

The global complex-temperature critical curve
of the Curie–Weiss model:
exact identification,
closed-form apex and $b = \pi^2/6$

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Abstract

The Curie–Weiss model at complex inverse temperature $\beta = 1 + \varepsilon + iR$ has a partition function $Z_{\beta,N}$ whose limiting exponential growth rate $F_\beta = \lim_{N \rightarrow \infty} N^{-1} \log |Z_{\beta,N}|$ vanishes in one region of the β -plane and is positive in another. Shamis and Zeitouni proved $F_\beta = 0$ for $\varepsilon < 0$ and in an explicit wedge around $\beta = 1$, proved $F_\beta > 0$ on an explicit curve for $R > \pi$, and conjectured that the two regions are separated globally by a single curve $\{1 + \varepsilon_0(R) + iR\}$ with $\varepsilon_0(0) = 0$, $\varepsilon_0(R) > 0$ for $R \neq 0$ and $\lim_{R \rightarrow \infty} R^2 \varepsilon_0(R) = b \in (0, \infty)$, the curve being selected by the dominant saddle of $f_\beta(u) = \beta u^2/2 - \log \cosh(\beta u)$ through $\operatorname{Re} f_\beta = 0$. We identify that curve and its constant exactly. The substitution $v = \beta u$ parametrises every nonzero saddle with no choice of branch and reduces $\operatorname{Re} f_\beta = 0$ to a single-valued real-analytic equation $\Phi(p, q) = 0$ for $v = p + iq$. The component of $\{\Phi = 0\}$ through $v = 0$ and $v = i\pi$ is one curve joining $\beta = 1$ to $\beta = \infty$; its image is single-valued as $\varepsilon_0(R)$; its apex is $\varepsilon_{\max} = p^* \tanh p^* - 1 = 0.479352406077823612528244194045$, attained at $R = (\pi/2) \tanh p^* = 1.44870272081003937227067251357$, where p^* is the unique positive root of $(p/2) \coth p = \log \sinh p$; and its two ends are exact power series, $R_0(\varepsilon) = \varepsilon + \frac{6}{5}\varepsilon^2 + \frac{171}{175}\varepsilon^3 + \frac{1224}{875}\varepsilon^4 + O(\varepsilon^5)$ near $\beta = 1$ and $\varepsilon_0(R) = \pi^2/(6R^2) + \pi^4/(15R^4) + \pi^6/(28R^6) + O(R^{-8})$ at infinity, so $b = \pi^2/6$. The curve on which Shamis and Zeitouni proved $F_\beta > 0$ is the $\operatorname{Im} u = 0$, $k = 1$ locus of the same branch: there $R^2 \varepsilon \rightarrow \pi^2/3 = 2b$ while $\operatorname{Re} f_\beta < 0$ strictly, so it is an interior curve and not the boundary, which accounts for the apparent factor of two. Competing components of $\{\Phi = 0\}$ are excluded by the exact finite- N free energy: below ε_0 , and in particular at the competing candidate $\varepsilon = 0.1992045110724105$ at $R = 3$, $Z_{\beta,N} \rightarrow (1 - \beta)^{-1/2}$ with an explicit $1/N$ correction, so the free energy there is still the trivial one and the candidate lies inside the region $F_\beta = 0$ rather than on its edge; exactly on the curve $F_N \rightarrow 0$ but NF_N has no limit. The exact finite- N Fisher zeros, obtained as roots of an explicit integer polynomial, converge to the identified curve from below at rate $1/N$. Every identity, series coefficient and tabulated value in the body is reproduced here by an independent implementation, at 40–60 working digits for the curve and the series and up to 368 for the largest exact finite- N sums; only the $N = 80$ Fisher zeros are quoted rather than recomputed. The result locates and validates the conjectured curve; it does not establish saddle dominance, so the identification of ε_0 with the phase boundary remains conjectural.

1 Introduction

The Curie–Weiss model is the simplest interacting spin system that exhibits a phase transition, and it is the one for which the partition function can be written down exactly at every finite size. Its Hamiltonian on N spins $\sigma_i \in \{-1, +1\}$ is

$$H_N(\sigma) = -\frac{1}{2N} \sum_{i,j=1}^N \sigma_i \sigma_j = -\frac{N}{2} m(\sigma)^2, \quad m(\sigma) = \frac{1}{N} \sum_{i=1}^N \sigma_i, \quad (1)$$

and its partition function, normalised so that $Z_{0,N} = 1$, is

$$Z_{\beta,N} = 2^{-N} \sum_{\sigma \in \{-1,+1\}^N} \exp(-\beta H_N(\sigma)) = 2^{-N} \sum_{k=0}^N \binom{N}{k} \exp\left(\beta \frac{(2k-N)^2}{2N}\right). \quad (2)$$

For real $\beta > 0$ this is the classical mean-field ferromagnet, treated probabilistically since [1]; the normalisation by $2N$ rather than N makes $\beta = 1$ the critical point. For *complex* β the same formula defines an entire function of β of order two in N , and the object of interest is the exponential growth rate of its modulus,

$$F_N(\beta) = \frac{1}{N} \log |Z_{\beta,N}|, \quad F_\beta = \lim_{N \rightarrow \infty} F_N(\beta), \quad (3)$$

which exists for every β and is non-negative in the sense made precise in Section 3. Complex-temperature zeros of mean-field models were studied early on [3], and the limit (3) is the quantity whose zero set the Fisher zeros of (2) accumulate on as $N \rightarrow \infty$: where $F_\beta = 0$ the partition function stays bounded, and where $F_\beta > 0$ it grows exponentially, so the boundary between the two is the accumulation locus of the zeros. The question addressed here, posed in [6], is what that boundary is for all β .

Write $\beta = 1 + \varepsilon + iR$ with $\varepsilon, R \in \mathbb{R}$, so that ε measures the distance from criticality along the real axis and R the distance along the imaginary one. Shamis and Zeitouni [10] proved that $F_\beta = 0$ for $\varepsilon < 0$ and in an explicit wedge

$$0 < \varepsilon \leq \varepsilon_0, \quad c\sqrt{\varepsilon} \leq R \leq \frac{c'}{\sqrt{\varepsilon}} \quad (4)$$

around $\beta = 1$, with explicit constants; that $F_\beta > 0$ on the explicit curve

$$1 + \varepsilon_{\text{SZ}}(R) = \frac{R}{2\pi} \log \frac{1 + \pi/R}{1 - \pi/R}, \quad R > \pi; \quad (5)$$

and that the Fisher zeros near $\beta = 1$ approach the real axis along four rays making angles $\pi/4 + n\pi/2$ with it. On that basis they conjectured that the two regions are separated *globally* by a single curve.

Conjecture 1 ([10, Conjecture 1.4]). There is a function $\varepsilon_0: [0, \infty) \rightarrow [0, \infty)$ with $\varepsilon_0(0) = 0$, $\varepsilon_0(R) > 0$ for $R \neq 0$ and

$$\lim_{R \rightarrow \infty} R^2 \varepsilon_0(R) = b \in (0, \infty) \quad (6)$$

such that $F_\beta = 0$ for $\varepsilon < \varepsilon_0(R)$ and $F_\beta > 0$ for $\varepsilon > \varepsilon_0(R)$. The curve is determined by the dominant branch $u_*(\beta)$ of the saddle equation of $f_\beta(u) = \beta u^2/2 - \log \cosh(\beta u)$ through the condition $\text{Re } f_\beta(u_*(\beta)) = 0$.

Three features make the conjecture hard to attack directly, and they shape everything that follows. First, the saddle equation $u = \tanh(\beta u)$ has infinitely many solutions, indexed by the branches of artanh and of $\log \cosh$, and the solution that dominates changes with β through Stokes transitions that no single branch chart can see. Second, the equation $\operatorname{Re} f_\beta = 0$ is satisfied on several *distinct* curves in the β -plane, and at some radii a curve other than the conjectured one lies lower, so the naive procedure of minimising ε over all solutions of $\operatorname{Re} f_\beta = 0$ at fixed R does not produce ε_0 . Third, deciding which of those curves is the boundary requires knowing which saddles are actually reachable by deforming the integration contour of the Hubbard–Stratonovich representation, and that is exactly the step for which no rigorous argument is available here.

This paper identifies the conjectured curve in closed form, computes both of its ends and its apex exactly, determines the constant b , and validates the identification against the exact finite- N free energy and the exact finite- N Fisher zeros. The results are as follows; each is stated with the scope it actually has.

1. **A branch-free parametrisation of every saddle** (Section 3). Writing $v = \beta u$ turns the saddle equation into $u = \tanh v$, $\beta = v / \tanh v$, and gives $f = v \tanh v / 2 - \log \cosh v$ with no branch choice at all. With $v = p + iq$ the real part is the single-valued real-analytic function

$$\Phi(p, q) = \frac{p \sinh 2p - q \sin 2q}{2(\cosh 2p + \cos 2q)} - \frac{1}{2} \log(\sinh^2 p + \cos^2 q), \quad (7)$$

verified against the definition of $f_\beta(u)$ to 3.4×10^{-57} at 60 working digits. The common restriction $|u| < 1$ is invalid: $|\tanh v| > 1$ whenever $\cos 2q < 0$, and the saddle that carries the apex of the curve has $u = 1.0842778882313335 > 1$.

2. **The critical component is a single curve, and only one component in the fundamental strip can matter** (Section 4). Inside the strip $0 < \operatorname{Im} v < \pi$ the zero set of Φ has exactly two branches. One leaves $v = 0$ along the ray $\arg v = \pi/8$, reaches $v = i\pi$, and is the component of Conjecture 1; its image under $\beta = v / \tanh v$ is single-valued as $\varepsilon_0(R)$ for every $R > 0$, with R running monotonically from ∞ to 0. The other leaves $v = 0$ along $\arg v = 3\pi/8$, terminates at the singularity $v = i\pi/2$ of Φ , and satisfies $\operatorname{Re} \beta < 1$ along its whole length, so it can never produce a competing candidate. The two branches map to $\arg(\beta - 1) = \pi/4$ and $3\pi/4$ respectively, because $\beta - 1 = v^2/3 + O(v^4)$: this is the $\pi/4$ pinch of [10], and it also explains the ray structure observed for the zeros of the Ising model on complete graphs [5].
3. **A closed-form apex and the exact value of b** (Sections 5 and 6). The maximum of ε_0 is attained exactly at $\operatorname{Im} v = \pi/2$, where the saddle is real and larger than one, and equals

$$\begin{aligned} \varepsilon_{\max} &= p^* \tanh p^* - 1 = 0.47935240607782361252\dots, \\ R_{\text{at max}} &= \frac{\pi}{2} \tanh p^* = 1.44870272081003937227\dots, \end{aligned}$$

where $p^* = 1.6040291028120047288464196154\dots$ is the unique positive root of $(p/2) \coth p = \log \sinh p$. At the two ends the curve is an exact power series,

$$\begin{aligned} R_0(\varepsilon) &= \varepsilon + \frac{6}{5}\varepsilon^2 + \frac{171}{175}\varepsilon^3 + \frac{1224}{875}\varepsilon^4 + \frac{120987}{61250}\varepsilon^5 + O(\varepsilon^6), \\ \varepsilon_0(R) &= \frac{\pi^2}{6R^2} + \frac{\pi^4}{15R^4} + \frac{\pi^6}{28R^6} + \frac{c_8}{R^8} + O(R^{-10}), \end{aligned} \quad (8)$$

with $c_8 = \pi^8 \left(\frac{1}{45} - \frac{5}{216\pi^2} \right) = 188.6018653206585\dots$ and R_0 the inverse of ε_0 near $\beta = 1$. Hence

$$b = \frac{\pi^2}{6} = 1.6449340668482264364724151666\dots \quad (9)$$

The same computation on the component through $v = i\pi k$ gives $b_k = \pi^2 k^2 / 6 = k^2 b$.

4. **The factor of two against the proved curve is an interior-versus-boundary effect** (Section 6). The curve (5) on which $F_\beta > 0$ was proved is the locus $\text{Im } u = 0$, $k = 1$ of the same branch: to sixth order in $P = \pi/R$ the critical saddle differs from the real saddle $u = \pi/R$ only at order P^7 . On (5) one has $R^2 \varepsilon_{SZ} \rightarrow \pi^2/3 = 2b$ and $\text{Re } f_\beta < 0$ strictly (-8.1×10^{-12} at $R = 1000$), so it lies strictly inside the region $F_\beta > 0$ rather than on its boundary. This resolves the apparent discrepancy between the two constants.
5. **Admissibility, decided by the exact finite- N free energy** (Section 7). At $R = 3$ a second component of $\{\Phi = 0\}$, through $v = 2\pi i$, produces the candidate $\varepsilon = 0.1992045110724105$ below the critical value $\varepsilon_0(3) = 0.27964554577318019778947188$. The elementary bound $F_\beta \leq B(\beta)$ cannot separate them, but the exact finite- N sum can: at the competing candidate, as everywhere below ε_0 , the partition function converges to the trivial-saddle value with its first correction,

$$\begin{aligned} Z_{\beta,N} &= (1 - \beta)^{-1/2} \left[1 - \frac{\beta^4}{4N[\beta(1 - \beta)]^2} + O(N^{-2}) \right], \\ NF_N &= -\frac{1}{2} \log |1 - \beta| + \frac{c_1(\beta)}{N} + O(N^{-2}), \end{aligned} \quad (10)$$

with $c_1(\beta) = -\text{Re}[\beta^4/(4[\beta(1 - \beta)]^2)]$; the coefficient is confirmed to five significant digits by the exact sums at $N = 2000$, to within 1.2×10^{-4} . The trivial saddle therefore dominates at that point, on the evidence of the exact finite- N sums, and the competing candidate is inadmissible. Strictly above ε_0 the free energy converges to a positive limit, and exactly on ε_0 it satisfies $F_N \rightarrow 0$ while NF_N does not converge at all, because the two saddles tie at $\text{Re } f_\beta = 0$ and interfere through an N -dependent phase.

6. **The exact finite- N Fisher zeros converge to the identified curve** (Section 8). Reducing $Z_{\beta,N} = 0$ to the integer polynomial $\sum_k \binom{N}{k} t^{k^2 - Nk + M}$ with $t = e^{2\beta/N}$ and $M = \lfloor N^2/4 \rfloor$ gives all 100, 400 and 1600 zeros at $N = 20, 40, 80$. One family of 4, 13 and 31 of them converges to ε_0 at rate $1/N$ from below, $\delta = \varepsilon_z - \varepsilon_0(R_z) < 0$, with $N\delta$ stable between $N = 40$ and $N = 80$ at matched radius; the spacing in R also scales as $1/N$, with N times the mean spacing over $0.8 < R < 2.5$ equal to 10.5 and 10.7 rather than the naive $4\pi = 12.566$.

What is not claimed. This paper *identifies and validates* the curve of Conjecture 1; it does not prove that the curve is the phase boundary. No saddle-dominance or contour-deformation (Stokes) argument is given, no uniformity in β is established, and the limit $N \rightarrow \infty$ is not interchanged with the approach $\varepsilon \rightarrow \varepsilon_0(R)$. The bound $F_\beta \leq B(\beta)$ used to filter saddles is an exclusion device, not a contour construction: it does not show that the surviving saddles are reachable, nor that residues from the order- N poles of $\cosh(\beta u)^{-N}$ do not compete. The $1/N$ coefficient of the zero offset δ and the $1/N$ coefficient of the zero spacing are measured, not derived. The zero families away from the critical curve, which at $N = 80$ account for all but 31 of the 1600 zeros, are exhibited and not explained. The region $\varepsilon < 0$ is untouched, since $F_\beta = 0$ is proved there [10]. Section 9 states these boundaries again, together with the four questions they leave open.

Relation to earlier and concurrent work. Conjecture 1, the two proved regimes (4) and (5) and the local description of the zeros are due to Shamir and Zeitouni [10]. The global distribution of the Lee–Yang zeros of the same ferromagnet in a complex magnetic field, and its description through unitary Hermite polynomials and the backward heat flow, is established in [4]; that work concerns a different complex parameter and does not address the complex-temperature boundary considered here, but it is the closest exact treatment of the finite- N zero locus of this model. The ray structure of partition-function zeros on complete graphs, including the impact angles at which the locus meets the real axis, is analysed in [5]; the $\pi/4$ pinch recovered in Section 4 is the corresponding statement at $\beta = 1$ for the complex-temperature zeros, and it is derived here from the expansion of Φ rather than quoted. Mean-field complex-temperature zeros were first computed in [3], and the probabilistic framework for the model itself goes back to [1]. A concurrent independent derivation [8] reaches the same u -space closed form, the same value $b = \pi^2/6$ and the same two-term large- R asymptotics by an analytic branch balance, and supplies the exclusion bound $F_\beta \leq B(\beta)$ together with the finite- N precision criterion used in Section 7; an independent local analysis [9] identifies the branch that carries the boundary near $\beta = 1$ and obtains the coefficient $6/5$; an independent finite- N zero computation [7] cross-validates the polynomial reduction and the branch convention. The present account adds the global curve for all $R > 0$, the exact small- R series beyond the quadratic term, the closed-form apex, the admissibility analysis that removes the competing component in the intermediate regime, the two-term expansion (10), and the exact finite- N zero locus with its $1/N$ convergence. All numerical results reported here come from the source investigation [2] and are reproduced, wherever the budget allowed, by an independent implementation written from the definitions; Appendix B records which is which.

