

Refutation of the free-diffusion bound for odd underdamped currents

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Abstract

For an underdamped Brownian particle driven around a periodic ring, the precision of a time-integrated current that is odd under time reversal is conjectured to be bounded below by the corresponding equilibrium free-diffusion value, at every observation time and for every odd power of the velocity. We show that the conjecture is false. In the one-mode potential family the short-time ratio of the uncertainty product to its free-diffusion reference equals one plus a coefficient times the square of the potential amplitude. We evaluate that coefficient as an equilibrium resolvent quantity, prove by an exact moment recurrence that it is negative for every odd order from seven to four hundred and one, and reduce the monotonicity the argument needs to a single sign property of an explicitly known boundary function, which we establish for the first six spatial modes by a three-region computer-assisted proof. Each of those orders therefore violates the bound at sufficiently short observation times and sufficiently weak potentials. For the first-order current we prove the opposite: an entropy-production decomposition yields a sharp stationary bound on the mean velocity, from which the bound holds near zero time, strictly unless the potential is constant. At finite times we resolve a homogeneous seventh-order violation out to fifty relaxation times, and an odd-weight violation that appears although the infinitesimal test passes. Third and fifth order remain open; several apparent fifth-order violations are truncation artifacts, and one row is reported as unresolved rather than retired.

1 Introduction

A current sustained by a nonequilibrium steady state cannot be both cheap and sharp. The thermodynamic uncertainty relation makes this precise for overdamped Markov dynamics: with σ the entropy production rate and Y_τ any time-integrated current, the dimensionless product $\text{Var}(Y_\tau)\sigma\tau/\langle Y_\tau \rangle^2$ is at least 2, both in the long-time limit [1, 12, 34] and at every finite observation time [35, 17], with affinity- and topology-dependent refinements [33]; see Seifert [40] for the inference side of the story. The bound is saturated only in special circumstances [6, 28], and it has been extended to periodically driven systems [20, 19], to arbitrary initial states [26], and to activity-based [7] and fluctuation-theorem-based [16] variants.

Underdamped dynamics does not fit that statement without modification. A current built from an odd power of the velocity is ballistic at short times: its mean grows like τ and its variance like τ^2 , so the uncertainty product vanishes linearly as $\tau \rightarrow 0$ and a constant lower bound of 2 cannot hold there. A family of modified relations restores a valid bound by bringing the kinetic degrees of freedom into the constant [45, 24, 25, 10, 23]. A different proposal keeps the form of the bound but replaces the number 2 by a τ -dependent reference: Fischer et al. [8] conjectured that for the currents

$$Y_n(\tau) = \int_0^\tau u_s^n ds, \quad n \text{ odd}, \quad (1)$$

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of a one-dimensional underdamped particle in an arbitrary potential, the uncertainty product is bounded below by its value for *free* diffusion, that is, by the exact equilibrium reference obtained by setting the potential to zero. The reference has the correct ballistic behaviour at short times, saturates at an explicit rational value at long times, and for $n = 1$ that value is exactly the number 2 of the thermodynamic uncertainty relation. The conjecture was supported by extensive one-dimensional numerics and, for $n \leq 5$, by closed forms; its published version is Fischer et al. [8], and the setting is that of a driven periodic potential, the standard model of a Brownian motor [21, 15, 37, 14, 5].

A universal, potential-independent finite-time reference for underdamped precision would be a useful tool. It bounds the cost of a current without solving for the steady state, and it feeds directly into the dissipation estimates that are inferred from observed current fluctuations [13, 30, 18] and into the coherence bounds for autonomous Brownian oscillators [2]. The same dynamics also admits upper bounds on entropy production [4, 31, 9, 29] and bounds relating dissipation to response [32, 36, 46, 27], so a matching lower bound on precision would close the picture from the other side. This paper determines how much of that survives.

Three results are proved here, and they do not point in the same direction.

At first order the bound holds near zero observation time, and we prove it. The short-time ratio of the uncertainty product to the free reference reduces exactly to the sign of $f - \langle u \rangle$, and we show that the stationary mean velocity of a driven periodic diffusion obeys $\langle u \rangle \leq \chi f \leq f$, where $\chi \leq 1$ is an explicit functional of the position marginal. The proof is a two-term entropy-production decomposition [4, 39, 41] plus one Cauchy–Schwarz step, and it identifies the equality case: saturation forces the potential to be constant. The bound is therefore strict for every non-constant potential, infinitesimally.

The sign reverses at seventh order and beyond. In the one-mode family $V(\theta) = V_0 \cos(2\pi k\theta)$ the short-time ratio obeys $R_{0,n}(V_0) = 1 + \kappa_n V_0^2 + O(V_0^4)$, and κ_n has an exact closed form as an equilibrium resolvent quantity. We evaluate it in two independent ways, one of them an exact moment recurrence run to every odd $n \leq 401$ for the modes $k = 1, \dots, 6$, and we find $\kappa_n < 0$ throughout that range, with the sign change occurring precisely between $n = 5$ and $n = 7$. The only non-algebraic ingredient is a monotonicity statement, which we reduce to the positivity of the derivative of a single explicitly known function $\beta(u)$ on the half-line; that positivity is established by a three-region argument whose outer region is analytic and whose inner regions are certified numerically. The outcome is a refutation: for each of those modes and orders there is a weak-potential, short-time window in which the conjectured inequality is reversed.

