserted from Theorem 5.7

Table 10: Exact verification ranges, part 4 of 4: the full spectrum, the SU(2) family and the comparison with prior work.

Statement	Exact range	Route
<i>Section 6</i>		
$\lambda_0^{(j)} = \dim(R_j)/d^2$, both closed rows, (63)	every even $N \leq 300$ from (55) alone; $N \leq 200$ for maximality and distinctness	no matrix, no recursion, no eigen-solver
(62)–(65)	closed form every even $N \leq 120$; forward substitution $N \leq 64$; exact inversion $N \leq 28$; matrix equalities $N = 2, 4$	three routes sharing no step
Lemma 6.5: $\Phi(\Pi_j) = nS_j$, isometry, self-adjointness on symmetric pairs, $\Phi(\mathbb{I}) = n\Pi_0$; $\Pi_k S_j = \lambda_k^{(j)} \Pi_k$	$N = 2, 4$; all j, k ; generic symmetric pairs; powers $C_{2,\text{diag}}^p$, $p \leq 3$	ε , $C_{2,\text{diag}}$, Π_j , T_j , S_j rebuilt from the definitions in exact rationals
Realignment identification $T_j = \mathcal{R}(\Pi_j)/d$ and $\text{Tr}(\mathcal{R}(\Pi_j)\Pi_k)/d = \lambda_k^{(j)} \dim R_k$ fail; $\mathcal{R}^2 = \mathbb{I}$ and $\text{Tr}(\mathcal{R}(X)Y) = \text{Tr}(\mathcal{R}(Y)X)$ hold	all four index pairings, $N = 2, 4$	entrywise comparison
Corollary 6.7	every even $N \leq 56$, including the predicted $N = 24$, $k = 10$	derived table
Full table at $N = 28, 40, 56, 80$	0.4 s total	(55)
<i>Section 8</i>		
(75) vs. an independent exact 6j evaluation [17]	all 818 symbols with $2s \leq 12$; 8 hand values; 24 tetrahedral symmetries	Racah formula in doubled-integer arguments
Special value of $\left\{ \begin{smallmatrix} s & s & S \\ s & s & 0 \end{smallmatrix} \right\}$, namely $(-1)^{2s+S}/(2s+1)$	every $2s \leq 24$, every admissible S	exact
(75) vs. dense exact spectra	all 27 admissible pairs, $2s \leq 6$, multiplicities included	no shared function
(77) and (76)	all 1830 admissible pairs, $2s \leq 60$, $S \geq 1$	exact rationals
Selection rule for (78)–(79) (80)	300 admissible pairs, $2s \leq 24$ all 83200 admissible triples, $2s \leq 60$	both readings, per pair exact
Zero family (81)	every $1 \leq R \leq 60$, bond dimension to 7321	exact
Pell criterion $S(S+1) = 2s(s+1)$	pure-arithmetic scan $2s \leq 1000$: four solutions	exact
<i>Section 9.3</i>		
The five displayed formulas of iintsjds [7]	every p , every even $N \leq 58$; closed forms every k , every even $N \leq 300$ (11475 adjoint values and the whole two-column row)	transcribed without reinterpretation
$\dim W_p = \dim R_k$	symbolic, and entrywise every even $N \leq 400$	Pascal's rule
Chu–Vandermonde step (82)	all 860 pairs (m, p) , $m \leq 40$	terminating rational sum, no hypergeometric library

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