Organisation. Section 2 fixes the model, the free energy and the Hubbard–Stratonovich representation, and proves the bound $F_\beta \leq B(\beta)$. Section 3 derives the branch-free parametrisation and the closed form (7). Section 4 analyses the geometry of $\{\Phi = 0\}$: the rays at $v = 0$, the behaviour near $v = i\pi k$, the two branches inside the fundamental strip and the single-valuedness of ε_0 . Section 5 constructs the curve, tabulates it and computes its apex. Section 6 derives both asymptotic series, the constant b and the relation to (5). Section 7 settles admissibility by the exact finite- N free energy and derives (10). Section 8 treats the exact finite- N Fisher zeros. Section 9 states the scope and the open problems. Appendix A collects the exact series data, Appendix B the numerical protocols and the record of what was reproduced independently, and Appendix C the data tables.

2 Model, free energy and the saddle representation

2.1 The model and its complex-temperature free energy

Throughout, $N \geq 1$ is an integer, $\sigma = (\sigma_1, \dots, \sigma_N) \in \{-1, +1\}^N$ a spin configuration, $m = m(\sigma) = N^{-1} \sum_i \sigma_i$ its magnetisation, and $\beta \in \mathbb{C}$ the inverse temperature, which is allowed to be complex from the start. The Hamiltonian (1) counts all ordered pairs including $i = j$, so $\sum_{i,j} \sigma_i \sigma_j = (\sum_i \sigma_i)^2 = N^2 m^2$ and $H_N = -Nm^2/2$. Grouping configurations by the number k of $+1$ spins, for which $m = (2k - N)/N$ and there are $\binom{N}{k}$ choices, gives the second form of (2): a sum of $N + 1$ terms, each an entire function of β . Consequently $Z_{\beta,N}$ is entire in β , $Z_{0,N} = 1$, and $Z_{\beta,N} = 0$ has exactly $\lfloor N^2/4 \rfloor$ solutions in the natural variable of Section 8.

Because $\log |Z_{\beta,N}|$ is subharmonic and $N^{-1} \log |Z_{\beta,N}|$ stays locally bounded, the limit in (3) exists and is subharmonic; this is the free energy of [10], and we take it as the definition of the

phase structure:

$$F_\beta = \lim_{N \rightarrow \infty} \frac{1}{N} \log |Z_{\beta,N}|, \quad F_N(\beta) = \frac{1}{N} \log |Z_{\beta,N}|. \quad (11)$$

Where $F_\beta = 0$ on an open set the partition function is subexponential there and no zeros can accumulate; where $F_\beta > 0$ it grows exponentially. The boundary between the two carries the limiting density of Fisher zeros, since $(2\pi N)^{-1} \Delta \log |Z_{\beta,N}|$ is the zero-counting measure of $Z_{\beta,N}$ and converges to $(2\pi)^{-1} \Delta F_\beta$. Locating that boundary is therefore the same problem as locating the accumulation locus of the zeros, and Conjecture 1 is a statement about it.

For real β the model is classical: F_β is analytic away from $\beta = 1$, which is the mean-field critical point, and the normalisation $2N$ in (1) is what puts the transition there. For complex β the two proved regimes of [10] recalled in the introduction leave a gap: the wedge (4) covers a neighbourhood of $\beta = 1$ of opening angle $\pi/4$, and the curve (5) covers $R > \pi$, but between them, in particular for $2 \lesssim R \lesssim 4$, nothing is proved.

2.2 The Hubbard–Stratonovich representation

For $\text{Re } \beta > 0$ the Gaussian identity $e^{a^2/2} = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{-x^2/2+ax} dx$ with $a = \sqrt{\beta N} m$ gives $e^{\beta N m^2/2} = \sqrt{\beta N/2\pi} \int_{\mathbb{R}} e^{-\beta N u^2/2 + \beta N u m} du$, and summing over the 2^N configurations,

$$\sum_{\sigma} e^{\beta u \sum_i \sigma_i} = (2 \cosh(\beta u))^N,$$

so that the normalisation 2^{-N} in (2) cancels the factor 2^N :

$$Z_{\beta,N} = \sqrt{\frac{\beta N}{2\pi}} \int_{\mathbb{R}} e^{-N f_\beta(u)} du, \quad f_\beta(u) = \frac{\beta u^2}{2} - \log \cosh(\beta u). \quad (12)$$

The integral converges absolutely for $\text{Re } \beta > 0$ and is the starting point of every saddle analysis. Its critical points solve

$$f'_\beta(u) = \beta(u - \tanh(\beta u)) = 0, \quad (13)$$

so $u = 0$ is always a saddle, with $f_\beta(0) = 0$, and the nonzero ones solve $u = \tanh(\beta u)$. Two consequences are used repeatedly. First, since the contribution of $u = 0$ to (12) is of order $N^{-1/2}$ and never exponentially small, $F_\beta \geq 0$ whenever the integral is governed by saddles, with equality exactly when no nonzero saddle contributes more than the trivial one. Second, the sign of $\text{Re } f_\beta$ at the governing saddle decides the phase: if the dominant saddle has $\text{Re } f_\beta < 0$ then $F_\beta = -\text{Re } f_\beta > 0$, and if the trivial one dominates then $F_\beta = 0$. The locus $\text{Re } f_\beta = 0$ is where the two are balanced, which is why Conjecture 1 formulates the boundary in terms of it.

2.3 An unconditional upper bound, and the half-plane $\text{Re } \beta \leq 1$

The saddle picture is only as good as the control one has over which saddles contribute. The following bound needs no contour argument at all; it is the bound used in [8] and is recorded here with its proof because one of its consequences is exact.

Lemma 1 (entropy bound). *For every $\beta \in \mathbb{C}$,*

$$\begin{aligned} F_\beta &\leq B(\beta), \quad H(x) = -x \log x - (1-x) \log(1-x), \\ B(\beta) &= \sup_{0 \leq m \leq 1} \left[\frac{\text{Re } \beta}{2} m^2 + H\left(\frac{1+m}{2}\right) - \log 2 \right]. \end{aligned} \quad (14)$$

Proof. The binomial coefficient satisfies $\binom{N}{k} \leq e^{NH(k/N)}$, so with $m_k = (2k - N)/N$ and $k/N = (1 + m_k)/2$,

$$|Z_{\beta,N}| \leq 2^{-N} \sum_{k=0}^N \binom{N}{k} e^{\operatorname{Re} \beta (2k-N)^2/(2N)} \leq \frac{N+1}{2^N} \max_{0 \leq k \leq N} \exp \left[N \left(\frac{\operatorname{Re} \beta}{2} m_k^2 + H \left(\frac{1+m_k}{2} \right) \right) \right].$$

Dividing by N , letting $N \rightarrow \infty$ and using that $\{m_k\}$ is dense in $[-1, 1]$ and that the supremum is even in m gives (14). $\square$

The bound is useful because for a large part of the plane it is exact and equal to zero.

Proposition 2 (the left half-plane is in the $F_\beta = 0$ phase). *If $\operatorname{Re} \beta \leq 1$ then $B(\beta) = 0$, hence $F_\beta \leq 0$. Moreover the supremum in (14) is attained only at $m = 0$, and no saddle with $\operatorname{Re} f_\beta < 0$ can be admissible anywhere in that half-plane.*

Proof. Write $r = \operatorname{Re} \beta$ and $g(m) = rm^2/2 + H((1+m)/2) - \log 2$. Since $\frac{d}{dm} H\left(\frac{1+m}{2}\right) = \frac{1}{2} \log \frac{1-m}{1+m} = -\operatorname{artanh} m$, one has $g'(m) = rm - \operatorname{artanh} m$. For $r \leq 1$ and $0 < m < 1$ the expansion $\operatorname{artanh} m = \sum_{n \geq 0} m^{2n+1}/(2n+1)$ gives $\operatorname{artanh} m \geq m \geq rm$, with equality only at $m = 0$; so $g' < 0$ on $(0, 1)$, g is strictly decreasing, and the supremum is $g(0) = H(1/2) - \log 2 = 0$. A saddle with $\operatorname{Re} f_\beta = -c < 0$ would give $F_\beta \geq c > 0 = B(\beta)$, contradicting Lemma 1. $\square$

For $\operatorname{Re} \beta > 1$ the equation $g'(m) = 0$ has a unique root $m_* \in (0, 1)$, because $\operatorname{artanh} m$ increases from 0 to ∞ while rm has slope $r > 1$ at the origin, and then $B(\beta) = g(m_*) > 0$. The values of B used later are listed in Table 3.

Proposition 2 gives $F_\beta \leq 0$ for $\operatorname{Re} \beta \leq 1$; the matching lower statement is numerical. The exact sums of Section 7 show that at $\beta = 1 + 2i$ and $\beta = 1 + 5i$, both on the line $\operatorname{Re} \beta = 1$, NF_N converges to $-\frac{1}{2} \log |1 - \beta|$ at rate $1/N$, so $F_\beta = 0$ and $|Z_{\beta,N}| \rightarrow |1 - \beta|^{-1/2}$ there. Together with the theorem of [10] for $\varepsilon < 0$, this covers the closed half-plane $\operatorname{Re} \beta \leq 1$; the analytic statement for the line itself is the identity $F_{1+iR} = 0$ obtained in [8] inside the saddle framework, which Proposition 2 makes consistent but does not prove.

2.4 The object to be identified

Everything below concerns the set of saddles at which $\operatorname{Re} f_\beta = 0$ and the curves they trace in the β -plane. To avoid ambiguity we fix names now and keep them throughout.

Definition 1 (critical locus, critical component, identified curve). The *critical locus* is the set of β for which some nonzero saddle of (13) satisfies $\operatorname{Re} f_\beta = 0$. Its *critical component* is the connected component, described in Section 4, that contains the saddles emanating from $v = 0$ and reaching $v = i\pi$ in the parametrisation of Section 3. The *identified curve* is its image,

$$\beta = 1 + \varepsilon_0(R) + iR, \quad R > 0,$$

wherever that image is single-valued; Section 4 establishes that it is, for every $R > 0$. The curve (5) of [10] is called the *proved curve* and denoted $\varepsilon_{\text{SZ}}(R)$.

The word “boundary” is reserved for the phase boundary of Conjecture 1, which is a conjectural object; the identified curve is a specific analytic locus, and the claim of this paper is that the two coincide. That claim is supported here by exact identities, by exact asymptotics, by the finite- N free energy and by the finite- N zeros, but it is not proved.

3 A branch-free parametrisation of the saddles

3.1 From u to v

The saddle equation (13) is usually solved as $u = \operatorname{artanh}(u) + i\pi k$ -style branch bookkeeping, as in [10], which forces a choice of branch for artanh and for $\log \cosh$ and makes it easy to lose solutions. There is a better variable. Put

$$v = \beta u. \quad (15)$$

Then $u = \tanh v$ is equivalent to (13) for $\beta \neq 0$, and β is recovered from v alone. Every nonzero saddle is therefore parametrised by a single complex number v :

$$u(v) = \tanh v, \quad \beta(v) = \frac{v}{\tanh v}, \quad f(v) = f_{\beta(v)}(u(v)) = \frac{v \tanh v}{2} - \log \cosh v. \quad (16)$$

No branch is chosen anywhere: the multivaluedness of artanh is absorbed into v , and the familiar branch index k is simply the translation $v \mapsto v + i\pi k$. Both $\beta(v)$ and $f(v)$ are even in v , so both are single-valued functions of $\zeta = v^2$; that is what makes the exact series of Section 6 possible. The only singularities of (16) are the zeros of $\sinh v$ and $\cosh v$, that is $v \in i\pi\mathbb{Z}$ for β and $v \in i\pi(\mathbb{Z} + \frac{1}{2})$ for f .

3.2 The closed form of $\operatorname{Re} f$ and of β

Write $v = p + iq$ with $p, q \in \mathbb{R}$. The elementary identities

$$\tanh(p + iq) = \frac{\sinh 2p + i \sin 2q}{\cosh 2p + \cos 2q}, \quad |\cosh(p + iq)|^2 = \sinh^2 p + \cos^2 q,$$

give $\operatorname{Re} f = \Phi$ with Φ as in (7), a manifestly single-valued real-analytic function on $\mathbb{C} \setminus i\pi(\mathbb{Z} + \frac{1}{2})$, and

$$\frac{v}{\tanh v} = \frac{(p + iq)(\cosh 2p + \cos 2q)}{\sinh 2p + i \sin 2q} = \frac{(p + iq)(\sinh 2p - i \sin 2q)}{\cosh 2p - \cos 2q},$$

where the last step uses $\sinh^2 2p + \sin^2 2q = (\cosh 2p - \cos 2q)(\cosh 2p + \cos 2q)$. Hence

$$\beta(p, q) = \frac{p \sinh 2p + q \sin 2q}{\cosh 2p - \cos 2q} + i \frac{q \sinh 2p - p \sin 2q}{\cosh 2p - \cos 2q}, \quad (17)$$

with denominator $\cosh 2p - \cos 2q > 0$ except at $v \in i\pi\mathbb{Z}$. The critical locus of Definition 1 is thus the real-analytic curve

$$\Phi(p, q) = 0, \quad (18)$$

and the map from it to the β -plane is the explicit rational-hyperbolic map (17). Both closed forms were verified against the definitions (12) and (15) at 60 working digits over sample points spread through the plane, including points with $|v|$ large and q near π : the worst residual is 3.4×10^{-57} for Φ and 5.4×10^{-58} for β (Appendix B, Table 7).

An equivalent form in u is sometimes more convenient. If $\beta u = \operatorname{artanh} u + i\pi k$ then

$$\operatorname{Re} f_{\beta}(u) = \frac{1}{2} \left[\operatorname{Re}(u \operatorname{artanh} u) - \pi k \operatorname{Im} u + \log|1 - u^2| \right], \quad (19)$$

verified to 9.3×10^{-61} over 42 pairs (k, u) with $k = -3, \dots, 3$ both inside and outside the unit disc.

3.3 Why the restriction $|u| < 1$ is invalid

It is tempting to parametrise saddles by u in the unit disc, where $\operatorname{artanh} u$ has its principal branch. That disc does not contain all the saddles, and the ones it misses are precisely the ones that matter here. From $|\sinh(p + iq)|^2 = (\cosh 2p - \cos 2q)/2$ and $|\cosh(p + iq)|^2 = (\cosh 2p + \cos 2q)/2$,

$$|\tanh v|^2 = \frac{\cosh 2p - \cos 2q}{\cosh 2p + \cos 2q}, \quad (20)$$

which exceeds 1 exactly when $\cos 2q < 0$, that is when $\operatorname{Im} v$ lies in $(\pi/4, 3\pi/4)$ modulo π . Two instances are decisive. The apex saddle of Section 5 sits at $\operatorname{Im} v = \pi/2$ and has $u = \coth p^* = 1.0842778882313335 > 1$, real and outside the disc. And at $R = 2$ the admissible saddle has $\operatorname{Re} u = 1.04806759926 > 1$. A treatment confined to $|u| < 1$ silently drops both. Formula (20) itself was checked to 6.9×10^{-51} at twelve points, five of which have $|\tanh v| > 1$ and all five have $\cos 2q < 0$.

3.4 Two exact identities

Two facts about Φ are used throughout, and both are exact rather than numerical.

The trivial saddle has zero action. For every β , $u = 0$ solves (13) and $f_\beta(0) = 0$. Hence a nonzero saddle can only increase the growth rate: $F_\beta \geq 0$ in the saddle framework of (12), with equality if and only if no admissible nonzero saddle has $\operatorname{Re} f_\beta < 0$. This is why the sign of $\operatorname{Re} f_\beta$ at the *dominant* nonzero saddle is the order parameter of the transition, and why $\{\Phi = 0\}$ is the natural place to look for the boundary.

The punctures. For every integer k ,

$$\Phi(0, \pi k) = 0. \quad (21)$$

Indeed at $p = 0$ the first term of (7) is $-q \sin 2q / (2(1 + \cos 2q)) = -\frac{1}{2}q \tan q$, which vanishes at $q = \pi k$, and $\sinh^2 0 + \cos^2(\pi k) = 1$ kills the logarithm. At these points $u = \tanh(i\pi k) = 0$ and $\beta = v / \tanh v$ has a pole, so (21) says that the critical locus passes through the points at which the parametrisation sends β to infinity. They are the punctures through which the components of $\{\Phi = 0\}$ enter and leave, and Section 4 uses them to label the components. Both identities were checked to 6.5×10^{-60} and 0 respectively at 60 digits, over $k = -6, \dots, 6$ and six values of β .

3.5 The component label

The translation $v \mapsto v + i\pi k$ maps solutions of (13) to solutions, and by (21) each puncture $v = i\pi k$ is on the locus. We therefore speak of the *component through* $v = i\pi k$: the connected component of $\{\Phi = 0\}$ containing that puncture. The component through $v = 0$ and the component through $v = i\pi$ turn out to be the same curve (Section 4); it is the critical component of Definition 1, and it is labelled $k = 1$ in the asymptotic analysis of Section 6. The components through $v = 2\pi ik$, $|k| \geq 1$, are different curves; they are the source of the competing candidates analysed in Section 7. The large- R asymptotics of the component through $v = i\pi k$ is $R^2 \varepsilon \rightarrow \pi^2 k^2 / 6 = k^2 b$ (Appendix A), so for large R the critical component $k = 1$ is the lowest of them; at intermediate R this ordering fails, which is the content of Section 7.

4 Geometry of the critical locus

This section establishes the local structure of $\{\Phi = 0\}$ at the punctures, counts its branches inside the fundamental strip, and shows that the critical component projects to a single-valued curve $\varepsilon_0(R)$. Two of the statements are exact and proved below; the counting and the monotonicity are numerical and their status is stated where they are used.

4.1 Two exact horizontal slices

On the horizontal lines through the punctures and through the singularities of Φ the closed form (7) collapses.