Nor is the failure confined to infinitesimal times. Resolving the full propagator, we track a homogeneous seventh-order violation out to fifty relaxation times without any sign of recovery, and we find a second, qualitatively different failure mode in which a positive spatially modulated weight passes the infinitesimal test and then crosses the bound at a finite observation time. The crossing is localised and is stable under raised spectral truncation.

The paper also reports what did not work. The orders $n = 3$ and $n = 5$ remain open: their coefficients κ_3, κ_5 are positive for every mode examined, and no converged violation was found. Of the 252 resolved fifth-order rows of the stationary sweep, 26 fall below the bound at the default truncation; a convergence ladder retires the representative uniform-weight rows as truncation artifacts, while one weighted row oscillates instead of settling and is reported as an unresolved floating-point candidate, used as no evidence at all. A ratchet potential that briefly appeared to violate the stationary inequality by a large margin collapses under refinement and under an independent Monte Carlo estimate. These negative and inconclusive results are part of the record.

Section 2 fixes the model, the currents and the conjecture, and derives the exact reference. Section 3 reduces the short-time bound to stationary quantities. Section 4 proves the first-order bound near zero time and identifies the equality case. Section 5 carries the refutation for $n \geq 7$, including the computer-assisted certificate. Sections 6–7 report the finite-time windows and the

inconclusive rows, Section 8 gives the weight-variational form of the bound, and Section 9 collects the limitations. The exact reference coefficients, the certificate constants and the numerical protocol are in the appendices.

2 Model, currents and the conjectured bound

2.1 The dimensionless ring

Mass, friction coefficient, temperature and spatial period are all set to one. The particle position is an angle $\theta \in \mathbb{T} = [0, 1)$ with periodic identification, u is the rescaled velocity, and τ is time measured in velocity-relaxation units. The dynamics is

$$d\theta = u d\tau, \quad du = [f - V'(\theta)]d\tau - u d\tau + \sqrt{2}dW_\tau, \quad (2)$$

where W is a standard Wiener process, $V: \mathbb{T} \rightarrow \mathbb{R}$ is smooth and 1-periodic, and $f > 0$ is a constant non-conservative drive. The backward generator acting on observables is

$$L = u \partial_\theta + (f - V'(\theta) - u) \partial_u + \partial_u^2, \quad (3)$$

and the forward Fokker–Planck operator is its L^2 adjoint on the ring. Because the noise acts only on u and the drift is smooth, (2) has a unique stationary law; we write $\rho(\theta, u) > 0$ for its density,

$$P(\theta) = \int_{\mathbb{R}} \rho(\theta, u) du, \quad \nu(\theta) = \frac{1}{P(\theta)} \int_{\mathbb{R}} u \rho(\theta, u) du, \quad m = \langle u \rangle, \quad (4)$$

for the position marginal, the local mean velocity and the stationary mean velocity, and $\langle \cdot \rangle$ for the stationary average. Integrating the stationary Fokker–Planck equation over u gives $\partial_\theta(\nu P) = 0$, so

$$\nu(\theta)P(\theta) = m \quad \text{for every } \theta \in \mathbb{T}, \quad (5)$$

which is the statement that the probability flux around the ring equals the mean velocity. Two further identities follow from $\langle L\phi \rangle = 0$ with $\phi = u$ and $\phi = \frac{1}{2}u^2 + V(\theta)$:

$$\langle V' \rangle = f - m, \quad \langle u^2 \rangle = 1 + fm. \quad (6)$$

The entropy production rate of the steady state is the work rate of the drive at unit temperature,

$$\sigma = f \langle u \rangle = fm > 0, \quad (7)$$

the standard energy balance for a driven Langevin particle [41, 39]; the positivity is proved in Lemma 1 below. Finally we shall use the harmonic mean of the position marginal,

$$\chi = \left(\int_0^1 \frac{d\theta}{P(\theta)} \right)^{-1} \in (0, 1], \quad (8)$$

which by (5) is the effective mobility of the ring: it equals 1 for a constant potential and decreases as the potential concentrates probability in the wells.

2.2 Currents, uncertainty products and the free reference

Let $w: \mathbb{T} \rightarrow (0, \infty)$ be smooth and 1-periodic, and let $n \geq 1$ be odd. The time-integrated current and its uncertainty product are

$$Y_n(\tau; w) = \int_0^\tau w(\theta_s) u_s^n ds, \quad Q_n(\tau; w) = \frac{\text{Var}(Y_n(\tau; w))}{\langle Y_n(\tau; w) \rangle^2} \sigma \tau. \quad (9)$$

Stationarity gives $\langle Y_n(\tau; w) \rangle = \tau \langle w u^n \rangle$, and the variance has the standard autocorrelation representation

$$\text{Var}(Y_n(\tau; w)) = 2 \int_0^\tau (\tau - t) C(t) dt, \quad C(t) = \text{Cov}(g, e^{tL} g), \quad g = w u^n. \quad (10)$$

The product (9) is dimensionless and invariant under rescaling of τ ; its inverse is the precision attained per unit of dissipated entropy. For overdamped Markov steady states the finite-time thermodynamic uncertainty relation asserts $Q_n \geq 2$ [35, 17]. That statement is not available here: for the currents (9) the variance grows like τ^2 while the mean grows like τ , so $Q_n(\tau; w) \rightarrow 0$ as $\tau \rightarrow 0$ for every potential, including the free one. A finite-time bound for underdamped velocity-odd currents must therefore be τ -dependent, which is what the modified relations of Van Vu and Hasegawa [45], Lee et al. [24, 25], Fu and Gingrich [10], Kwon and Lee [23] supply and what the free-diffusion reference supplies in a potential-independent form.

The reference is obtained by evaluating (9) for $V \equiv 0$ and $w \equiv 1$, where u is an Ornstein–Uhlenbeck process with stationary law $\mathcal{N}(f, 1)$ [44, 3]. Write

$$Q_n^0(\tau) := Q_n(\tau; 1)|_{V \equiv 0}. \quad (11)$$

Both the numerator and the denominator of (9) scale as f^2 for odd n , so Q_n^0 is independent of the drive and is a universal function of τ . Appendix A derives its exact closed form; the result is

$$Q_n^0(\tau) = q_n - \frac{s_n}{\tau} + \frac{1}{\tau} \sum_{\substack{k=1 \\ k \text{ odd}}}^n d_{n,k} e^{-k\tau}, \quad q_n = \sum_k k d_{n,k}, \quad s_n = \sum_k d_{n,k}, \quad c_n := \frac{1}{2} \sum_k k^2 d_{n,k}, \quad (12)$$

with rational coefficients $d_{n,k} > 0$ given explicitly in Appendix A and tabulated in Table 7. The short-time slope is

$$Q_n^0(\tau) = c_n \tau + O(\tau^2), \quad c_n = \frac{(2n-1)!!}{n^2((n-2)!!)^2}, \quad (13)$$

and the long-time limit is $Q_n^0(\tau) \rightarrow q_n$. For $n = 1$,

$$Q_1^0(\tau) = \frac{2}{\tau} (\tau - 1 + e^{-\tau}), \quad (14)$$

which increases monotonically from 0 to $q_1 = 2$: the first-order reference interpolates between the ballistic regime and the constant value of the thermodynamic uncertainty relation, and it is the only odd order for which the saturation value coincides with that constant. For $n = 3$ the saturation value is $q_3 = 22/9 > 2$, and q_n grows with n .

Fischer et al. [8] conjectured that the reference bounds the uncertainty product from below for every potential:

$$Q_n(\tau; 1) \geq Q_n^0(\tau) \quad \text{for every odd } n \geq 1, \tau > 0, \text{ and smooth periodic } V. \quad (15)$$

The conjecture concerns the homogeneous weight $w \equiv 1$ and the currents (1). Weighted currents are not covered by (15); we treat them separately, both because a positive weight is the natural generalisation and because the weighted rows isolate which part of a current carries a violation (Sections 6 and 8). It is convenient to work with the two ratios

$$R_n(\tau; w) = \frac{Q_n(\tau; w)}{Q_n^0(\tau)}, \quad R_{0,n}(V) = \lim_{\tau \rightarrow 0} R_n(\tau; 1), \quad (16)$$

so that (15) is the statement $R_n(\tau; 1) \geq 1$ and $R_{0,n} \geq 1$ is its infinitesimal version.

3 The short-time ratio is a stationary quantity

Expanding the autocorrelation in (10), $C(t) = \text{Var}(g) + t \text{Cov}(g, Lg) + O(t^2)$, gives

$$\text{Var}(Y_n(\tau; w)) = \text{Var}(g) \tau^2 + \frac{1}{3} \text{Cov}(g, Lg) \tau^3 + O(\tau^4), \quad (17)$$

and hence

$$Q_n(\tau; w) = A_n(w) \tau + B_n(w) \tau^2 + O(\tau^3), \quad A_n(w) = \frac{\sigma \text{Var}(g)}{\langle g \rangle^2}, \quad B_n(w) = \frac{\sigma \text{Cov}(g, Lg)}{3 \langle g \rangle^2}. \quad (18)$$

Comparing with (13),

$$R_{0,n} = \frac{A_n(1)}{c_n}, \quad \text{and} \quad R_{0,n} < 1 \implies R_n(\tau; 1) < 1 \text{ for all sufficiently small } \tau > 0. \quad (19)$$

The infinitesimal bound is thus a statement about stationary expectations alone; no propagator is needed to test it. Two cases are worth separating.

First order. For $n = 1$ and $w \equiv 1$ we have $g = u$, $\langle g \rangle = m$, $\sigma = fm$, and by (6) $\text{Var}(u) = 1 + fm - m^2$. Hence

$$A_1 = \frac{f(1 + fm - m^2)}{m}, \quad c_1 = 1, \quad R_{0,1} - 1 = \frac{(f - m)(1 + fm)}{m}. \quad (20)$$

For $m > 0$ and $f > 0$ the factor $1 + fm$ is positive, so an infinitesimal violation of (15) at $n = 1$ is *equivalent* to $m > f$. Since $\text{Var}(u) = 1 + m(f - m)$, the same condition is equivalent to $\text{Var}(u) < 1$: the bound can fail infinitesimally only if the stationary velocity distribution is narrower than its thermal width. Section 4 excludes both.