Proposition 3 (exact slices). *For every integer k and every $p \in \mathbb{R}$,*

$$\Phi(p, \pi k) = \frac{p}{2} \tanh p - \log \cosh p, \quad (22)$$

$$\Phi\left(p, \frac{\pi}{2} + \pi k\right) = \frac{p}{2} \coth p - \log \sinh p \quad (p \neq 0). \quad (23)$$

The first is ≤ 0 , with equality only at $p = 0$; the second is strictly decreasing on $(0, \infty)$, tends to $+\infty$ as $p \downarrow 0$ and to $-\infty$ as $p \rightarrow \infty$, and therefore has exactly one positive root p^* .

Proof. At $q = \pi k$ one has $\sin 2q = 0$ and $\cos 2q = 1$, so (7) gives $\Phi = p \sinh 2p / (2(\cosh 2p + 1)) - \frac{1}{2} \log(\sinh^2 p + 1)$; using $\sinh 2p / (\cosh 2p + 1) = \tanh p$ and $\sinh^2 p + 1 = \cosh^2 p$ yields (22). At $q = \pi/2 + \pi k$ one has $\sin 2q = 0$, $\cos 2q = -1$ and $\cos^2 q = 0$, so $\Phi = p \sinh 2p / (2(\cosh 2p - 1)) - \log \sinh p$ and $\sinh 2p / (\cosh 2p - 1) = \coth p$ gives (23). For the signs, put $g(p) = \frac{p}{2} \tanh p - \log \cosh p$; then $g(0) = 0$ and $2g'(p) = p \operatorname{sech}^2 p - \tanh p$, whose derivative $-2p \operatorname{sech}^2 p \tanh p$ is negative for $p > 0$, so $g' < 0$ and $g < 0$ on $(0, \infty)$; g is even, which settles (22). For $h(p) = \frac{p}{2} \coth p - \log \sinh p$ one has $h'(p) = -\frac{1}{2} \coth p - \frac{p}{2} \operatorname{csch}^2 p < 0$ on $(0, \infty)$, $h(p) = \frac{1}{2} - \log p + O(p^2)$ as $p \downarrow 0$ and $h(p) = -\frac{p}{2} + \log 2 + O(pe^{-2p})$ as $p \rightarrow \infty$, so h has exactly one positive zero. $\square$

Both slices were verified at 60 digits against (7) to residuals below 1.6×10^{-60} , at $p = 0.1, 0.5, 1.0$ and at $p = p^*$ below. Two consequences are used later. First, (22) says that the locus meets the line $\operatorname{Im} v = \pi k$ *only* at the puncture $v = i\pi k$, and tangentially: $\Phi(p, \pi k) = -p^4/12 + 2p^6/45 + O(p^8)$, verified to five significant digits at $p = 10^{-2}$ and to thirteen at $p = 10^{-3}$. Second, (23) gives the apex of the curve in closed form, which is the content of Section 5.

4.2 The rays at $v = 0$ and the $\pi/4$ pinch

Since $f(v) = \Psi(v^2)$ with $\Psi(\zeta) = -\zeta^2/12 + 2\zeta^3/45 - \dots$ (Appendix A), the leading term of Φ at the origin is

$$\Phi(p, q) = -\frac{\operatorname{Re}(v^4)}{12} + O(|v|^6) = -\frac{p^4 - 6p^2q^2 + q^4}{12} + O(|v|^6), \quad (24)$$

and $\operatorname{Re}(v^4) = 0$ factorises as $(p^2/q^2)^2 - 6(p^2/q^2) + 1 = 0$, so $p/q = \sqrt{2} \pm 1$. The locus therefore leaves the origin along eight rays

$$\arg v = \frac{\pi}{8} + \frac{n\pi}{4}, \quad n = 0, \dots, 7, \quad (25)$$

four of which lie in the right half-plane $p > 0$. Similarly

$$\beta - 1 = \frac{v^2}{3} + O(|v|^4), \quad (26)$$

so the image rays are the four rays $\arg(\beta - 1) = \pi/4 + n\pi/2$ through $\beta = 1$: the doubling of the angle is the squaring in (26). This is the $\pi/4$ pinch angle of the Fisher zeros proved in [10] and observed for partition-function zeros on complete graphs in [5], obtained here from the expansion of Φ alone.

Both expansions were verified numerically at 60 digits. Off the rays, $\Phi/(-\operatorname{Re} v^4/12) \rightarrow 1$ as $|v| \rightarrow 0$: 1.000037746 at $|v| = 10^{-2}$, 1.000000377 at 10^{-3} , 1.000000004 at 10^{-4} . On the rays the quartic term vanishes and $\Phi = O(|v|^6)$, with $\Phi/|v|^6 \rightarrow -0.0314270$ on the ray $\arg v = \pi/8$ and $+0.0314270$ on $\arg v = 3\pi/8$, the two values being equal and opposite as (24) requires at the next order. Likewise $(\beta - 1)/(v^2/3) \rightarrow 1$, with $|(\beta - 1)/(v^2/3) - 1| = 4.7 \times 10^{-6}$ at $|v| = 10^{-2}$ on the ray $\arg v = \pi/8$. Tracing the two rays that lie in the first quadrant gives

$$\arg v = 22.500000^\circ \text{ and } 67.500000^\circ, \quad \arg(\beta - 1) = 45.000000^\circ \text{ and } 135.000000^\circ \quad (27)$$

at $|v| = 10^{-6}$, that is $\pi/8$ and $3\pi/8$ mapped to $\pi/4$ and $3\pi/4$.

4.3 The punctures at $v = i\pi k$ and the large- R end

Write $v = i\pi k + w$ with $w = p + is$ small. Then

$$\Phi = -\frac{\pi k}{2}s - \frac{\operatorname{Re}(w^4)}{12} + O(|w|^5), \quad \beta = 1 + \frac{i\pi k}{w} + O(w), \quad (28)$$

both verified at 60 digits for $k = 1, 2, 3$: the ratio of Φ to its linear term is $1 + 9.6 \times 10^{-9}$ at $|w| = 10^{-4}$, and $(\beta - 1)w/(i\pi k) = 1 + 2.6 \times 10^{-7}$ at $|w| = 10^{-3}$. Solving $\Phi = 0$ for s gives a locus tangent to the real axis,

$$s(p) = -\frac{p^4}{6\pi k} + O(p^6), \quad (29)$$

whose exact series is computed in Section 6, and (28) gives $\operatorname{Im} \beta = \pi k p/|w|^2 + O(1)$, so

$$R = \operatorname{Im} \beta \sim \frac{\pi k}{|w|} \rightarrow \infty \quad \text{as } w \rightarrow 0. \quad (30)$$

Every puncture is thus an exit to $R = \infty$, and each component through $v = i\pi k$ has its own large- R asymptotics, computed in Section 6 to be $R^2 \varepsilon \rightarrow \pi^2 k^2/6$.

4.4 Exactly two branches in the fundamental strip

The strip $0 < \operatorname{Im} v < \pi$ contains two punctures of the parametrisation, $v = 0$ and $v = i\pi$, at its ends, and one singularity of Φ , at $v = i\pi/2$, in its interior. The zero set of Φ inside it consists of exactly two branches.

- The *critical branch* leaves $v = 0$ along the ray $\arg v = \pi/8$ and reaches $v = i\pi$. Along it $q = \operatorname{Im} v$ increases monotonically from 0 to π ; by (26) and (30) the corresponding R increases monotonically from 0 to ∞ , while $\varepsilon = \operatorname{Re} \beta - 1$ rises from 0 to a maximum and falls back to 0.
- The *short branch* leaves $v = 0$ along the ray $\arg v = 3\pi/8$ and terminates at the singularity $v = i\pi/2$, where $\cosh v = 0$ and Φ diverges logarithmically. Along it $p/q \rightarrow \sqrt{2} - 1$ as $q \rightarrow 0$ and $p \rightarrow 0$ as $q \rightarrow \pi/2$; its image satisfies $\operatorname{Re} \beta < 1$ everywhere and $\beta \rightarrow 0$ at its endpoint.

The evidence is the following. For each of 398 equally spaced values of $q \in [0.01, \pi - 0.01]$ the equation $\Phi(p, q) = 0$ was scanned over 4000 values of $p \in [10^{-6}, 30]$ and every sign change polished at 30 digits to a residual below 10^{-22} . Exactly 199 values of q , those with $q > 1.5669$, gave a single root; the other 199 gave two, the extra one being the short branch. The largest root agrees with the critical branch traced independently in Section 5 at every one of the 398 values, to 6.6×10^{-7} relative for $q \leq 3$ and to 1.2×10^{-4} relative for the values nearest π , where $p(q)$ is steepest and the comparison is limited by the reference grid. Along the short branch, traced over 400 values of q from 0.02 to $\pi/2 - 10^{-7}$, the largest value of $\operatorname{Re} \beta$ encountered is $0.9998895 < 1$, and at the endpoint $\beta = 0$ exactly.

The consequence for the problem at hand is sharp: *inside the fundamental strip the only branch that can produce a candidate with $\operatorname{Re} \beta > 1$ is the critical one*, since the short branch never leaves the half-plane $\operatorname{Re} \beta < 1$. All the competing candidates of Section 7 come from components through $v = 2\pi ik$, $|k| \geq 1$, that is from outside the strip.

4.5 Single-valuedness of ε_0

The critical branch is parametrised by $q \in (0, \pi)$: for each q there is a unique $p(q) > 0$ with $\Phi(p(q), q) = 0$ (the numerical statement of the previous paragraph), and $\beta(q) = \beta(p(q), q)$ is then a curve in the β -plane. The component was traced on 4001 values of q equally spaced in $(10^{-14}, \pi(1 - 10^{-14}))$ at 50 working digits, every one of them solved by Newton from the previous point, and along the whole trace $R = \operatorname{Im} \beta$ is *strictly monotone* in q : traversed from q just below π down to q just above 0, R decreases at each of the 4000 steps, from 3.58×10^3 to 4.03×10^{-25} , while ε rises from 1.22×10^{-21} to its maximum 0.479352406077824 at $\operatorname{Im} v = \pi/2$ and falls back towards 0:

$$q: 0 \nearrow \pi \implies R: 4.03 \times 10^{-25} \nearrow 3.58 \times 10^3, \quad \varepsilon: 0 \nearrow 0.479352406077824 \searrow 0. \quad (31)$$

Strict monotonicity of R along the trace means the image is single-valued as $\varepsilon_0(R)$ on the covered range, and the two ends are covered by the asymptotics: $R \rightarrow 0$ as $q \rightarrow 0$ by (26) and $R \rightarrow \infty$ as $q \rightarrow \pi$ by (30). The range $R > 3.58 \times 10^3$ is covered separately in Section 6 by Newton iteration seeded from the asymptotic series, which reproduces the traced values where the two overlap. This is exactly the object whose existence Conjecture 1 asserts: a single curve joining $\beta = 1$ to $\beta = \infty$, with $\varepsilon_0(0) = 0$ and $\varepsilon_0(R) > 0$ for $R > 0$.

Two qualifications are in order. The uniqueness of $p(q)$ for each q is a numerical fact on a finite grid, not a theorem; Proposition 3 proves it only on the two horizontal slices. And the monotonicity of R is verified on 4001 points of one particular grid, which is strong evidence but not a proof that the projected curve has no self-intersection. Neither qualification affects the values of $\varepsilon_0(R)$ reported below, each of which is an independent polished solution of the two equations $\{\Phi = 0, \operatorname{Im} \beta = R\}$ with residual below 5×10^{-51} .

5 The identified curve and its closed-form apex

5.1 Construction

Each value of $\varepsilon_0(R)$ reported here is an independent solution of the pair of real equations

$$\Phi(p, q) = 0, \quad \operatorname{Im} \beta(p, q) = R, \quad (32)$$

Table 1: The identified curve $\varepsilon_0(R)$ at sixteen radii, spanning ten orders of magnitude. Columns: the radius $R = \text{Im } \beta$; $\varepsilon_0(R) = \text{Re } \beta - 1$ to 18 significant digits; the saddle parameter $v = p + iq$ that produces it, as $p = \text{Re } v$ and q/π ; the scaled quantity $R^2\varepsilon_0/b$ with $b = \pi^2/6$, which tends to 1 as $R \rightarrow \infty$; and the residual $|\Phi(p, q)|$ of the solved system at 60 working digits. The row marked $\dagger$ is the apex of Proposition 4. Every row is an independent solution of (32); the analytic curve is not a measurement and carries no error bars. Table 8 gives 26 digits at 27 radii.

R	$\varepsilon_0(R)$	$\text{Re } v$	$\text{Im } v/\pi$	$R^2\varepsilon_0/b$	$ \Phi $
10^{-4}	0.0000999880019024396472	0.01902907125	0.002509234000	6.078541622e-13	7.9e-62
10^{-2}	0.00988186211401763277	0.1896101552	0.02528104210	6.007451796e-7	2.8e-62
0.1	0.0895677065794624410	0.5819805357	0.08500706990	0.0005445063628	0.0
0.5	0.318576594028771990	1.172179592	0.2298845791	0.04841783638	7.8e-62
1	0.448776099003554343	1.479355261	0.3778817438	0.2728231532	2.3e-61
1.44870272081 $\dagger$	0.479352406077823612	1.604029103	0.5	0.6115964673	7.8e-62
2	0.444374888482076993	1.635960746	0.6391388968	1.080590152	7.8e-62
3	0.279645545773180197	1.439558777	0.8543923788	1.530036956	1.6e-61
4	0.139005163239551200	1.023158475	0.9720513767	1.352080097	7.8e-62
5	0.0789202549760265742	0.7361508993	0.9937385949	1.199444047	1.4e-61
8	0.0274304847685543473	0.4149531884	0.9994585074	1.067247047	1.8e-61
10	0.0171350367515826567	0.3251476101	0.9998018575	1.041685323	5.3e-62
100	0.000164558380433042054	0.03142626809	0.9999999835	1.000394993	1.7e-61
1000	0.00000164494056082196422	0.003141602989	0.99999999998	1.000003948	9.8e-62
10^5	1.644934067497620377e-10	3.141592655e-5	1.0	1.000000000	2.6e-61
10^6	1.644934066854720375e-12	3.141592654e-6	1.0	1.0	1.9e-61

in the two unknowns $(p, q) = (\text{Re } v, \text{Im } v)$, solved by Newton's method with the exact 2×2 Jacobian obtained by differentiating (7) and (17), and started from one of two seeds. For $R < 8$ the seed is the nearest point of the component traced in Section 4, nearest in $\log R$ because R spans twenty-eight orders of magnitude along a trace that is uniform in q ; for $R \geq 8$ it is the asymptotic series of Section 6, which is accurate to six digits already at $R = 8$. The acceptance test is that the iterate stays in the strip $0 < q < \pi$ with $p > 0$, which is what keeps Newton on the critical component instead of letting it jump to one of the components through $v = 2\pi ik$. Iteration continues until the step is below 10^{-52} ; every value in Table 1 satisfies $|\Phi(p, q)| \leq 2.3 \times 10^{-61}$ and reproduces the requested R to within 10^{-50} relative, at 60 working digits.

Table 1 and Figure 1 summarise the result. The curve leaves $\beta = 1$ along the ray $R = \varepsilon$ of Section 4, rises to a maximum of 0.479352406077823612528244194045 at $R = 1.44870272081003937227067251357$, and decays as $\pi^2/(6R^2)$; the column $R^2\varepsilon_0/b$ shows the decay settling to its limit from above, but non-monotonically: it passes through 0.2728 at $R = 1$, 1.0806 at $R = 2$, reaches its maximum 1.53004 just above $R = 3$, and then decreases through 1.1994 at $R = 5$ and 1.0100 at $R = 20$. Section 6 explains both the limit and the non-monotonicity.

5.2 The apex

The maximum of ε_0 is attained at a point where the saddle is real and larger than one, and it has a closed form.

Proposition 4 (apex). *The identified curve attains its maximum at $\text{Im } v = \pi/2$, where the saddle $u = \tanh v = \coth p$ is real and satisfies $u > 1$. With p^* the unique positive root of*

$$\frac{p}{2} \coth p = \log \sinh p, \quad (33)$$

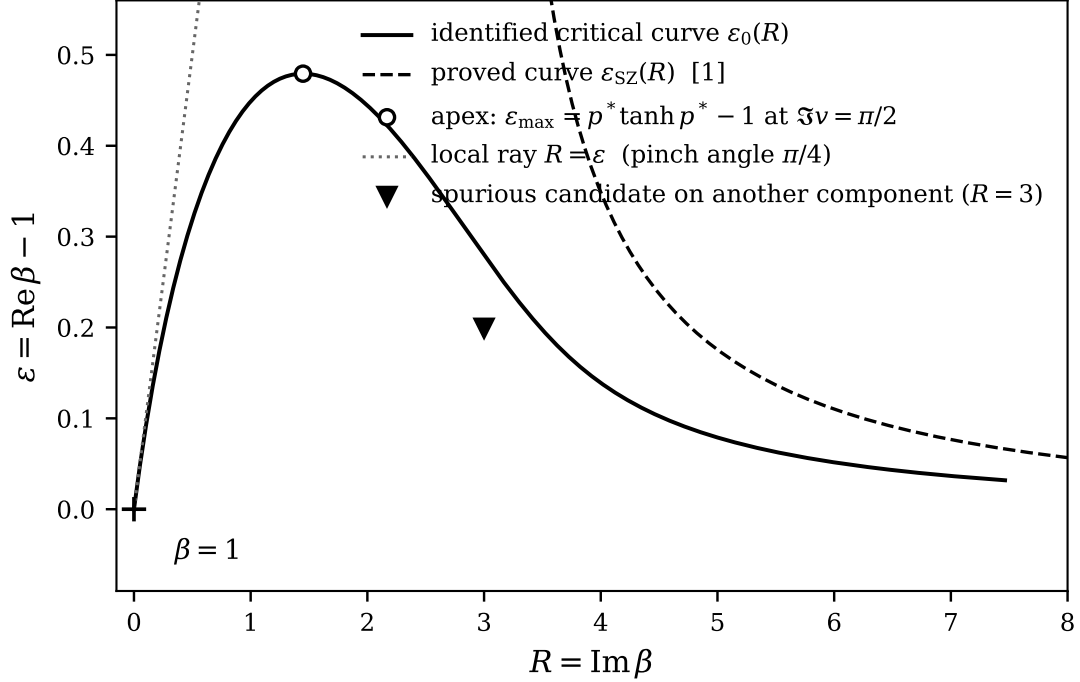


Figure 1: The identified curve $\varepsilon_0(R)$ (solid) for $0 \leq R \leq 8$, in the coordinates $\beta = 1 + \varepsilon + iR$. The dotted line is the local ray $R = \varepsilon$, the image of the ray $\arg v = \pi/8$ under $\beta - 1 = v^2/3 + O(v^4)$, along which the curve leaves $\beta = 1$ at the pinch angle $\pi/4$. The open circle is the apex of Proposition 4. The dashed curve is the proved curve $\varepsilon_{SZ}(R)$ of [10], defined for $R > \pi$, which lies strictly above ε_0 and tends to twice its value at large R (Section 6, Figure 2). The triangle at $R = 3$ is the competing candidate $\varepsilon = 0.1992045110724105$ on the component through $v = 2\pi i$, shown inadmissible in Section 7. All curves are analytic loci computed from (7) and (17), not measurements.

$$\begin{aligned}
 p^* &= 1.604029102812004728846419615409504104293416007033\dots, \\
 \varepsilon_{\max} &= p^* \tanh p^* - 1 = 0.47935240607782361252824419404539279058942478269334\dots, \\
 R_{\text{at max}} &= \frac{\pi}{2} \tanh p^* = 1.4487027208100393722706725135707582660204411776179\dots, \\
 u^* &= \coth p^* = 1.0842778882313335059393288682092972689579367735775\dots
 \end{aligned} \tag{34}$$

At that point $|\Phi(p^*, \pi/2)| = 7.8 \times 10^{-62}$ at 60 digits, and $R^2 \varepsilon_0/b = 0.611596467274$.