Higher odd orders. For $n \geq 3$ the ratio has a useful equilibrium limit. At $f = 0$ the velocity marginal is $\mathcal{N}(0, 1)$ for every potential, because the potential reshapes only the θ -dependence; hence $\langle u^n \rangle = 0$ for odd n , $\langle u^{2n} \rangle = (2n - 1)!!$ exactly, and both m and $m_n := \langle u^n \rangle$ are odd functions of f . Writing the linear-response coefficients

$$\mu' = \left. \frac{dm}{df} \right|_{f=0}, \quad M_{1,n} = \left. \frac{dm_n}{df} \right|_{f=0}, \quad (21)$$

a short computation from (18) gives

$$R_{0,n}(0) = \frac{\mu' (n!!)^2}{M_{1,n}^2}, \quad (22)$$

where both responses are integrated equilibrium velocity correlations in the Green–Kubo sense [22],

$$\mu' = \int_0^\infty \langle u(t)u(0) \rangle_0 dt, \quad M_{1,n} = \int_0^\infty \langle u^n(t)u(0) \rangle_0 dt, \quad (23)$$

with free values 1 and $n!!$. Both are positive, so $R_{0,n}(0) \geq 1$ is equivalent to $M_{1,n} \leq n!!\sqrt{\mu'}$: the bound holds infinitesimally if and only if the n th velocity response is not more strongly suppressed by the potential than the mobility is. For free diffusion both sides are equal, and $R_n(\tau; 1) \equiv 1$ identically.

4 First order: the bound holds near zero time

The equivalence (20) reduces the infinitesimal bound at $n = 1$ to a stationary inequality, and that inequality is true.

Lemma 1 (Positivity of the stationary flux). *Let V be smooth and 1-periodic and let $f > 0$. Then the stationary mean velocity of (2) satisfies $m > 0$.*

Proof. Suppose $m = 0$. Then $\sigma = fm = 0$, and by Lemma 2 below $\sigma = \sigma^{\text{lm}} + \sigma^{\text{vf}}$ is a sum of two non-negative terms, so both vanish. In particular $\sigma^{\text{lm}} = \int_0^1 \nu^2 P d\theta = 0$ forces $\nu \equiv 0$, and $\sigma^{\text{vf}} = 0$ forces $\partial_u \ln \rho = -(u - \nu)$ pointwise, so $\rho(\theta, u) = P(\theta)\varphi(u)$ with $\varphi(u) = (2\pi)^{-1/2}e^{-u^2/2}$. Substituting this product form into the stationary Fokker–Planck equation $0 = -u\partial_\theta \rho - \partial_u[(f - V' - u)\rho] + \partial_u^2 \rho$ and using $\varphi' = -u\varphi$, $\varphi'' = (u^2 - 1)\varphi$ gives $u[P(f - V') - P']\varphi = 0$, hence $(\ln P)' = f - V'(\theta)$. Integrating over one period and using the periodicity of V yields $0 = f$, contradicting $f > 0$. $\square$

Lemma 2 (Exact entropy identities). *Let ρ be the stationary density of (2) and $\psi = \partial_u \ln \rho$. Then*

$$\int_{\mathbb{T}} \int_{\mathbb{R}} \frac{(\partial_u \rho)^2}{\rho} du d\theta = 1, \quad \int_{\mathbb{T}} \int_{\mathbb{R}} \rho(u + \psi)^2 du d\theta = fm = \sigma. \quad (24)$$

Proof. Write $F = \int \int (\partial_u \rho)^2 / \rho = \int \int \rho \psi^2$. The first identity in (6) applied to $\phi = u^2/2$ gives $\langle u^2 \rangle = 1 + fm$. Integration by parts in u gives $\int \int u \partial_u \rho = -\int \int \rho = -1$, that is, $\langle u\psi \rangle = -1$. It remains to show $F = 1$. Multiply the stationary Fokker–Planck equation by $\ln \rho$ and integrate over $\mathbb{T} \times \mathbb{R}$. The θ -term vanishes: $-\int \int \ln \rho \partial_\theta(u\rho) = \int \int u \partial_\theta \rho = \int_{\mathbb{R}} u [\rho]_{\theta=0}^{\theta=1} du = 0$ by periodicity. For the u -terms, integrating by parts twice,

$$0 = \int \int \psi(f - V' - u)\rho - F = f\langle \psi \rangle - \langle V'\psi \rangle - \langle u\psi \rangle - F.$$

Now $\langle \psi \rangle = \int \int \partial_u \rho = 0$, $\langle V'\psi \rangle = \int_{\mathbb{T}} V'(\theta) [\int_{\mathbb{R}} \partial_u \rho du] d\theta = 0$ because V' does not depend on u , and $\langle u\psi \rangle = -1$. Hence $F = 1$. Combining, $\int \int \rho(u + \psi)^2 = \langle u^2 \rangle + 2\langle u\psi \rangle + F = (1 + fm) - 2 + 1 = fm$. $\square$

The second identity in (24) is the entropy-production rate written as a Fisher-type quadratic form; it is the starting point of the decomposition used by Dechant [4], and Lemma 2 derives it here without reference to that work.