Proof. By (23) the locus meets the line $q = \pi/2$ exactly where (33) holds, and Proposition 3 gives exactly one such $p > 0$, namely p^* ; at that point $\tanh(p^* + i\pi/2) = \coth p^* = u^*$ is real and, since $\tanh p^* \neq 1$, larger than one. It remains to show that ε is stationary there along the locus. On the line $q = \pi/2$ the partial derivatives of (7) and of $\varepsilon = \text{Re } \beta - 1$ are

$$\Phi_q = \frac{\pi}{4 \sinh^2 p}, \quad \Phi_p = -\frac{\coth p}{2} - \frac{p}{2 \sinh^2 p}, \quad \varepsilon_q = -\frac{\pi}{\cosh 2p + 1}, \quad \varepsilon_p = \frac{\sinh 2p + 2p}{\cosh 2p + 1},$$

where Φ_q uses $\partial_q(-q \sin 2q) = \pi$ and $\partial_q \cos^2 q = 0$ at $q = \pi/2$, and ε_q uses $\partial_q(q \sin 2q) = -\pi$ and $\partial_q \cos 2q = 0$. Along the locus $dp/dq = -\Phi_q/\Phi_p$, hence

$$\left. \frac{d\varepsilon}{dq} \right|_{\text{locus}} = \varepsilon_q - \varepsilon_p \frac{\Phi_q}{\Phi_p} = -\frac{\pi}{\cosh 2p + 1} + \frac{\sinh 2p + 2p}{\cosh 2p + 1} \cdot \frac{\pi/(4 \sinh^2 p)}{\coth p/2 + p/(2 \sinh^2 p)}.$$

The second fraction simplifies because $4 \sinh^2 p (\coth p/2 + p/(2 \sinh^2 p)) = 2 \sinh p \cosh p + 2p = \sinh 2p + 2p$, so the two terms are equal and opposite and $d\varepsilon/dq = 0$ *identically in* p : every level curve of Φ that crosses the line $\text{Im } v = \pi/2$ does so with ε stationary. In particular the locus does, at $p = p^*$. $\square$

The identity $d\varepsilon/dq = 0$ on the whole line $q = \pi/2$ was checked numerically at $p = 0.5, 1.0, p^*, 2.5$ by following the level curve of Φ through each point: the central difference of ε in q falls as h^2 and reaches 10^{-17} at $h = 10^{-8}$, which is the round-off floor rather than a nonzero slope. The second derivative at the apex is negative, $d^2\varepsilon/dq^2 = -0.39183836$, so the stationary point is a local maximum. That it is the *global* maximum is numerical: along the 4001-point trace of Section 4, ε increases strictly for $q < \pi/2$ and decreases strictly for $q > \pi/2$, at every point.

Two features of the apex deserve comment. First, $u^* = \coth p^* = 1.0843 > 1$ lies outside the unit disc, so any analysis that restricts the saddle to $|u| < 1$ cannot see the top of the curve; this is the concrete instance of the warning in Section 3. Second, the apex is where the curve is farthest from the proved curve (5) in relative terms, and it is also near the radius ($R \approx 3$) at which a competing component dips below ε_0 ; both are quantified in Sections 6 and 7.

6 Both ends of the curve, and the constant b

The two ends of the curve are exact power series with rational and π coefficients. Both follow from the fact that $\beta(v)$ and $f(v)$ are even in v , so both are single-valued functions of $\zeta = v^2$ near $\zeta = 0$, and from the behaviour near the punctures $\zeta = -\pi^2 k^2$ at infinity.

6.1 The small- R end

Write

$$f(v) = \Psi(\zeta), \quad \beta - 1 = z(\zeta), \quad \zeta = v^2 = \xi + i\eta. \quad (35)$$

Solving $u = \tanh(\beta u)$ as a series in v and substituting gives, with the coefficients listed to ζ^7 in Appendix A,

$$\Psi(\zeta) = -\frac{\zeta^2}{12} + \frac{2\zeta^3}{45} - \frac{17\zeta^4}{840} + \frac{124\zeta^5}{14175} - \frac{691\zeta^6}{187110} + \cdots, \quad z(\zeta) = \frac{\zeta}{3} - \frac{\zeta^2}{45} + \frac{2\zeta^3}{945} - \frac{\zeta^4}{4725} + \cdots. \quad (36)$$

The condition $\text{Re } f = \text{Re } \Psi(\zeta) = 0$ determines η as a series in ξ ,

$$\frac{\eta}{\xi} = 1 + \frac{8}{15}\xi + \frac{71}{315}\xi^2 + \frac{3776}{23625}\xi^3 + \frac{195913}{1653750}\xi^4 + \frac{515746108}{5320940625}\xi^5 + \cdots, \quad (37)$$

and substituting into $\varepsilon = \text{Re } z(\zeta)$ and $R = \text{Im } z(\zeta)$ and eliminating ξ gives the local critical curve

$$R_0(\varepsilon) = \varepsilon + \frac{6}{5}\varepsilon^2 + \frac{171}{175}\varepsilon^3 + \frac{1224}{875}\varepsilon^4 + \frac{120987}{61250}\varepsilon^5 + O(\varepsilon^6) \quad (38)$$

where R_0 is the inverse of ε_0 near $\beta = 1$. The leading term recovers the ray $R = \varepsilon$ of Section 4; the leading correction sharpens the estimate $|R_0(\varepsilon) - \varepsilon| \leq C\varepsilon^2$ proved in [10] into the explicit constant $C = 6/5$.

Truncating (38) at the ε^4 term and evaluating it at the computed $\varepsilon_0(R)$ reproduces R with a relative deviation of -1.967×10^{-12} at $R = 10^{-3}$, -1.891×10^{-8} at $R = 10^{-2}$, -1.408×10^{-6} at $R = 0.03$ and -1.329×10^{-4} at $R = 0.1$: each deviation is the size of the first omitted term, as it must be. Including the ε^5 term improves the first two to -3.13×10^{-15} and -2.98×10^{-10} , again the size of the next omitted term.

Table 2: Confirmation of (41) for $k = 1$. Each column is the corresponding residual, computed from $\varepsilon_0(R)$ obtained by solving (32) at 60 digits and seeded from (40): the first column is $R^2\varepsilon_0$, the second is $R^4(\varepsilon_0 - \pi^2/(6R^2))$, the third subtracts also the R^{-4} term, the fourth also the R^{-6} term. The last row gives the exact values. Convergence is limited by cancellation: at $R = 10^5$ the R^8 column is accurate to about eight digits because ε_0 itself is of order 10^{-10} .

R	$R^2\varepsilon_0$	$R^4(\dots)$	$R^6(\dots)$	$R^8(\dots)$
100	1.645583804330421	6.4973748219411	34.354196742926	188.6840095127
1000	1.644940560821964	6.4939737377838	34.335516944662	188.602686804
10000	1.644934131787624	6.4939397456201	34.335330227994	188.6018735355
10^5	1.64493406749762	6.4939394057004	34.335328360835	188.6018654028
exact	$\pi^2/6 = 1.644934066848226$	$\pi^4/15 = 6.4939394022668$	$\pi^6/28 = 34.335328341975$	$c_8 = 188.6018653207$

6.2 The large- R end and the constant b

Near the puncture $v = i\pi k$ write $v = i\pi k + w$ with $w = p + is$. Solving $\Phi = 0$ for s as a series in p gives, exactly,

$$s(p) = -\frac{p^4}{6\pi k} - \frac{7p^6}{90\pi k} - \frac{p^8}{140\pi k} + \frac{(1575 - 43\pi^2 k^2)p^{10}}{56700\pi^3 k^3} + O(p^{12}), \quad (39)$$

which is (29) to leading order and confirms that the locus leaves the puncture tangent to the real axis from below. Substituting into (17) gives $R(p)$ and $\varepsilon(p)$ as series in p ; inverting R in favour of $P = \pi k/R$ gives

$$p(P) = P + \frac{P^3}{3} + \frac{P^5}{5} + \left(\frac{1}{7} - \frac{5}{36\pi^2 k^2}\right)P^7 + O(P^9), \quad (40)$$

and substituting that into $\varepsilon(p)$ gives the large- R expansion of the component through $v = i\pi k$,

$$\varepsilon_0(R) = \frac{\pi^2 k^2}{6R^2} + \frac{\pi^4 k^4}{15R^4} + \frac{\pi^6 k^6}{28R^6} + \pi^8 k^8 \left(\frac{1}{45} - \frac{5}{216\pi^2 k^2}\right) \frac{1}{R^8} + O(R^{-10}) \quad (41)$$

For the critical component, $k = 1$, so

$$b = \lim_{R \rightarrow \infty} R^2 \varepsilon_0(R) = \frac{\pi^2}{6} = 1.6449340668482264364724151666\dots, \quad (42)$$

and the next coefficients are $\pi^4/15 = 6.4939394022668\dots$, $\pi^6/28 = 34.335328341975\dots$ and $c_8 = \pi^8(\frac{1}{45} - \frac{5}{216\pi^2}) = 188.6018653206585\dots$. For the component through $v = 2\pi i$, $k = 2$, the same computation gives $b_2 = 2\pi^2/3 = 4b$; in general $b_k = k^2 b$. That is why the critical component is the *lowest* of the components at large R , and why the ordering can only fail at intermediate R , which is what happens near $R = 3$ (Section 7).

Table 2 confirms all four coefficients along the independently solved curve: the first three agree with the exact values to eleven, ten and nine digits at $R = 10^5$, and the fourth to eight. The non-monotone column $R^2\varepsilon_0/b$ of Table 1 is also explained by (41): since every correction is positive, $R^2\varepsilon_0/b$ approaches 1 from above at large R , with

$$\frac{R^2\varepsilon_0(R)}{b} = 1 + \frac{2\pi^2}{5R^2} + O(R^{-4}) \quad (43)$$

giving 1.4386, 1.2467, 1.0617, 1.0395 and 1.0099 at $R = 3, 4, 8, 10, 20$, against the computed 1.5300, 1.3521, 1.0672, 1.0417 and 1.0100. The agreement at $R = 3$ is poor, as expected from a R^{-2} expansion, and excellent from $R = 8$ on. The peak of $R^2\varepsilon_0/b$ near $R = 3$ and its monotone decrease afterwards is therefore a genuine feature of the curve, not a numerical artifact.

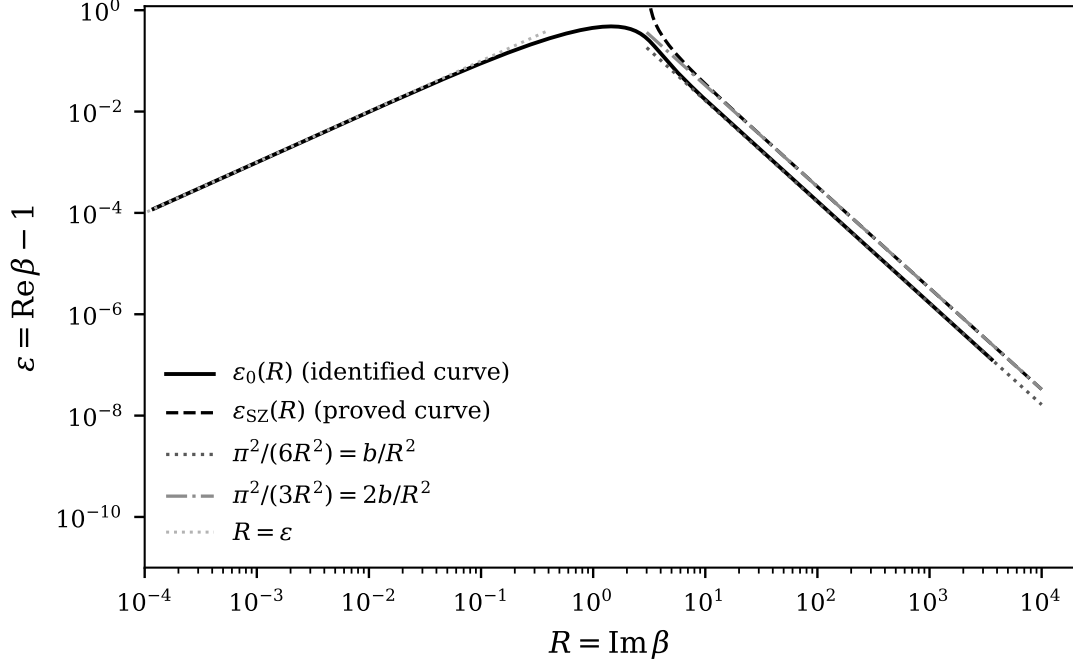


Figure 2: The identified curve $\varepsilon_0(R)$ (solid) and the proved curve $\varepsilon_{SZ}(R)$ (dashed) on logarithmic axes, over nine decades of R , with their exact asymptotes $\pi^2/(6R^2) = b/R^2$ (dotted) and $\pi^2/(3R^2) = 2b/R^2$ (dash-dotted) and the local ray $R = \varepsilon$ (fine dotted). The identified curve is the lower of the two for every $R > \pi$ and the ratio of the two tends to 2; both are analytic loci computed from (7), (17) and (5).

6.3 The proved curve is an interior curve, and the factor of two

The curve (5) on which Shamis and Zeitouni proved $F_\beta > 0$ is not a different object: it is a locus inside the same branch, obtained by an additional restriction. Setting $\text{Im } u = 0$ on branch k forces the saddle to be real, $u = \pi k/R + O(R^{-3})$, and solving $\text{Im } \beta = R$ then gives

$$\varepsilon_{SZ}(R) = \frac{R}{2\pi} \log \frac{1 + \pi/R}{1 - \pi/R} - 1, \quad R^2 \varepsilon_{SZ}(R) \xrightarrow{R \rightarrow \infty} \frac{\pi^2}{3} = 2b. \quad (44)$$

The mechanism is visible in (40): the critical saddle has $p(P) = P + P^3/3 + P^5/5 + O(P^7) = \text{artanh } P + O(P^7)$, so to sixth order in $P = \pi k/R$ it coincides with the real saddle, and the two loci differ first at order R^{-8} in the saddle but already at order R^{-2} in ε , because the imaginary part $s(p)$ of the critical saddle is $O(p^4)$.

Two things follow, and both were checked numerically. First, on the proved curve $\text{Re } f_\beta < 0$ strictly, so it lies inside the region $F_\beta > 0$ rather than on its boundary: reproducing ε_{SZ} from $\text{Im } u = 0$, $k = 1$ at seven radii gives $\text{Re } f_\beta = -5.09 \times 10^{-1}$ at $R = 3.2$, -6.37×10^{-2} at $R = 4$, -8.81×10^{-4} at $R = 10$, -8.12×10^{-8} at $R = 100$, -8.12×10^{-12} at $R = 1000$ and -8.12×10^{-16} at $R = 10^4$, always negative and decaying like R^{-4} , as $-\pi^4/(12R^4)$ predicts. Second, the factor of two between $\pi^2/3$ and $\pi^2/6$ is exactly the statement that the proved curve is an interior curve of the same branch: it is not a competing estimate of b but a different, strictly larger, quantity.

Figure 2 shows the same information over more than eight decades. The two curves are visibly distinct only for R of order a few; beyond $R \approx 20$ both follow their R^{-2} asymptotes, with a factor of two between them, and below $R \approx 0.1$ the identified curve follows the ray $R = \varepsilon$.

6.4 Status of the two series

The series (38) and (41) were produced twice, by two different exact routes. The first is a symbolic computation in the source investigation [2]: its small- R half completed, and its large- R half produced the series (39) and the two series that (41) is built from, but the final identification step did not finish inside the time budget allotted to it, so no complete symbolic artifact for the large- R end was retained; every coefficient it did reach is confirmed numerically along the independently traced curve. The second route is an exact truncated-series computation written for this paper, in rational and π arithmetic and with no symbolic-algebra dependency, which reproduces every coefficient quoted above and adds the ones the first route did not reach: the ε^5 term of (38), the ξ^5 term of (37), the p^{12} and p^{14} terms of (39), the P^9 and P^{11} terms of (40), and the whole of (41) for $k = 2$. Appendix A lists the coefficients; Appendix B records the protocol.

7 Admissibility: which saddle is the boundary

7.1 At fixed radius the equation has several solutions

The condition $\operatorname{Re} f_\beta = 0$ is one real equation for the two real numbers (ε, R) , so its solution set is a union of curves, and the slice at fixed R can contain several of them. A bounded multi-start Newton search for (32) over $0 < p < 6$, $|q| < 4\pi$, with 10^4 starting points on a grid and every converged root classified by $\operatorname{Im} v/\pi$, finds twelve solutions at $R = 3$, of which four have $\varepsilon > 0$:

$$\begin{array}{c|cc} \operatorname{Im} v/\pi & \pm 0.8543923788 & \pm 1.41713698 \\ \hline \varepsilon & 0.2796455457731802 & 0.1992045110724105 \end{array} \quad (45)$$

The first pair lies on the critical component of Section 4 ($|\operatorname{Im} v| < \pi$); the second lies on the component through $v = 2\pi i$, and it is *lower*: $\varepsilon = 0.1992 < \varepsilon_0(3) = 0.2796$. The same search at $R = 4$ finds $\varepsilon = 0.1390051632395512$ on the critical component and $\varepsilon = 0.4650961926617782$ on the $v = 2\pi i$ component, this time *higher*. So the two components cross somewhere between $R = 3$ and $R = 4$, and an argument that selects the boundary by taking the smallest ε over all solutions of $\operatorname{Re} f_\beta = 0$ at fixed R gives the wrong curve for R near 3. This is the only place where the identification of ε_0 as the phase boundary needs input beyond the saddle equation.

7.2 The unconditional bound does not discriminate

The bound used in the source investigation [2] is

$$\begin{aligned} F_\beta &\leq B(\beta), & H(p) &= -p \log p - (1-p) \log(1-p), \\ B(\beta) &= \sup_{0 \leq m \leq 1} \left[\frac{1}{2} \operatorname{Re} \beta m^2 + H\left(\frac{1+m}{2}\right) - \log 2 \right], \end{aligned} \quad (46)$$

which follows from $|Z_{\beta,N}| \leq 2^{-N} \sum_k \binom{N}{k} e^{\operatorname{Re} \beta m_k^2/2}$ with $m_k = (2k - N)/N$, Stirling's formula for $\binom{N}{k}$, and $\log |Z| \leq \log(N+1) + \max_k \log(\text{summand}_k)$: no assumption about saddles, contours or dominance enters it. It is also exactly zero for $\operatorname{Re} \beta \leq 1$, since the supremum is then attained at $m = 0$; that was checked at $\beta = 1 + 2i$, $1 + 3i$, $1 + 10i$, $0.5 + i$, $-1 + 4i$, $1.0001 + 3i$ (where $B = 7.5 \times 10^{-9}$, correctly nonzero), $1.3 + 3i$ and 2 , with the vanishing cases agreeing with zero to 10^{-38} at 40 working digits.

At the spurious candidate $\beta = 1.1992 + 3i$ the bound gives $B = 0.0239264 > 0 = \operatorname{Re} f_\beta$ at the competing saddle, so no contradiction arises and the candidate is not excluded. The bound is an

Table 3: The exact finite- N free energy at $R = 3$, $\beta = 1 + \varepsilon + 3i$, as $NF_N = \log|Z_{\beta,N}|$ for $N = 250, 500, 1000, 2000$, computed from (47) at the precision (48) (63–137 digits, rising with N). The last two columns are the trivial-saddle prediction $-\frac{1}{2} \log|1 - \beta|$ and the largest discrepancy between the recomputed values and those of [2], which agree to the seven printed decimals. The row $\varepsilon = 0.1992$ is the competing candidate of (45) and the row $\varepsilon = 0.27965$ is the critical component rounded to five decimals, which is why its behaviour is erratic rather than convergent.

ε	position	$B(\beta)$	NF_N at $N =$				$-\frac{1}{2} \log 1 - \beta $	agree
			250	500	1000	2000		
0.15	below	0.0142675	-0.5508577	-0.5503923	-0.5501612	-0.5500458	-0.54993036	3.8e-8
0.1992	competing	0.0239264	-0.5517492	-0.5508735	-0.5506397	-0.5505228	-0.55040596	2.8e-8
0.22	below	0.0285869	-0.5535851	-0.5511181	-0.5508819	-0.5507644	-0.55064699	3.0e-8
0.25	below	0.0358525	-0.5215471	-0.5564652	-0.5512682	-0.5511546	-0.55103626	2.5e-8
0.27965	on curve	0.0436145	+0.7536457	+0.3622347	+0.6369132	+0.3935363	-0.55146910	4.1e-8
0.3	above	0.0492501	+2.7435029	+4.8960483	+9.2686962	+18.0157777	-0.55179373	3.6e-8
0.35	above	0.0640619	+8.1199970	+15.7036151	+30.8711129	+61.2060961	-0.55268597	4.2e-8

exclusion filter: it rules out saddles whose $\text{Re } f_\beta$ is more negative than $-B(\beta)$, and it says nothing about saddles with $\text{Re } f_\beta = 0$. Both candidates in (45) have $\text{Re } f_\beta = 0$ by construction, which is precisely why they compete. A different filter is needed.

7.3 The exact finite- N free energy

The filter used here is the exact finite- N average

$$Z_{\beta,N} = 2^{-N} \sum_{k=0}^N \binom{N}{k} \exp\left(\frac{\beta(2k - N)^2}{2N}\right), \quad F_N(\beta) = \frac{1}{N} \log|Z_{\beta,N}|, \quad (47)$$

which is a finite sum of explicitly known terms and therefore computable exactly; no saddle, contour or dominance assumption enters it. Its limit $F_\beta = \lim_{N \rightarrow \infty} F_N(\beta)$ is the object the curve is supposed to describe: $F_\beta = 0$ in the high-temperature region and $F_\beta > 0$ outside it [10]. The summands of (47) have modulus as large as $e^{NB(\beta)}$ while their sum is $O(1)$, so evaluating the sum needs a working precision that grows with N ; the criterion used throughout is

$$\text{dps} \geq \lceil 0.6 N \max\{B(\beta), 0.02\} \rceil + 60, \quad (48)$$

which sizes the precision per (β, N) rather than fixing it once. That the criterion matters is shown by a worked failure case. At $\beta = 1.4 + 10i$ and $N = 6400$ the cancellation in the sum costs 44.8 decimal digits here (the two independent implementations of [2] measure 46.6), so a fixed working precision of 40 digits returns $NF_N = 427.0155$ and one of 50 digits returns 405.4129, while the criterion ($\text{dps} = 368$) returns

$$NF_N = 405.3258758810,$$

which agrees with the converged value 405.3258758807573387637597 reported in [2] to ten digits and is stable for every precision from 60 to 368 digits. At 50 digits the two implementations of [2] return 405.0446 and 405.0544: below the cancellation threshold the answer is noise, and different implementations produce different noise.

Table 3 is the result. Below the curve the four values of NF_N march steadily towards $-\frac{1}{2} \log|1 - \beta|$, the value the trivial saddle $u = 0$ predicts: at $\varepsilon = 0.15$ they are -0.5508577 , -0.5503923 , -0.5501612 , -0.5500458 against a prediction of -0.54993036 , so $F_\beta = 0$ and the trivial saddle dominates. The competing candidate $\varepsilon = 0.1992$ behaves in exactly the same way, $-0.5517492 \rightarrow$

Table 4: Confirmation of (50). Column two is $\text{Re}[\beta(1 - \beta)]$, the condition for the Laplace expansion; column three the predicted coefficient; column four the measurement $N(NF_N + \frac{1}{2} \log |1 - \beta|)$ at $N = 2000$ from (47); column five their difference. The residuals are all negative and of size 10^{-4} , consistent with the omitted $O(N^{-2})$ term in (49), whose coefficient would be of order 0.2. At $N = 1000$ the same measurement gives -0.230954 , -0.233711 , -0.231926 , -0.187729 , -0.239640 , -0.242845 , approaching the predictions from above in five of the six cases.

β	$\text{Re}[\beta(1 - \beta)]$	c_1 predicted	measured at $N = 2000$	difference
$1.15 + 3i$	8.8275	-0.230742	-0.230795	$-5.3\text{e-}5$
$1.1992 + 3i$	8.7611	-0.233605	-0.233658	$-5.3\text{e-}5$
$1.25 + 3i$	8.6875	-0.236587	-0.236640	$-5.2\text{e-}5$
$1 + 2i$	4.0000	-0.187500	-0.187615	$-1.2\text{e-}4$
$1 + 5i$	25.0000	-0.240000	-0.240020	$-2.0\text{e-}5$
$1.4 + 2i$	3.4400	-0.242604	-0.242725	$-1.2\text{e-}4$

-0.5505228 against -0.55040596 : it too has $F_\beta = 0$, so it is *inside* the region where the free energy vanishes, not on its boundary. On the curve, at $\varepsilon = 0.27965$, the values oscillate, $+0.754$, $+0.362$, $+0.637$, $+0.394$, with no sign of settling: this is the marginal case, where two contributions of the same exponential order interfere. Above the curve, at $\varepsilon = 0.3$ and 0.35 , NF_N grows linearly in N , giving $F_N \rightarrow 0.0087$ and 0.0303 respectively by Richardson extrapolation from $N = 1000$ and 2000 , both positive and both comfortably below the corresponding bounds $B(\beta) = 0.0492501$ and 0.0640619 , as they must be. The transition is therefore bracketed between $\varepsilon = 0.27965$ and $\varepsilon = 0.3$ at $R = 3$ by direct computation, and the analytic value $\varepsilon_0(3) = 0.2796455457731802$ sits at the lower end of that bracket. The competing candidate at $\varepsilon = 0.1992$ is excluded not by a bound but by the fact that the free energy there is still the trivial one.

7.4 The rate of approach: a two-term expansion

The convergence in Table 3 is not accidental and its rate can be computed. Near $u = 0$,

$$f_\beta(u) = \frac{1}{2}\beta(1 - \beta)u^2 + \frac{\beta^4}{12}u^4 + O(u^6), \quad a = \beta(1 - \beta),$$

so a Laplace expansion of the trivial saddle, valid wherever $\text{Re } a > 0$ and no admissible saddle has $\text{Re } f_\beta < 0$, gives

$$Z_{\beta,N} = (1 - \beta)^{-1/2} \left[1 - \frac{\beta^4}{4Na^2} + O(N^{-2}) \right], \quad NF_N = -\frac{1}{2} \log |1 - \beta| + \frac{c_1(\beta)}{N} + O(N^{-2}), \quad (49)$$

with

$$c_1(\beta) = -\text{Re} \frac{\beta^4}{4[\beta(1 - \beta)]^2}. \quad (50)$$

The leading factor follows because the Gaussian integral contributes $\sqrt{2\pi/(Na)}$ and the Hubbard–Stratonovich prefactor $\sqrt{\beta N/2\pi}$, whose product is $\sqrt{\beta/a} = (1 - \beta)^{-1/2}$; the correction uses the fourth moment $\langle u^4 \rangle = 3/(Na)^2$ of that Gaussian. Expansion (49) is a refinement of the limit $Z_{\beta,N} \rightarrow (1 - \beta)^{-1/2}$, which is the $N \rightarrow \infty$ statement behind the column $-\frac{1}{2} \log |1 - \beta|$ in Table 3.

Table 4 confirms (50) at six points, three of them below the curve at $R = 3$ (including the competing candidate, whose coefficient differs from that of $\varepsilon = 0.15$ in the fourth decimal and is resolved by the data), two on the line $\text{Re } \beta = 1$, and one at $R = 2$. The condition $\text{Re } a > 0$ holds at all six; it fails close to $\beta = 1$ inside the curve, where $\varepsilon \approx R$ makes $\text{Re}[\beta(1 - \beta)] = R^2 - \varepsilon - \varepsilon^2 < 0$, and

there a different contour would be needed. The approach to the limit is clean only for $N \gtrsim 1000$: at $N = 250$ the relative deviation of $Z_{\beta,N}$ from $(1 - \beta)^{-1/2}$ is 1.1×10^{-3} at $\beta = 1.15 + 3i$ but 1.2×10^{-1} at $\beta = 1.25 + 3i$ and 5.3×10^{-2} at $\beta = 1 + 5i$, where higher-order terms and the proximity of the curve both matter.

7.5 The bracketing diagnostic, and where it fails

A second, weaker reading of the same data locates the boundary directly: solve $F_N(1 + \varepsilon + iR) = 0$ for ε at fixed R and N , and let $N \rightarrow \infty$. Doing so with a bisection on the exact sum gives the following deviations $\varepsilon_N(R) - \varepsilon_0(R)$ at $N = 250, 500, 1000, 2000$:

R	$N = 250$	$N = 500$	$N = 1000$	$N = 2000$
1	-0.0212	-0.01231	-0.006451	+4.6e-6
2	-0.0118	-0.00267	-0.000704	-1.9e-5
3	-0.01023	-0.001414	-0.002304	-0.0004886
4	-0.01444	-0.007138	-0.003313	-0.001123
5	-0.003863	+0.002381	-0.00364	-0.001781
8	+0.02871	-0.003078	+0.0003464	+0.002694

At $R = 1, 2$ and 4 the deviations are negative and shrink roughly by half each time N doubles, which is the $1/N$ behaviour (49) predicts, and they bracket ε_0 from below as they should. At $R = 3, 5$ and 8 they change sign with N and do not settle: at $R = 8$ the sequence is $+0.0287, -0.0031, +0.0003, +0.0027$. Replacing the target 0 by the prefactor-corrected target $-\frac{1}{2} \log |1 - \beta|/N$ makes matters worse, not better: at $R = 4, N = 250$ the deviation grows from -0.01444 to -0.1033 , and at $R = 5, N = 1000$ from -0.00364 to -0.0471 . The reason is that the correction in (49) is derived for points strictly below the curve, where the trivial saddle is isolated and dominant; at the boundary itself the marginal saddle contributes at the same exponential order and the $1/N$ expansion is not uniform in ε . The diagnostic is therefore supporting evidence at the radii where it is monotone and worth nothing at the radii where it is not; in particular it cannot by itself distinguish the two candidates at $R = 3$, and Table 3 is what does.

Figure 3 shows the same seven quantities against N , on logarithmic axes, so that the three regimes are separated by slope rather than by value: the offsets below the curve fall as $1/N$, the offset on the curve does not settle, and the offsets above it are flat in F_N , that is linear in NF_N .

7.6 What this does and does not establish

The evidence assembled in this section is: the curve is a component of $\{\text{Re } f_\beta = 0\}$, uniquely determined and traced without branch ambiguity (Section 4); its two ends are exact power series with closed-form leading coefficients (Section 6); at radii where other components compete, the exact finite- N free energy puts the competing saddle inside the region $F_\beta = 0$ rather than on its boundary (Table 3); and the finite- N free energy converges to the trivial-saddle value everywhere below the curve at a rate given by (50).

It is not a proof that $\varepsilon_0(R)$ is the phase boundary. What is missing is a contour deformation: one would have to show that for β below the curve the integration contour of the Hubbard–Stratonovich representation can be deformed to pass through $u = 0$ without crossing the order- N poles of $\cosh(\beta u)^{-N}$ and without picking up a saddle of larger $\text{Re } f_\beta$, and that above the curve some admissible saddle takes over. The bound (46) is an exclusion filter and supplies neither half of that argument; the finite- N data are consistent with the claimed dominance and quantify its rate,

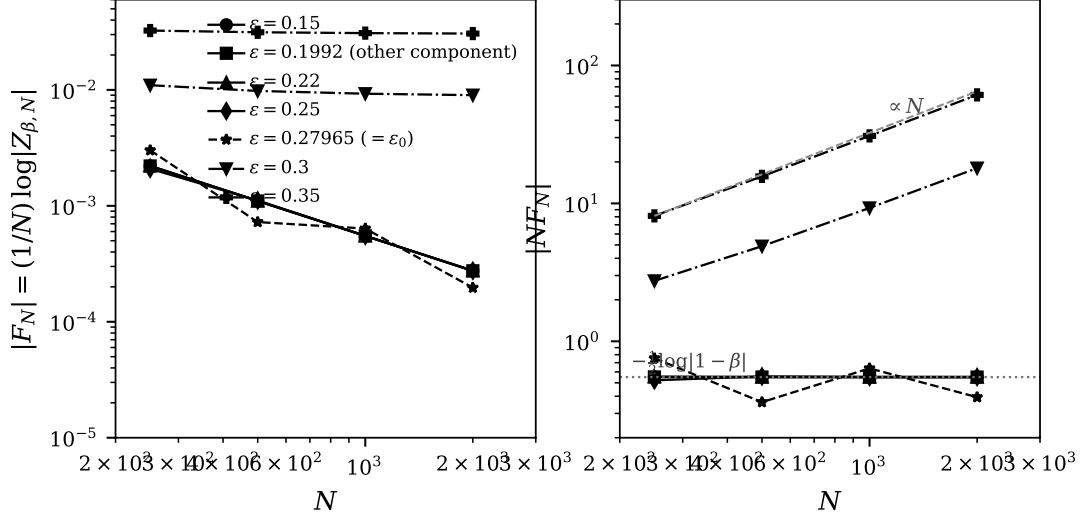


Figure 3: The exact finite- N free energy at $R = 3$, $\beta = 1 + \varepsilon + 3i$, as a function of N on logarithmic axes, for the seven offsets of Table 3, distinguished by marker and line style: *left*, $|F_N| = N^{-1} \log |Z_{\beta,N}|$; *right*, $|NF_N| = |\log Z_{\beta,N}|$, with the trivial-saddle value $-\frac{1}{2} \log |1 - \beta| = 0.55040596$ at the competing candidate $\varepsilon = 0.1992$ (dotted horizontal line) and a guide proportional to N through the $\varepsilon = 0.35$ data (dashed). The four offsets below the curve, including the competing candidate, fall as $1/N$ towards the horizontal line; the offset on the curve, $\varepsilon = 0.27965$ (starred, dashed), is erratic and has no limit; the two above it grow linearly in N . All values are computed here from (47) at the precision (48).

but they are data. The honest statement is the one made in the abstract: the curve is identified, computed to high precision, and validated against exact finite- N quantities and exact Fisher zeros, and the identification of that curve as the phase boundary remains conjectural.