Theorem 3 (Sharp stationary velocity bound). *Let V be smooth and 1-periodic, $f > 0$, and let m, P, χ be as in (4) and (8). Then*

$$0 < m \leq \chi f \leq f. \quad (25)$$

Equality $m = f$ holds if and only if $V' \equiv 0$.

Proof. Decompose $u = \nu(\theta) + \varpi$ with $\varpi = u - \nu(\theta)$. By Lemma 2,

$$\sigma = \int \int \rho(u + \psi)^2 = \underbrace{\int_0^1 \nu(\theta)^2 P(\theta) d\theta}_{\sigma^{\text{lm}}} + \underbrace{\int \int \rho(\varpi + \psi)^2 du d\theta}_{\sigma^{\text{vf}}},$$

because the two cross terms vanish: $\int \int \rho \nu \varpi = \int_0^1 \nu [\int u \rho du - \nu P] d\theta = 0$ by the definition of ν , and $\int \int \rho \nu \psi = \int_0^1 \nu [\int \partial_u \rho du] d\theta = 0$. Both terms are non-negative. The first is local-mean-velocity production, the second velocity-fluctuation production. Using the flux identity (5),

$$\sigma^{\text{lm}} = \int_0^1 \frac{m^2}{P(\theta)} d\theta = \frac{m^2}{\chi},$$

so $fm = m^2/\chi + \sigma^{\text{vf}} \geq m^2/\chi$. Dividing by $m > 0$ (Lemma 1) gives $m \leq \chi f$. Finally, Cauchy–Schwarz in the measure $d\theta$ gives $(\int_0^1 P d\theta)(\int_0^1 P^{-1} d\theta) \geq (\int_0^1 1 d\theta)^2 = 1$, and $\int_0^1 P d\theta = 1$, so $\chi \leq 1$.

For the equality case, let $m = f$. Then $f^2 = f^2/\chi + \sigma^{\text{vf}}$ with $\chi \leq 1$ and $\sigma^{\text{vf}} \geq 0$, which forces $\chi = 1$ and $\sigma^{\text{vf}} = 0$. Cauchy–Schwarz equality gives $P \equiv 1$, and $\sigma^{\text{vf}} = 0$ gives $\psi = -(u - \nu)$, hence $\rho(\theta, u) = (2\pi)^{-1/2} e^{-(u-\nu(\theta))^2/2}$ with $\nu = m/P = m$. Substituting the θ -independent Gaussian $\varphi(u) = (2\pi)^{-1/2} e^{-(u-m)^2/2}$ into the stationary Fokker–Planck equation, and using $\varphi' = -(u - m)\varphi$, $\varphi'' = [(u - m)^2 - 1]\varphi$, gives

$$0 = (u - m) \varphi(u) [f - m - V'(\theta)]$$

for every (θ, u) , hence $V'(\theta) = f - m$ is constant. A smooth periodic potential has $\int_0^1 V' d\theta = 0$, so $f = m$ and $V' \equiv 0$. The converse is free diffusion, for which $u \sim \mathcal{N}(f, 1)$ and $m = f$. $\square$

Corollary 4 (Infinitesimal first-order bound). *For every smooth 1-periodic V and every $f > 0$,*

$$R_{0,1} - 1 = \frac{(f - m)(1 + fm)}{m} \geq 0, \quad (26)$$

with equality if and only if $V' \equiv 0$. Consequently there is $\tau_1 > 0$ such that $Q_1(\tau; 1) \geq Q_1^0(\tau)$ for all $0 < \tau < \tau_1$, strictly unless the potential is constant.

Proof. The formula is (20); Theorem 3 gives $m \leq f$ and the equality case, and (19) turns $R_{0,1} \geq 1$ into the finite- τ statement near $\tau = 0$. $\square$

Corollary 4 is an infinitesimal statement only. It says that (15) cannot fail at $n = 1$ by a slope argument; whether $Q_1(\tau; 1) \geq Q_1^0(\tau)$ holds at every finite τ remains open, and the numerical evidence for it is reported in Section 7. Two further consequences of Theorem 3 are worth recording. First, $\langle V' \rangle = f - m \geq 0$: the potential always opposes the drive on average. Second, $\text{Var}(u) = 1 + m(f - m) \geq 1$, so the stationary velocity distribution of a driven periodic diffusion is never narrower than the thermal one, and the deficit $1 - \chi$ measures how much of the entropy production is spent on velocity fluctuations rather than on the mean drift.

Table 1 collects the decomposition on eight probe potentials, from the free case to a strongly pinned cosine and to an asymmetric four-mode potential. Every row satisfies $m \leq \chi f \leq f$, $\sigma^{\text{vf}} \geq 0$ and $R_{0,1} \geq 1$; the free row sits exactly at $\chi = 1$, $\sigma^{\text{vf}} = 0$, $R_{0,1} = 1$, and the weak-potential rows approach it, as the equality case of Theorem 3 requires. The values of σ^{lm} printed in the table reproduce m^2/χ to a relative 2.6×10^{-3} at worst, and the printed $R_{0,1}$ reproduce $f \text{Var}(u)/m$ to a relative 1.4×10^{-2} at worst; both deviations are discretisation and printing tolerances of the spectral solve, not failures of the identities, which are exact.

5 Odd orders $n \geq 7$: the bound fails

5.1 The quadratic law and its exact coefficient

For the one-mode family

$$V(\theta) = V_0 \cos(\omega\theta), \quad \omega = 2\pi k, \quad k \in \mathbb{N}, \quad (27)$$

the equilibrium limit (22) has a clean expansion in the amplitude. Perturbing the equilibrium resolvent once in V_0 gives

$$R_{0,n}(V_0) = 1 + \kappa_n V_0^2 + O(V_0^4), \quad \mu' = 1 - \mu_2 V_0^2 + O(V_0^4), \quad M_{1,n} = n!! (1 - a_n V_0^2 + O(V_0^4)), \quad (28)$$

and therefore

$$\kappa_n = 2a_n - \mu_2. \quad (29)$$

Table 1: Entropy-production decomposition and the infinitesimal first-order ratio on eight stationary probe states, evaluated with the Fourier–Hermite solver of Appendix C. Columns: drive f ; stationary mean velocity $m = \langle u \rangle$; mobility χ of (8); the bound χf ; the local-mean and velocity-fluctuation parts of $\sigma = fm$; and $R_{0,1} = f \text{Var}(u)/m$ from (20). The four-mode potential is $V = 0.7 \cos 2\pi\theta + 0.5 \cos(4\pi\theta + 1) + 0.3 \cos(6\pi\theta + 2)$. Every row obeys $m \leq \chi f \leq f$ and $R_{0,1} \geq 1$, with equality only in the free row.

probe	f	$m = \langle u \rangle$	χ	χf	σ^{lm}	σ^{vf}	R_0
$V \equiv 0$	0.70	0.700000	1.000000	0.700000	0.490000	+0.000000	1.0000
$0.1 \cos 2\pi\theta$	0.70	0.679083	0.996155	0.697308	0.462933	+0.012425	1.0454
$0.25 \cos 2\pi\theta$	0.70	0.591717	0.977093	0.683965	0.358340	+0.055862	1.2584
$0.5 \cos 2\pi\theta$	0.70	0.422013	0.908666	0.636066	0.196004	+0.099405	1.8539
$4 \cos 2\pi\theta$	3.00	0.014634	0.023470	0.070410	0.009142	+0.034761	213.5937
$1.5 \cos 4\pi\theta$	1.50	0.208244	0.463067	0.694600	0.093893	+0.218473	9.2672
four-mode asymmetric, $f = 1.5$	1.50	1.329460	0.984687	1.477030	1.794988	+0.199201	1.3772
four-mode asymmetric, $f = 0.5$	0.50	0.380891	0.967705	0.483853	0.149901	+0.040544	1.3693

The parity of the perturbation is what makes the law quadratic rather than linear: every first-order term in $M_{1,n}$ vanishes by the central symmetry of the equilibrium velocity law and by $\langle \cos \omega \theta \rangle_0 = 0$. The coefficients have a closed form that involves no stationary spectral solve. Let φ solve the single scalar ordinary differential equation

$$M_\omega \varphi = 1, \quad M_\omega := \partial_u^2 - u \partial_u - i\omega u, \quad (30)$$

and let $\ell_n = L_{\text{free}}^{-1}(u^n)$ be the explicit degree- n polynomial for the free generator $L_{\text{free}} = u \partial_\theta - u \partial_u + \partial_u^2$; the first three are $\ell_1 = -u$, $\ell_3 = -u^3/3 - 2u$ and $\ell_5 = -u^5/5 - 4u^3/3 - 8u$. Then

$$\mu_2 = -\frac{\omega^2}{2} \text{Re} \langle \varphi \rangle, \quad a_n = \frac{1}{n!!} \left[\frac{\omega}{2} \text{Im} \langle u^n \varphi \rangle - \frac{\omega^2}{2} \text{Re} \langle \ell_n \varphi' \rangle \right], \quad (31)$$

where $\langle \cdot \rangle$ is the expectation against the standard Gaussian $d\mu = (2\pi)^{-1/2} e^{-u^2/2} du$ and $\varphi' = \partial_u \varphi$. In the orthonormal probabilists' Hermite basis e_j ,

$$M_\omega e_j = -j e_j - i\omega (\sqrt{j+1} e_{j+1} + \sqrt{j} e_{j-1}), \quad (32)$$

so (30) is the resolvent of a tridiagonal operator and every projection in (31) is a finite matrix solve. The mobility coefficient is the $n = 1$ case, $a_1 = \mu_2$, so $\kappa_1 = a_1 = \mu_2 > 0$: the first-order coefficient is positive for every mode, in agreement with Corollary 4, which proves the same statement without perturbation theory.