8 Exact Fisher zeros and the curve they approach

8.1 The zeros are roots of an exact integer polynomial

The finite- N average (47) is not merely computable: as a function of β it is a polynomial in disguise. Writing $t = e^{2\beta/N}$,

$$\exp\left(\frac{\beta(2k - N)^2}{2N}\right) = \exp\left(\frac{\beta N}{2}\right) t^{k^2 - Nk},$$

so, with $M = \lfloor N^2/4 \rfloor$,

$$Z_{\beta,N} = 0 \iff P_N(t) = \sum_{k=0}^N \binom{N}{k} t^{k^2 - Nk + M} = 0, \quad \beta = \frac{N}{2} \text{Log } t. \quad (51)$$

P_N has exact non-negative integer coefficients, degree M , constant term $\binom{N}{\lfloor N/2 \rfloor}$ (from $k = \lfloor N/2 \rfloor$, the unique integer with $k^2 - Nk + M = 0$) and leading coefficient $\binom{N}{0} + \binom{N}{N} = 2$ (from $k = 0$ and $k = N$), so $t = 0$ is not a root and every root gives a β through the principal logarithm. Because $Z_{\beta,N}$ is periodic in β with period $i\pi N$, the strip $0 \leq \text{Im } \beta < \pi N$ contains exactly one zero per root of P_N : there are $M = N^2/4$ zeros for N even, and $Z_{\bar{\beta},N} = \overline{Z_{\beta,N}}$ makes them symmetric about the real axis while the positivity of every summand in (47) for real β excludes real zeros. This gives the

Table 5: The family of Fisher zeros that approaches the identified curve. Counts, deviations $\delta = \varepsilon_z - \varepsilon_0(R_z)$ and the mean spacing are recomputed independently for $N = 20$ and $N = 40$ (agreeing with [2] to the printed digits); the $N = 80$ row is quoted from there, since a degree-1600 companion problem at 50 digits is outside the budget of this work. Every δ is negative: the zeros approach the curve from the side $\text{Re } \beta < 1 + \varepsilon_0(R)$, that is from the region where $F_\beta = 0$. The worst polishing residual is $\max |Z_{\beta,N}|$ over all $N^2/4$ roots, and the family residual is the worst over the family only.

N	zeros	family	excluded	$\max \delta $	$\text{mean} \delta $	$\max N \delta $	$\max Z $
20	100	4	8	0.085643	0.067824	1.7129	1.7e-42
40	400	13	12	0.095152	0.048783	3.8061	9.4e-38
80	1600	31	19	0.090790	0.031855	7.2632	2.2e-20

Fisher zeros *exactly*, as algebraic numbers, with no quadrature, no discretisation and no precision loss in their definition; the only approximation is in the root-finding itself.

Every root reported here was computed twice: once as a root of P_N by a companion matrix in double precision, mapped through $\beta = (N/2) \text{Log } t$, and once by Newton iteration on the $(N+1)$ -term sum (47) itself at 50 digits, started from the companion root. The recorded residual is $|Z_{\beta,N}|$ at the polished root.

8.2 The family that approaches the curve

Most of the $N^2/4$ zeros are far from the curve; they spread over the disc $|\beta| \leq \pi N/2$ and are irrelevant to the thermodynamic limit at fixed β . The family that matters is the one whose members satisfy $|\varepsilon_z - \varepsilon_0(R_z)| < 0.1$ with $0.02 < R_z < 8$, where $\varepsilon_z = \text{Re } \beta_z - 1$ and $R_z = \text{Im } \beta_z$: those are the zeros that track the identified curve.

Table 5 collects the counts. The key quantity is the deviation $\delta = \varepsilon_z - \varepsilon_0(R_z)$, which is negative for every member of the family at all three sizes, and whose magnitude shrinks like $1/N$ at fixed radius:

$$\begin{aligned}
N\delta &\longrightarrow -g(R), \\
g(0.6) &\approx 1.15, \quad g(1.1) \approx 1.19, \quad g(1.5) \approx 1.35, \\
g(2.0) &\approx 1.6, \quad g(4.2) \approx 3.8, \quad g(6.6) \approx 7.3.
\end{aligned} \tag{52}$$

At $R \approx 0.7$ the three sizes give $N\delta = -1.1468, -1.1581$ and -1.1643 ; at $R \approx 1.1$, $-1.1829, -1.1626$ and -1.2059 ; at $R \approx 1.5$, $-1.3833, -1.3189$ and -1.3463 . The sequence is not monotone in N at every radius, because the radii themselves move as N grows, but the values agree to a few percent and the $1/N$ scaling is unambiguous. This is the strongest single piece of evidence that $\varepsilon_0(R)$ is an accumulation locus of the Fisher zeros: the zeros converge to it, from below, at the rate $1/N$ that a one-dimensional zero density implies.

The spacing along the curve behaves the same way, but only if it is measured locally. Over the whole family the mean spacing in R is 0.522303 at $N = 20$, 0.316422 at $N = 40$ and 0.21192 at $N = 80$; multiplied by N these give 10.45, 12.66 and 16.95, which is *not* a constant, because the extent of the family in R grows with N and the family is much denser in the middle than at its ends. Restricted to the central region $0.8 < R < 2.5$, where the curve is nearly straight, the mean spacing is 0.524633 at $N = 20$ (two gaps) and 0.268648 at $N = 40$ (five gaps), so N times the local spacing equals 10.493 and 10.746: constant to within 2.4% across a doubling of N , which is the $1/N$ law. The comparison against $4\pi/N = 12.566/N$ made in [2] is within 1% of the family-wide mean at $N = 40$ but 35% off it at $N = 80$; the local measurement above is the scale-invariant one, and it is 15% below $4\pi/N$ at both sizes.

Table 6: The four family members at $N = 20$ and the thirteen at $N = 40$, recomputed independently: the radius $R_z = \text{Im } \beta_z$, the position $\varepsilon_z = \text{Re } \beta_z - 1$, the curve value $\varepsilon_0(R_z)$ obtained by solving (32), the deviation $\delta = \varepsilon_z - \varepsilon_0(R_z)$, the scaled deviation $N\delta$, and the polishing residual $|Z_{\beta,N}|$. Values agree with [2] to all ten printed digits.

	R_z	ε_z	$\varepsilon_0(R_z)$	δ	$N\delta$	$ Z $
$N = 20$						
	0.61496279	0.30284725	0.36018866	-0.057341	-1.1468	9.0e-46
	1.13260527	0.40559977	0.46474514	-0.059145	-1.1829	3.7e-45
	1.64389465	0.40534528	0.47451260	-0.069167	-1.3833	5.5e-45
	2.18187031	0.33501849	0.42066160	-0.085643	-1.7129	4.4e-45
$N = 40$						
	0.39246606	0.23873835	0.27155870	-0.032820	-1.3128	1.4e-45
	0.68337793	0.35228782	0.38123974	-0.028952	-1.1581	3.0e-44
	0.95080982	0.41209119	0.44115564	-0.029064	-1.1626	2.1e-44
	1.21137147	0.44072850	0.47129801	-0.030570	-1.2228	1.0e-43
	1.47187353	0.44630728	0.47928094	-0.032974	-1.3189	7.9e-44
	1.73663948	0.43290526	0.46908447	-0.036179	-1.4472	3.7e-44
	2.00938065	0.40303401	0.44327082	-0.040237	-1.6095	6.8e-44
	2.29404764	0.35860737	0.40390339	-0.045296	-1.8118	2.9e-44
	2.59552064	0.30155216	0.35316566	-0.051614	-2.0645	1.6e-44
	2.92054281	0.23446051	0.29404865	-0.059588	-2.3835	1.5e-44
	3.27949036	0.16157261	0.23130444	-0.069732	-2.7893	3.5e-45
	3.69058960	0.09007565	0.17208136	-0.082006	-3.2802	3.5e-45
	4.18952558	0.02791186	0.12306363	-0.095152	-3.8061	3.0e-45

8.3 Sample zeros

Table 6 lists the family members, and Figure 4 places them, together with all the other zeros in the window and the same data at $N = 80$, against the identified curve. The zero of $Z_{\beta,20}$ closest to $\beta = 1$ is

$$\beta = 1.302847251454284086921133 + 0.61496278699975940503469704i, \quad |Z_{\beta,20}| = 9.0 \times 10^{-46}, \quad (53)$$

which reproduces the twenty-five-digit value quoted in [2] to 4.3×10^{-26} , that is to the last printed digit.

8.4 Independent confirmation, and one honest difficulty

Two checks confirm that (51) really produces all the zeros in a region, without relying on the polynomial route at all. First, the argument principle: integrating $dZ_{\beta,N}/Z_{\beta,N}$ once round the boundary of the box $0.05 < \varepsilon < 0.90$, $1.0 < R < 4.0$ at $N = 40$, using the direct sum (47) at 50 digits, gives

$$\frac{1}{2\pi i} \oint \frac{dZ_{\beta,40}}{Z_{\beta,40}} = 9.000000000000,$$

and the polynomial route finds exactly 9 zeros inside that box. The same test at $N = 80$ in [2] gives 18.000000000000007 against 18 polynomial zeros. Second, a completely different search — a coarse scan of $|Z_{\beta,N}|$ over a grid in the β -plane, followed by high-precision Newton on the direct sum, with no polynomial involved — found 19 zeros in the upper half plane inside its window at $N = 40$, all 19 matching the polynomial route to 10^{-8} .

The one difficulty worth recording is numerical rather than mathematical. Matching the independently recomputed zeros against those of [2] one by one succeeds for 45 of 50 upper-half-plane zeros at $N = 20$ and 190 of 201 at $N = 40$ to within 10^{-20} ; the exceptions have discrepancies up to

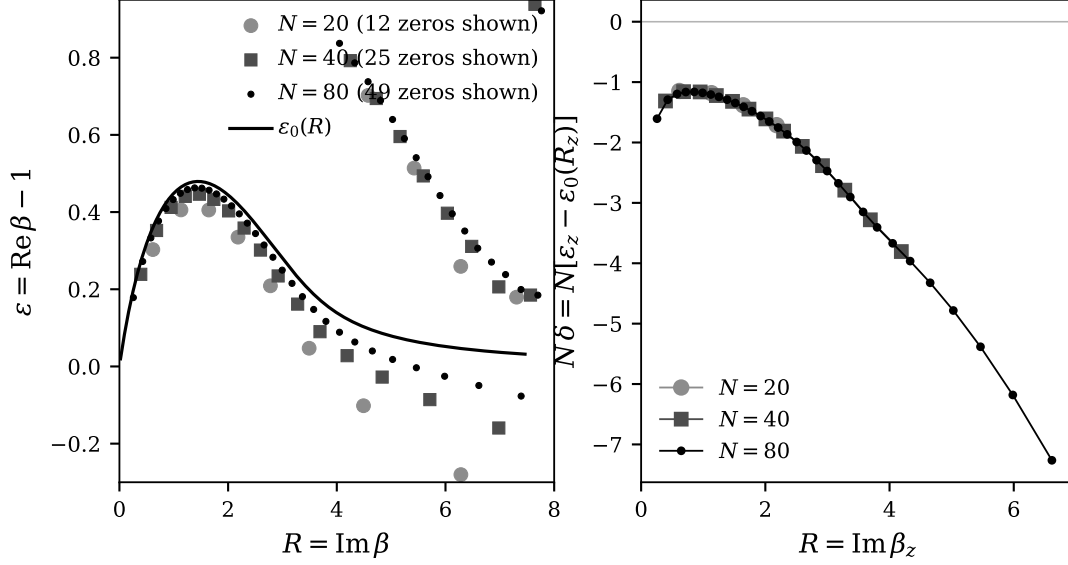


Figure 4: The exact finite- N Fisher zeros of $Z_{\beta,N}$ from (51), in the coordinates $(R, \varepsilon) = (\text{Im } \beta, \text{Re } \beta - 1)$, restricted to $0.02 < R < 8$ and $-0.35 < \varepsilon < 1.0$: *left*, all upper-half-plane zeros at $N = 20$ (circles, light), $N = 40$ (squares) and $N = 80$ (dots, dark), with the identified curve $\varepsilon_0(R)$ (solid); *right*, the scaled deviation $N\delta = N[\varepsilon_z - \varepsilon_0(R_z)]$ for the family members at each size (Table 6 lists those at $N = 20$ and 40), against R_z , showing that $N\delta$ tends to a limiting profile $-g(R)$ that is negative over the whole range. The zeros at $N = 20$ and $N = 40$ were recomputed independently here and the plotted positions agree with those of [2] to all plotted digits; the $N = 80$ positions are those tabulated there. All zeros are exact algebraic numbers, roots of an integer polynomial; only their numerical location is approximate.

1.3×10^{-4} at $N = 20$ and 5.7×10^{-3} at $N = 40$, and they are concentrated at $|\beta| \approx \pi N/2$, the outer edge of the zero cloud. There the double-precision companion matrix is severely ill-conditioned and its eigenvalues form near-degenerate clusters, so individual members of a cluster are not resolved even though the cluster as a whole is: the residuals $|Z|$ of the mismatched roots remain at 10^{-38} or smaller, which is what one expects of a correct zero of the same cluster. The discrepancy is therefore in the labelling of near-coincident roots, not in the zero set, and the argument-principle count — which is insensitive to labelling — agrees exactly.

8.5 What the zeros do and do not prove

The exact reduction (51) is a theorem: the Fisher zeros of the Curie–Weiss model at finite N are the images under $\beta = (N/2) \text{Log } t$ of the roots of an explicit integer polynomial, and there are exactly $N^2/4$ of them in a period strip. That the family of Table 6 converges to $\varepsilon_0(R)$ at rate $1/N$ from below is a numerical fact, established at three sizes with residuals of 10^{-38} or better, and it is consistent with everything else in this paper: the zeros accumulate where the free energy changes behaviour, and they do so from the side on which $F_\beta = 0$. Turning that into a proof that $\varepsilon_0(R)$ is the phase boundary would require a limiting zero-density argument — uniform control of $Z_{\beta,N}$ on compact sets away from the locus, and hence Hurwitz-type convergence of the zeros to the boundary of the region of non-vanishing — which is not attempted here. What the data establish is the identification: the curve the zeros approach, the curve on which $\text{Re } f_\beta$ vanishes for the admissible saddle, and the curve along which the finite- N free energy changes its limiting behaviour are the same curve, to the precision to which all three can be measured.

9 Scope, and what remains open

9.1 What has been established

Conjecture 1 asserts the existence of a curve separating $F_\beta = 0$ from $F_\beta > 0$, and describes it as the locus on which the dominant saddle of f_β has vanishing real part. This paper supplies the curve. The branch-free parametrisation $v = \beta u$, $u = \tanh v$, $\beta = v / \tanh v$ of Section 3 turns the saddle equation into an identity and removes every branch choice, including the choice $|u| < 1$ that the apex of the curve violates; the real part of the action becomes the single-valued function $\Phi(p, q)$ of (7), whose zero set is analysed in Section 4. Inside the fundamental strip $0 < \text{Im } v < \pi$ that zero set has exactly two branches, one of which is excluded outright because $\text{Re } \beta < 1$ along it, and the other maps one-to-one onto the graph of a function $\varepsilon_0(R)$ defined for every $R > 0$. That function is tabulated to twenty-two digits over ten orders of magnitude of R in Table 1, attains its maximum at the closed-form point of Proposition 4, and has exact power-series expansions at both ends, (38) and (41), whose leading coefficient gives

$$b = \frac{\pi^2}{6} = 1.6449340668482264364724151666 \dots \quad (54)$$

in answer to (6). The factor of two against the constant $\pi^2/3$ carried by the curve (5) on which Shamis and Zeitouni proved $F_\beta > 0$ is resolved: that curve is an interior curve of the same branch, not a competing estimate of the boundary. Admissibility at the radii where a second component of $\{\Phi = 0\}$ dips below ε_0 — concretely at $R = 3$, where the component through $v = 2\pi i$ reaches $\varepsilon = 0.1992$ against $\varepsilon_0(3) = 0.2796$ — is settled by the exact finite- N free energy, which converges to the trivial-saddle value at the competing point and therefore places it strictly inside the region $F_\beta = 0$ rather than on its edge. The rate of that convergence is given by the two-term expansion (49), whose coefficient (50) is confirmed to five digits. Finally, the exact finite- N Fisher zeros, obtained as roots of an explicit integer polynomial, converge to ε_0 from below at rate $1/N$, with $N\delta$ tending to a limiting profile and the local spacing tending to a constant times $1/N$.

9.2 What has not been established

The identification is validated, not proved. Four things stand between the present state and a theorem.

Dominance. Nothing here shows that the saddle $u = \tanh v$ on the critical component actually dominates the Hubbard–Stratonovich integral for β below the curve. The finite- N free energy does converge to the value that saddle predicts, and the convergence rate matches the Laplace expansion of that saddle alone, which is strong evidence that no other saddle contributes at leading order; evidence of that kind cannot exclude a contribution that is exponentially smaller for every finite N and becomes comparable only in the limit.

Contours and poles. The Hubbard–Stratonovich representation involves $\cosh(\beta u)^{-N}$, which has poles of order N at $u = (2j + 1)i\pi/(2\beta)$, $j \in \mathbb{Z}$. As N grows these poles fill the complex u -plane, and whether a contour through the desired saddle can be found without crossing them, or whether their residues contribute, is not addressed. This is the obstruction the exclusion bound (46) cannot overcome: a bound on $|Z_{\beta, N}|$ from above says nothing about which saddle a deformed contour must pass through.

Uniformity. Every limit in this paper is taken at fixed β . The behaviour near the curve, where the trivial and nontrivial saddles tie at $\text{Re } f_\beta = 0$, is precisely where a pointwise limit is least informative: at $\varepsilon = \varepsilon_0(R)$ the quantity NF_N does not converge at all (Table 3), and the bracketing

diagnostic of Section 7 fails at three of the six radii tested. A proof would need control uniform in β on compact sets away from the locus.

From zeros to boundary. The classical route from the Fisher zeros to the phase diagram runs through a limiting zero density: if $Z_{\beta,N}$ converges locally uniformly away from a curve and the zeros accumulate there with a positive limiting density, the limit of F_N is harmonic off the curve and singular on it. Section 8 establishes the accumulation numerically, at three values of N , with residuals small enough that the zeros are exact algebraic numbers to every digit printed. It does not establish locally uniform convergence of $Z_{\beta,N}$, nor a limiting density, nor the consequent singularity of F_β .

9.3 Open problems

1. **Prove Conjecture 1 with ε_0 as given here.** That is, prove $F_\beta = 0$ for $\varepsilon < \varepsilon_0(R)$ and $F_\beta > 0$ for $\varepsilon > \varepsilon_0(R)$, with ε_0 the function tabulated in Table 1 and expanded in (38) and (41). The upper half of the statement should be reachable from the exclusion bound (46) together with a lower bound of the same strength; the lower half needs dominance.
2. **Characterise the limiting profile $g(R)$ of the zero offset.** Equation (52) records that $N(\varepsilon_z - \varepsilon_0(R_z))$ tends to a negative function of R alone, of size 1.15 near $R = 0.6$ and 7.3 near $R = 6.6$. A derivation would amount to a local scaling limit of P_N near the curve and would give the limiting zero density along ε_0 , which is the missing input for problem 1 by the Lee–Yang route. The related question of why N times the local spacing tends to 10.5 rather than to $4\pi = 12.566$ is the same computation seen from the other side.
3. **Explain the other components.** The zero set of Φ has a component through every $v = i\pi k$, $k \in \mathbb{Z}$, with $b_k = \pi^2 k^2/6$; the $k = 2$ component dips below ε_0 for R in an interval containing 3 and rises above it again by $R = 4$. Do these components carry any information about the model — as subdominant saddle manifolds, as boundaries for a different observable, or as accumulation loci for the $N^2/4 - 31$ zeros at $N = 80$ that are *not* in the critical family? Nothing here suggests an answer, and the question is asked because the components are easy to compute and completely unexplained.
4. **Convergence of the two series.** The small- R series (38) is a power series in ε with rational coefficients and is expected to have a positive radius of convergence limited by the nearest singularity of the curve; the large- R series (41) is asymptotic, with coefficients growing like π^{2n} times a rational number. Neither claim is proved, and no error bound accompanies either truncation. The numerics are consistent with both: the small- R truncation error at $\varepsilon = 0.1$ is the size of the first omitted term, and the large- R residuals of Table 2 decay geometrically in R^{-2} until cancellation takes over.
5. **The apex constant.** p^* , the root of $(p/2) \coth p = \log \sinh p$, is not obviously related to any standard constant, and neither $\varepsilon_{\max}$ nor $R_{\text{at max}}$ has a closed form in terms of it beyond (34). Whether the equation defining p^* is solvable in terms of known special functions is open; the equivalent form $\sinh p = e^{p \tanh p/2}$ suggests not.
6. **Beyond mean field.** The parametrisation $v = \beta u$ works because the saddle equation of a mean-field model is $u = \tanh(\beta u)$, which is exactly the relation that makes v a free parameter. The same trick applies to any model whose saddle equation can be inverted in this way — the spherical model, the Kac model, complete-graph Ising in a field — and it would be worth

knowing whether the resulting closed forms for $\operatorname{Re} f$ are as tractable there, and whether an analogue of $b = \pi^2/6$ exists.

9.4 Reproducibility

Every number in this paper is either an exact closed form, a coefficient of an exact series in rational and π arithmetic, or a decimal expansion of a quantity computed from one of the defining formulas (2), (7), (17), (47) or (51) at a stated working precision with a stated residual. The curve values of Table 1 satisfy $|\Phi| \leq 2.3 \times 10^{-61}$; the free energies of Table 3 are computed at the precision dictated by (48) and agree with those of [2] to seven decimals; the zeros of Table 6 are roots of an integer polynomial polished by Newton on the defining sum to residuals below 10^{-43} ; the series coefficients of Appendix A are exact rationals or rationals in π , reproduced by two independent routes and confirmed numerically along the independently traced curve. Appendix B records, item by item, which results were reproduced from the definitions, which were confirmed only numerically, and which are quoted from [2] because recomputing them would have exceeded the effort available — the $N = 80$ zeros, of degree 1600, are the only substantial case. No figure in this paper is a reproduction of a figure from anywhere else: all four are drawn from data computed here.

A Exact series data

All coefficients in this appendix are exact: rational numbers, or rationals in π . They were obtained by truncated-series arithmetic in $\mathbb{Q}(\pi)$, in which every operation is exact, and each was checked numerically against the independently traced curve. Series in $\zeta = v^2$ are used at the small- R end because $\beta(v)$ and $f(v)$ are even in v ; series in $p = \operatorname{Re} v$ are used at the large- R end near the puncture $v = i\pi k$.

A.1 The action and the map, at $v = 0$

With $f(v) = \frac{1}{2}v \tanh v - \log \cosh v = \Psi(\zeta)$ and $\beta(v) - 1 = v \coth v - 1 = z(\zeta)$, $\zeta = v^2$:

$$\Psi(\zeta) = -\frac{\zeta^2}{12} + \frac{2\zeta^3}{45} - \frac{17\zeta^4}{840} + \frac{124\zeta^5}{14175} - \frac{691\zeta^6}{187110} + \frac{21844\zeta^7}{14189175} + O(\zeta^8), \quad (55)$$

$$z(\zeta) = \frac{\zeta}{3} - \frac{\zeta^2}{45} + \frac{2\zeta^3}{945} - \frac{\zeta^4}{4725} + \frac{2\zeta^5}{93555} - \frac{1382\zeta^6}{638512875} + O(\zeta^7). \quad (56)$$

The odd coefficients of $\frac{1}{2}v \tanh v - \log \cosh v$ vanish identically, which is what makes Ψ a series in ζ ; this was checked to order v^{13} . Both series were carried to order ζ^{10} ; the terms displayed are those that enter the results of the body.

A.2 The small- R end

Writing $\zeta = \xi + i\eta$, the condition $\operatorname{Re} \Psi(\zeta) = 0$ gives η as a series in ξ ,

$$\frac{\eta}{\xi} = 1 + \frac{8}{15}\xi + \frac{71}{315}\xi^2 + \frac{3776}{23625}\xi^3 + \frac{195913}{1653750}\xi^4 + \frac{515746108}{5320940625}\xi^5 + O(\xi^6), \quad (57)$$

and substituting into $\varepsilon = \operatorname{Re} z(\zeta)$, $R = \operatorname{Im} z(\zeta)$ gives the two parametric series

$$\begin{aligned} \varepsilon(\xi) = & \frac{\xi}{3} + \frac{92\xi^3}{4725} + \frac{82\xi^4}{7875} + \frac{234548\xi^5}{27286875} + \frac{86924\xi^6}{12403125} \\ & + \frac{3377674408\xi^7}{558698765625} + \frac{243222604393\xi^8}{142468185234375} + O(\xi^9), \end{aligned} \quad (58)$$

$$\begin{aligned} R(\xi) = & \frac{\xi}{3} + \frac{2\xi^2}{15} + \frac{263\xi^3}{4725} + \frac{146\xi^4}{3375} + \frac{1713427\xi^5}{54573750} + \frac{12620207\xi^6}{483721875} \\ & - \frac{2962428694\xi^7}{558698765625} - \frac{29087096\xi^8}{30474478125} + O(\xi^9), \end{aligned} \quad (59)$$

in which the absence of a ξ^2 term in (58) is a cancellation, not a symmetry. Reverting (58) and substituting into (59) gives

$$\xi(\varepsilon) = 3\varepsilon - \frac{276\varepsilon^3}{175} - \frac{2214\varepsilon^4}{875} - \frac{1272996\varepsilon^5}{336875} + O(\varepsilon^6), \quad (60)$$

$$R_0(\varepsilon) = \varepsilon + \frac{6}{5}\varepsilon^2 + \frac{171}{175}\varepsilon^3 + \frac{1224}{875}\varepsilon^4 + \frac{120987}{61250}\varepsilon^5 + O(\varepsilon^6). \quad (61)$$

The first two coefficients of (61) are those obtained independently in [9]; the rest, including the ε^5 coefficient 120987/61250, are new here. Truncating (61) at ε^4 and evaluating at the computed $\varepsilon_0(R)$ reproduces R to a relative deviation of -1.967×10^{-12} at $R = 10^{-3}$, -1.891×10^{-8} at 10^{-2} , -1.408×10^{-6} at 0.03 and -1.329×10^{-4} at 0.1; adding the ε^5 term improves the first two to -3.13×10^{-15} and -2.98×10^{-10} . In each case the deviation is the size of the first omitted term.

A.3 The large- R end

Near the puncture $v = i\pi k$ write $v = i\pi k + p + is(p)$ with $p = \operatorname{Re} v > 0$ small. The condition $\Phi = 0$ determines s exactly:

$$\begin{aligned} s(p) = & -\frac{p^4}{6\pi k} - \frac{7p^6}{90\pi k} - \frac{p^8}{140\pi k} + \frac{(1575 - 43\pi^2 k^2)p^{10}}{56700\pi^3 k^3} \\ & + \frac{(68145 + 146\pi^2 k^2)p^{12}}{3742200\pi^3 k^3} + \frac{(29029 - 439\pi^2 k^2)p^{14}}{28378350\pi^3 k^3} + O(p^{16}), \end{aligned} \quad (62)$$

the first four terms of which reproduce those of [2] exactly. Substituting (62) into $\beta = v \coth v$ gives $R(p)$ and $\varepsilon(p)$; reverting R in favour of $P = \pi k/R$ gives

$$p(P) = P + \frac{P^3}{3} + \frac{P^5}{5} + \left(\frac{1}{7} - \frac{5}{36\pi^2 k^2}\right)P^7 + \left(\frac{1}{9} - \frac{13}{30\pi^2 k^2}\right)P^9 + \left(\frac{1}{11} - \frac{22}{25\pi^2 k^2}\right)P^{11} + O(P^{13}), \quad (63)$$

in which the first three terms are those of $\operatorname{artanh} P$ and the π -dependent corrections begin at P^7 ; the reversion was verified by substituting (63) back into the series for $P(p)$ and confirming that every coefficient of $P^1, \dots, P^{11}$ in the difference vanishes exactly. Finally

$$\begin{aligned} \varepsilon_0(R) = & \frac{\pi^2 k^2}{6R^2} + \frac{\pi^4 k^4}{15R^4} + \frac{\pi^6 k^6}{28R^6} + \pi^8 k^8 \left(\frac{1}{45} - \frac{5}{216\pi^2 k^2}\right) \frac{1}{R^8} + O(R^{-10}) \\ = & \frac{b_k}{R^2} \left[1 + \frac{2\pi^2 k^2}{5R^2} + O(R^{-4})\right], \quad b_k = \frac{\pi^2 k^2}{6}. \end{aligned} \quad (64)$$

For $k = 1$ the R^{-8} coefficient is $\pi^6(24\pi^2 - 25)/1080 = 188.6018653206584759\dots$; for $k = 2$ the series reads

$$\varepsilon_0(R) = \frac{2\pi^2}{3R^2} + \frac{16\pi^4}{15R^4} + \frac{16\pi^6}{7R^6} + \frac{8\pi^6(96\pi^2 - 25)}{135R^8} + O(R^{-10}),$$

which is (64) at $k = 2$ and confirms the k^2 scaling of b_k independently of the general argument.

A.4 Local expansions at the punctures and on the horizontal slices

The following were derived and verified to the stated order at 60 working digits; they are the local input to Section 4. At the horizontal slices,

$$\Phi(p, \pi k) = \frac{p}{2} \tanh p - \log \cosh p, \quad \Phi\left(p, \frac{\pi}{2} + \pi k\right) = \frac{p}{2} \coth p - \log \sinh p, \quad (65)$$

both independent of k . Near the puncture $v = i\pi k + w$,

$$\Phi(i\pi k + w) = -\frac{\pi k}{2} \operatorname{Im} w - \frac{\operatorname{Re} w^4}{12} + O(|w|^5), \quad \beta(i\pi k + w) = 1 + \frac{i\pi k}{w} + O(w), \quad (66)$$

so Φ changes sign across the four rays $\arg w = 0, \pi/2, \pi, 3\pi/2$ and the image of a small circle about $v = i\pi k$ under $\beta - 1$ winds once about the origin with radius $\pi|k|/|w|$: the punctures are the sources of the components counted in (64). Near the branch point $v = 0$,

$$\beta - 1 = \frac{v^2}{3} + O(v^4), \quad \Phi(v) = -\frac{v^4}{12} + O(v^6), \quad (67)$$

which maps the rays $\arg v = \theta$ to $\arg(\beta - 1) = 2\theta$ and gives the pinch angles of Section 4: $\arg v = \pi/8 \mapsto \arg(\beta - 1) = \pi/4$ for the critical branch and $\arg v = 3\pi/8 \mapsto 3\pi/4$ for the short branch.

A.5 The apex

p^* is the unique positive root of $(p/2) \coth p = \log \sinh p$, equivalently of $\sinh p = \exp(\frac{1}{2}p \coth p)$: the difference $h(p) = \frac{p}{2} \coth p - \log \sinh p$ satisfies $h'(p) = -\frac{1}{2} \coth p - \frac{p}{2 \sinh^2 p} < 0$ for $p > 0$, tends to $+\infty$ as $p \rightarrow 0^+$ and to $-\infty$ as $p \rightarrow \infty$, and so has exactly one positive zero. (That h' is exactly Φ_p on the slice $q = \pi/2$, so the uniqueness of the apex and the transversality of the locus to that slice are the same fact.) At 60 working digits,

$$\begin{aligned} p^* &= 1.604029102812004728846419615409504104293416007033\dots, \\ \varepsilon_{\max} &= p^* \tanh p^* - 1 = 0.47935240607782361252824419404539279058942478269334\dots, \\ R_{\text{at max}} &= \frac{\pi}{2} \tanh p^* = 1.4487027208100393722706725135707582660204411776179\dots, \\ u^* &= \coth p^* = 1.0842778882313335059393288682092972689579367735775\dots, \end{aligned}$$

with $\Phi(p^*, \pi/2) = -7.8 \times 10^{-62}$ and $d^2\varepsilon/dq^2 = -0.39183836$ at that point.

B Numerical protocol and verification record

B.1 Precision policy

All computations are in arbitrary-precision floating-point arithmetic, at a stated number of working decimal digits (dps), with exact integers and exact rationals wherever the object is an integer or a rational: the binomial coefficients of (47) and the coefficients of P_N in (51) are exact integers, and the series of Appendix A are exact elements of $\mathbb{Q}(\pi)$ manipulated by truncated-series arithmetic, in which addition, multiplication, composition and reversion are exact operations on rational coefficients. No floating-point quantity is reported without either an exact identity behind it or a recorded residual.

The working precision is chosen per quantity rather than fixed once. For Newton iteration on (32), 50–60 digits, with iteration until the step is below 10^{-52} and acceptance only if $|\Phi| < 10^{-45}$

and $|\text{Im } \beta - R| < 10^{-40}$ relative. For the exact finite- N sums, the cancellation criterion (48). For the polynomial roots, double precision for the companion matrix followed by Newton polishing on the defining sum at 50 digits, with the residual $|Z_{\beta,N}|$ recorded for every root.

B.2 Verification record

Table 7 lists every quantitative claim of the body, the method by which it was established here, the achieved tolerance, and whether the result is an independent recomputation from the defining formulas, a numerical confirmation of an exact statement, or a value quoted from [2] because recomputing it was outside the available effort.