5.2 An exact moment recurrence to order 401

The projections in (31) satisfy a three-term recurrence that can be run symbolically. Let $z_j = \langle u^j \varphi \rangle$. Multiplying (30) by u^j , integrating against $d\mu$ and integrating by parts gives, for every $j \geq 0$,

$$j(j-1)z_{j-2} - j z_j - i\omega z_{j+1} = \mu_j, \quad \mu_j = \begin{cases} 0, & j \text{ odd}, \\ (j-1)!!, & j \text{ even}, \end{cases} \quad (33)$$

with $z_{-2} = z_{-1} = 0$. The recurrence needs two boundary constants: $z_1 = i/\omega$ and $z_0 = \langle \varphi \rangle$. The second is obtained from the backward continued fraction of the tridiagonal matrix (32), that is, as the minimal solution rather than by forward iteration, which is numerically unstable [38, 47]. With

$$\langle \ell_n \varphi' \rangle = - \sum_j h_j^{(n)} (z_{n+1-2j} - (n-2j)z_{n-1-2j}), \quad (34)$$

Table 2: The weak-potential coefficient κ_n of (28) for the one-mode family (27), from the exact recurrence (33)–(35) at 110-digit precision. Positive values support the infinitesimal bound, negative values violate it. The sign changes between $n = 5$ and $n = 7$ for every mode shown.

n	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
1	+4.112763	+8.045722	+11.981687	+15.918378	+19.855356	+23.792477
3	+1.704254	+3.015241	+4.327229	+5.639459	+6.951785	+8.264159
5	+0.414061	+0.641275	+0.891298	+1.147419	+1.406046	+1.665943
7	−0.326321	−0.677205	−1.016931	−1.355047	−1.692746	−2.030302
9	−0.790963	−1.525385	−2.260992	−2.997500	−3.734406	−4.471513
11	−1.111076	−2.131355	−3.157519	−4.184935	−5.212784	−6.240830
13	−1.349827	−2.594236	−3.844558	−5.095897	−6.347587	−7.599440
15	−1.538448	−2.963392	−4.392984	−5.823407	−7.254139	−8.685020
17	−1.693273	−3.266764	−4.843873	−6.421740	−7.999904	−9.578214
19	−1.823598	−3.521769	−5.223046	−6.925067	−8.627380	−10.329839
21	−1.935252	−3.739989	−5.547682	−7.356117	−9.164844	−10.973716
401	−3.617708	−7.058110	−10.501517	−13.945651	−17.390072	−20.834635

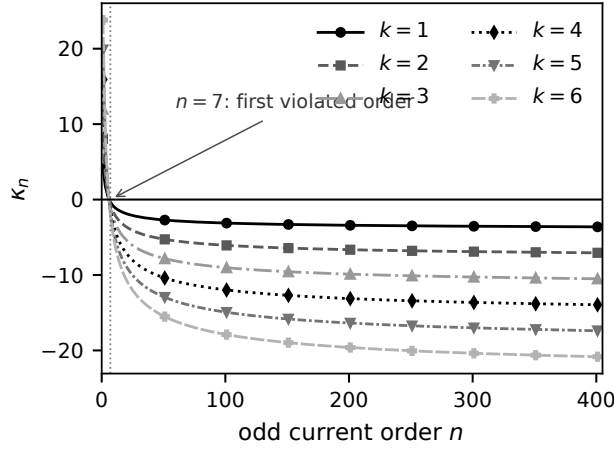


Figure 1: The coefficient κ_n against the odd current order n for the six modes $k = 1, \dots, 6$ of the one-mode family. The horizontal line is the free-diffusion value $\kappa_n = 0$; above it the infinitesimal bound holds, below it the bound is violated for sufficiently weak potentials. The sign change between $n = 5$ and $n = 7$ is common to all six modes.

$$h_0^{(n)} = \frac{1}{n}, \quad h_j^{(n)} = \frac{(n-1)(n-3)\cdots(n-2j+1)}{n-2j}, \quad (35)$$

equation (31) becomes a purely algebraic evaluation of κ_n . Running it at 110-digit working precision produces κ_n for every odd $n \leq 401$ and every mode $k = 1, \dots, 6$, with a maximum relative recurrence residual of 8.6×10^{-60} . Two independent checks were carried out: the Hermite resolvent route of Section 5.1 reproduces the tabulated coefficients for $n \leq 21$ to the digits shown, and the first three orders agree with the closed forms of the equilibrium reference problem, $a_1 = 2$, $a_3 = 22/9$, $a_5 = 898/225$ in the free-diffusion normalisation.

Table 2 and Figure 1 give the result. The sign pattern is the same for every mode: $\kappa_1, \kappa_3, \kappa_5 > 0$ and $\kappa_n < 0$ for every odd $n \geq 7$, with $|\kappa_n|$ increasing slowly with n and roughly linearly with k . The first violated order is $n = 7$, exactly as the earlier spectral fits suggested: fitting the quadratic law to four small amplitudes gives $\kappa_1 = +4.0937$, $\kappa_3 = +1.6857$, $\kappa_5 = +0.4028$, $\kappa_7 = -0.3312$ at $k = 1$, which differs from the exact values of Table 2 only by the $O(V_0^4)$ contamination of the fit.