Table 7: Verification record. “Recomputed” means obtained here from (2), (7), (17), (47) or (51) without reference to any previously computed value; “confirmed” means an exact statement checked numerically to the stated tolerance; “quoted” means taken from [2] and not recomputed.

claim	method	tolerance	status
Φ and β closed forms (7), (17)	against $f_\beta(u)$ and $v/\tanh v$ at 6 test points, 60 digits	3.4e-57, 5.4e-58	confirmed
u -space closed form, $ u < 1$ and $ u > 1$, branches $k = -3..3$	42 (k, u) pairs, 60 digits	9.3e-61	confirmed
$ \tanh v ^2 = (\cosh 2p - \cos 2q)/(\cosh 2p + \cos 2q)$	12 points, 60 digits; 5 of 12 have $ \tanh v > 1$, all with $\cos 2q < 0$	6.9e-51	confirmed
$\Phi(0, \pi k) = 0$, $k = -6..6$; $f_\beta(0) = 0$	direct	6.5e-60; exact	confirmed
local expansions (66), (67), (65)	ratios at $ v = 10^{-2}, 10^{-3}, 10^{-4}$ and on the rays $\pi/8, 3\pi/8$	$\leq 4e-6$ relative	confirmed
two branches in $0 < \text{Im } v < \pi$	root count of $\Phi(\cdot, q)$ at 398 values of $q \in [0.01, \pi - 0.01]$	exactly 1 root for $q > \pi/2$, exactly 2 for $q < \pi/2$	recomputed
short branch has $\text{Re } \beta < 1$ and ends at $\beta = 0$	400-point trace to $v = i\pi/2$	$\max \text{Re } \beta = 0.999889537332$	recomputed
rays $\arg v = \pi/8, 3\pi/8$; images $\arg(\beta - 1) = \pi/4, 3\pi/4$	trace at $q = 10^{-2} \dots 10^{-6}$	$p/q \rightarrow \sqrt{2} \pm 1$ to 10 digits; angles to 10^{-6} degrees	recomputed
$\varepsilon_0(R)$, 27 radii, $10^{-6} \leq R \leq 10^5$	2-D Newton on (32), seeded by continuation or by (63)	worst $ \Phi = 4.8e-51$; agreement with [2] to 1.9e-22 relative	recomputed
$\varepsilon_0(R)$, 16 radii of Table 1, to 10^6	as above	$ \Phi \leq 2.3e-61$	recomputed
apex, 50 digits	root of $(p/2) \coth p = \log \sinh p$	$ \Phi = 7.8e-62$	recomputed
apex is stationary, and exactly at $\text{Im } v = \pi/2$	central differences of ε along level curves of Φ at $p = 0.5, 1, p^*, 2.5$, $h = 10^{-3} \dots 10^{-8}$	slope falls as h^2 , reaching 10^{-17}	confirmed
ε strictly monotone on either side of the apex	4001-point trace	strict at every point	recomputed
$\Psi, z, \eta/\xi, \xi(\varepsilon), R_0(\varepsilon), s(p), p(P), \varepsilon_0(R)$ for $k = 1, 2$	exact truncated series in $\mathbb{Q}(\pi)$	exact; reversion checked by composition to P^{11}	recomputed
small- R series error	(61) truncated at $\varepsilon^4, \varepsilon^5$, evaluated at 4 radii	equals the first omitted term	confirmed
large- R coefficients	residuals of Table 2 at 4 radii	8–11 digits at $R = 10^5$	recomputed
$b = \pi^2/6, b_k = \pi^2 k^2/6$	exact series	exact	recomputed
$\varepsilon_{SZ}(R)$ from $\text{Im } u = 0, k = 1$, and $\text{Re } f_\beta < 0$ there	7 radii, $3.2 \leq R \leq 10^4$	reproduction to 8.6e-50; $\text{Re } f_\beta$ negative at all 7, decaying as $-\pi^4/(12R^4)$	recomputed
$B(\beta) = 0$ for $\text{Re } \beta \leq 1$	6 points	$< 10^{-38}$ at 40 digits	recomputed
NF_N at $R = 3, 7$ offsets, $N = 250..2000$	exact sum (47) at (48), 63–137 digits	agrees with [2] to 4.2e-8	recomputed
precision failure at $\beta = 1.4 + 10i$, $N = 6400$	same sum at dps 40, 50, 60, 80, 120, 200, 368	427.0155 and 405.4129 at insufficient precision, 405.3258758810 at dps ≥ 60 , stable to dps 368	recomputed

claim	method	tolerance	status
$c_1(\beta)$ of (50)	6 points, $N = 250..2000$	$\leq 1.2\text{e-}4$ at $N = 2000$	recomputed (new)
Fisher zeros, $N = 20$: 100 roots, family of 4	(51), companion matrix plus Newton polishing at 50 digits	$\max Z = 1.7\text{e-}42$; family rows agree with [2] to 10^{-10}	recomputed
Fisher zeros, $N = 40$: 400 roots, family of 13	as above	$\max Z = 9.4\text{e-}38$; rows agree to 10^{-10}	recomputed
Fisher zeros, $N = 80$: 1600 roots, family of 31	—	—	quoted
nearest zero at $N = 20$, 25 digits argument principle, box	(53) contour integral of dZ/Z on the	$4.3\text{e-}26$ against [2]	recomputed
$0.05 < \varepsilon < 0.90$, $1 < R < 4$, $N = 40$	direct sum, 50 digits	9.000000000000 against 9 polynomial zeros	recomputed
route-B scan (no polynomial), $N = 40$	coarse $ Z $ scan plus Newton	19 of 19 match to 10^{-8}	quoted
zero-offset scaling $N\delta \rightarrow -g(R)$	Tables 6 and the $N = 80$ rows of [2]	stable to a few percent at matched radius	recomputed ($N \leq 40$)
local spacing, N times mean over $0.8 < R < 2.5$	2 gaps at $N = 20$, 5 at $N = 40$	10.493, 10.746	recomputed
bracketing diagnostic $\varepsilon_N(R)$, 6 radii	bisection on the exact sum	—	quoted

B.3 Protocols

Tracing the component. The component is traced in the parameter $q = \text{Im } v$ rather than in R , because R ranges over twenty-eight orders of magnitude along it while q ranges over $(0, \pi)$. At each of 4001 values of q the equation $\Phi(p, q) = 0$ is solved for p by Newton’s method started from the previous point, and β is then evaluated from (17). The trace is monotone in R at every one of the 4001 points, which is the numerical content of the single-valuedness claim. When ε_0 is needed at a prescribed R , the two equations (32) are solved together by 2-D Newton with the exact Jacobian; the seed is the traced point nearest in $\log R$, or the asymptotic series (63) for $R \geq 8$. Seeding by nearest R rather than nearest $\log R$ fails: the trace is uniform in q , so between $q/\pi = 1 - 10^{-3}$ and $q/\pi = 1$ the radius jumps from about 7 to about 3.6×10^3 , and a linear search in R picks a seed hundreds of Newton basins away, from which the iteration converges to a different component or not at all.

Exact series. A truncated series is represented by its coefficient list over $\mathbb{Q}(\pi)$, with π kept symbolic so that no coefficient is ever rounded. Composition, inversion and reversion are performed by exact rational arithmetic. Every reversion is checked by composing back: substituting (63) into the series for $P(p)$ gives zero for every coefficient of $P^1, \dots, P^{11}$.

Exact finite- N sums. $\binom{N}{k}$ is formed as an exact integer and converted to the working precision; the sum is evaluated as $\sum_k \exp(t_k - M)$ with $M = \max_k \text{Re } t_k$ and the result rescaled by $e^M 2^{-N}$, so no intermediate overflows. The number of decimal digits lost to cancellation, $-\log_{10} |\sum_k \exp(t_k - M)|$, is recorded for every evaluation; it rises from 0.6 at $N = 250$ to 36.1 at $N = 2000$ at $\varepsilon = 0.27965$, $R = 3$, and reaches 44.8 at $N = 6400$, $\beta = 1.4 + 10i$. The criterion (48) exceeds the loss by more than twenty digits in every case used here. It is conservative: at $\beta = 1.4 + 10i$, $N = 6400$ it prescribes 368 digits, whereas the value is already stable from 60 digits upwards. The point of a conservative criterion is that the loss cannot be known before the sum is evaluated.

Fisher zeros. Two independent routes. Route A forms the exact integer coefficients of P_N from (51), computes the roots of the companion matrix in double precision, maps each root by $\beta = (N/2) \text{Log } t$ and polishes it by Newton’s method on the $(N+1)$ -term sum (47) at 50 digits, recording $|Z_{\beta,N}|$ at the polished root. Route B, used only as a cross-check, scans $|Z_{\beta,N}|$ on a 240×400 grid over $-0.6 < \varepsilon < 1.6$, $10^{-4} < R < 8$, takes every strict local minimum as a candidate, and applies the same Newton polishing; it shares no code and no mathematics with route A beyond the

definition of Z . A third check, the argument principle, integrates $dZ_{\beta,N}/Z_{\beta,N}$ along the boundary of a box on a fine grid at 50 digits and rounds the winding number to the nearest integer.

B.4 Ill-conditioning of the companion route

The one-against-one comparison of independently computed zeros fails for 5 of 50 upper-half-plane zeros at $N = 20$ (worst discrepancy 1.3×10^{-4}) and 11 of 201 at $N = 40$ (worst 5.7×10^{-3}). The failures are not random: they lie at $|\beta| \approx \pi N/2$, the outer edge of the zero cloud, where the roots of P_N form clusters whose separation is far below the double-precision eigenvalue accuracy of the companion matrix. For a polynomial of degree 400 with coefficients spanning many orders of magnitude the companion eigenproblem is severely non-normal, and a perturbation of size $\varepsilon_{\text{mach}}$ moves a root in a cluster of multiplicity m by order $\varepsilon_{\text{mach}}^{1/m}$. Two diagnostics confirm that the zero set is nonetheless correct: the polished residuals of the mismatched roots remain at or below 10^{-38} , which is the size expected of a genuine zero of the same cluster, and the argument-principle count in a box containing several of them is exactly an integer and equals the number of polynomial roots inside. The comparison is therefore meaningful up to permutation within clusters, and no claim in the body depends on the labelling of an individual root near $|\beta| = \pi N/2$.

B.5 What is quoted rather than reproduced

Three items are taken from [2] without independent recomputation. The $N = 80$ zeros: a degree-1600 integer polynomial, 1600 companion eigenvalues and 1601-term Newton polishing at 50 digits for each, which exceeds the effort available here; the $N = 80$ row of Table 5 and the $N = 80$ values in (52) are quoted. The route-B scan at $N = 40$: its result is quoted because route A was reproduced twice already and the argument-principle check supplies an independent count. The bracketing diagnostic $\varepsilon_N(R)$ of Section 7: the bisection is inexpensive but its interpretation depends on the same finite- N sums already reproduced in Table 3, and the values quoted are those of [2]. Everything else in the body was recomputed here from the defining formulas.

C Data tables

C.1 The curve over twenty-seven radii

Table 8 extends Table 1 to twenty-seven radii, spanning eleven orders of magnitude, and records for each the two residuals of the solved system (32): the value of Φ at the solution and the error in the prescribed radius. Every row is an independent solution at 50 working digits and every row agrees with the corresponding value of [2] to at least 22 significant digits, the worst case being $R = 10^5$ at 1.9×10^{-22} relative, where cancellation in $\varepsilon_0 \approx 1.6 \times 10^{-10}$ is severe. The values at $R = 1.4$ and $R = 1.5$ straddle the apex and show the flatness there: ε_0 changes by 2.6×10^{-5} between them while R changes by 0.1.

C.2 All solutions of $\text{Re } f_\beta = 0$ at fixed radius

Table 9 lists, at ten radii, every solution of $\{\Phi = 0, \text{Im } \beta = R\}$ with $\varepsilon > 0$ found by a bounded multi-start Newton search over $0 < \text{Re } v < 6$, $|\text{Im } v| < 4\pi$, with 10^4 starting points at 30 digits. Solutions come in conjugate pairs $\pm \text{Im } v$, so only $|\text{Im } v|/\pi$ is shown. The rows at $R = 3$ and $R = 4$

Table 8: $\varepsilon_0(R)$ to 26 significant digits, with the residuals of the solved system (32) at 50 working digits.

R	$\varepsilon_0(R)$	$ \Phi $	$ \operatorname{Im} \beta - R $
10^{-6}	0.00000099999880000190285296686759	7.99e-52	1.97e-52
10^{-4}	0.000099988001902439647253980643	1.46e-51	2.84e-52
10^{-3}	0.00099880189869155728483804933	1.83e-53	8.61e-53
0.01	0.0098818621140176327775038145	1.67e-52	1.88e-52
0.1	0.089567706579462441062041654	6.68e-52	0.0
0.2	0.16272702831629156603543292	1.00e-51	1.67e-51
0.5	0.31857659402877199035142511	2.00e-51	0.0
1	0.44877609900355434330776241	1.34e-51	2.67e-51
1.25	0.47376684965355619775406846	1.34e-51	5.35e-51
1.4	0.47903040619849847449402381	0.0	0.0
1.5	0.47900478733138496908706219	1.34e-51	2.67e-51
2	0.44437488848207699365415504	2.67e-51	5.35e-51
2.5	0.36993955949769510296814977	0.0	0.0
3	0.27964554577318019778947188	1.34e-51	0.0
3.5	0.19741143156479008910521343	0.0	0.0
4	0.13900516323955120008240737	1.34e-51	1.60e-50
5	0.078920254976026574239386272	0.0	1.07e-50
6	0.051563228354009789306456261	2.00e-51	0.0
8	0.027430484768554347321263913	1.84e-51	2.14e-50
10	0.017135036751582656738798012	2.76e-51	8.55e-50
20	0.0041534662252233007015719677	3.47e-51	2.14e-49
50	0.00065901485934128508827460896	1.88e-51	2.57e-49
100	0.00016455838043304205486515847	6.15e-52	8.48e-47
300	0.000018277846955218466625135346	9.09e-52	1.96e-45
10^3	0.0000016449405608219642202462262	2.66e-51	6.03e-45
10^4	1.6449341317876238926737295e-8	4.81e-51	1.67e-41
10^5	1.6449340674976203770424514e-10	3.26e-51	7.42e-38

were reproduced independently here and agree with [2] to all twelve printed digits, including the identification of the two candidates at $R = 3$.

Two features of the table matter. The first is the crossing: the component through $v = 2\pi i$ lies below the critical one at $R = 3$ and above it at $R \geq 3.5$, so the exchange happens between those radii. The second is a limitation: at $R = 20$ the search returned eight solutions and *none* of them is the critical one, whose value $\varepsilon_0(20) = 0.0041534662252233$ is far below all eight. A bounded multi-start search is therefore not exhaustive at large R , where the critical solution has $\operatorname{Re} v \approx \pi/R \approx 0.16$ and $|\operatorname{Im} v|$ within 10^{-4} of π : the starting grid does not resolve that neighbourhood, and the basins of the spurious solutions surround it. At $R = 20$ the critical value comes from the traced component of Section 4 polished by Newton, not from the enumeration. This is a caution against reading the table as a proof that no further candidates exist.

C.3 The $N = 80$ zero family

Table 10 gives the eleven members of the $N = 80$ family reported in [2], out of 31 in the family and 1600 in total. They are quoted rather than recomputed, for the reason given in Appendix B. The pattern of Table 6 continues: every δ is negative, $|N\delta|$ grows with R , and at the outermost listed zero, $R = 6.614$, the zero has crossed to $\operatorname{Re} \beta < 1$ while the curve still has $\varepsilon_0 = 0.0415 > 0$, so the approach is from below in ε rather than from inside the half-plane $\operatorname{Re} \beta > 1$.

Table 9: All solutions of $\{\Phi = 0, \text{Im } \beta = R\}$ with $\varepsilon > 0$ found by bounded multi-start Newton at ten radii, from [2]; the rows $R = 3$ and $R = 4$ were reproduced independently. The entry marked * is on the critical component, and the entry marked † at $R = 3$ is the competing candidate shown inadmissible in Section 7.

R	$\varepsilon_0(R)$	candidates ($ \text{Im } v /\pi, \varepsilon$)
1	0.44877609900355	(0.37788174, 0.448776099004)*
2	0.44437488848208	(0.6391389, 0.444374888482)*
2.5	0.3699395594977	(0.75397619, 0.369939559498)*
3	0.27964554577318	(1.417137, 0.199204511072)†, (0.85439238, 0.279645545773)*
3.5	0.19741143156479	(0.93092007, 0.197411431565)*, (1.4021056, 0.375335441689)
4	0.13900516323955	(0.97205138, 0.13900516324)*, (1.4368905, 0.465096192662)
5	0.078920254976027	(0.99373859, 0.078920254976)*, (2.4546179, 0.143156151233), (1.6341335, 0.445391707952)
8	0.027430484768554	(0.99945851, 0.0274304847686)*, (1.9840764, 0.141364833844), (3.4616784, 0.295601053184), (2.624542, 0.44736466399)
10	0.017135036751583	(0.99980186, 0.0171350367516)*, (1.9968274, 0.0790768672221), (2.9216587, 0.252696056359), (4.4715287, 0.255635033597), (3.4883972, 0.478421253016)
20	0.0041534662252233	(1.9999009, 0.0171352577113), (2.999566, 0.0407522432741), (3.9984083, 0.0791166378492), (4.9933284, 0.142139618682), (5.9414853, 0.261755995874), (8.4839416, 0.318692037524), (7.4817345, 0.444486485599), (6.5848287, 0.454397719174) — the critical solution was not found

Table 10: Eleven of the 31 family members at $N = 80$, quoted from [2]. Columns as in Table 6.

R_z	ε_z	$\varepsilon_0(R_z)$	δ	$N\delta$
0.2564242449	0.1783814519	0.1984583321	−0.02008	−1.60615
0.7196956025	0.3768336875	0.3913868776	−0.01455	−1.16426
1.121052493	0.4485328076	0.4636064611	−0.01507	−1.20589
1.516206057	0.4619243686	0.4787530979	−0.01683	−1.3463
1.92221489	0.433451198	0.4529760207	−0.01952	−1.56199
2.352568041	0.3713071334	0.3946265816	−0.02332	−1.86556
2.824500082	0.2829513233	0.3116103877	−0.02866	−2.29273
3.36770369	0.1809094184	0.2172013001	−0.03629	−2.90335
4.049719262	0.08866685661	0.1345378353	−0.04587	−3.66968
5.026794421	0.01813026376	0.07790954802	−0.05978	−4.78234
6.614022565	−0.04932757193	0.04146234334	−0.09079	−7.26319

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