5.3 Reducing the sign to one monotonicity statement

The recurrence shows that $\kappa_7 < 0$ exactly. To reach every odd $n \geq 7$ we need a_n to decrease with n , because then

$$\kappa_n = \kappa_7 - 2(a_7 - a_n) < \kappa_7 < 0 \quad \text{for every odd } n \geq 7. \quad (36)$$

That monotonicity is a single sign question about an explicitly known function. The argument of φ is even, since the even Hermite coefficients of φ are real and the odd ones purely imaginary; define the boundary function

$$\beta(u) = (I - L_{\text{free}})^{-1} \text{Re } \varphi(u), \quad \mathcal{M}_j = \int_{\mathbb{R}} u^j \beta(u) d\mu(u). \quad (37)$$

Theorem 5 (Monotonicity reduction). *For the one-mode family (27),*

$$a_n = -\frac{\omega^2}{2(n-2)!!} \mathcal{M}_{n-1} \quad (38)$$

for every odd $n \geq 1$, and

$$a_{j+1} - a_{j+3} = \frac{\omega^2}{2(j+1)!!} \int_{\mathbb{R}} u^{j+1} \beta'(u) d\mu(u) = \frac{\omega^2}{2(j+1)!!} [\mathcal{M}_{j+2} - (j+1)\mathcal{M}_j] \quad (39)$$

for every even $j \geq 0$. If $\beta'(u) > 0$ for all $u > 0$, then a_n is strictly decreasing in n over the odd integers.

Proof. On θ -independent functions the free generator reduces to the Ornstein–Uhlenbeck generator $L_{\text{OU}} = \partial_u^2 - u\partial_u$, which is self-adjoint in $L^2(d\mu)$ and satisfies $\langle L_{\text{OU}} F \rangle = 0$ whenever the boundary terms vanish. Write $\varphi = \psi_e + i\psi_o$ with $\psi_e = \text{Re } \varphi$ and $\psi_o = \text{Im } \varphi$; separating (30) into real and imaginary parts gives

$$L_{\text{OU}}\psi_e + \omega u\psi_o = 1, \quad L_{\text{OU}}\psi_o = \omega u\psi_e. \quad (40)$$

The polynomials ℓ_n and the function β are θ -independent, so $\beta = (I - L_{\text{OU}})^{-1}\psi_e$ and three elementary identities suffice. First, $\langle \psi_e \rangle = \langle (I - L_{\text{OU}})\beta \rangle = \langle \beta \rangle = \mathcal{M}_0$. Second, integration by parts against $d\mu$ gives the adjointness relation

$$\langle F, (u - \partial_u)G \rangle = \langle F', G \rangle, \quad (41)$$

so that $\langle \ell_n(u\psi_e - \psi'_e) \rangle = \langle \ell'_n, \psi_e \rangle$. Third, since $L_{\text{OU}}(F') = (L_{\text{OU}}F)' + F'$,

$$(I - L_{\text{OU}})\ell'_n = \ell'_n - [(L_{\text{OU}}\ell_n)' + \ell'_n] = -(u^n)' = -n u^{n-1}, \quad (42)$$

and therefore $\langle \ell'_n, \psi_e \rangle = \langle (I - L_{\text{OU}})\ell'_n, \beta \rangle = -n\mathcal{M}_{n-1}$.

Now evaluate (31). Self-adjointness together with (40) gives $\langle u^n \psi_o \rangle = \langle L_{\text{OU}}\ell_n, \psi_o \rangle = \langle \ell_n, L_{\text{OU}}\psi_o \rangle = \omega \langle \ell_n u \psi_e \rangle$, so the two terms of a_n combine into a single pairing:

$$\begin{aligned} n!! a_n &= \frac{\omega}{2} \langle u^n \psi_o \rangle - \frac{\omega^2}{2} \langle \ell_n \psi'_e \rangle = \frac{\omega^2}{2} \langle \ell_n (u\psi_e - \psi'_e) \rangle \\ &= \frac{\omega^2}{2} \langle \ell'_n, \psi_e \rangle = -\frac{\omega^2 n}{2} \mathcal{M}_{n-1}, \end{aligned} \quad (43)$$

which is (38), because $n!! = n(n-2)!!$ for odd n .

For (39), the first equality is integration by parts against $d\mu$: $\int u^{j+1} \beta' d\mu = \mathcal{M}_{j+2} - (j+1)\mathcal{M}_j$, and the second follows from (38) by writing $(j+1)!! = (j+1)(j-1)!!$ and collecting terms.

Finally, the even Hermite coefficients of φ are real and the odd ones purely imaginary, so ψ_e is even, β is even and β' is odd; $u^{j+1}\beta'(u)$ is then even for even j and the integral in (39) equals $2 \int_0^\infty u^{j+1}\beta'(u)d\mu(u)$, which is strictly positive whenever $\beta' > 0$ on the half-line. Strict positivity of every difference $a_{j+1} - a_{j+3}$ is exactly strict decrease of a_n over the odd integers. $\square$

5.4 The boundary function and its exact integral representation

The function β is not merely a numerical object: it has a closed integral representation. By Feynman–Kac for the Ornstein–Uhlenbeck process $dX = -X dt + \sqrt{2} dW$ started at u , the resolvent of (30) is

$$\varphi(u) = - \int_0^\infty \exp\left[-i\omega u(1 - e^{-t}) - \frac{\omega^2}{2}\varsigma_t^2\right] dt, \quad \varsigma_t^2 = 2t - 3 + 4e^{-t} - e^{-2t}, \quad (44)$$

so that $\operatorname{Re} \varphi(u) = - \int_0^\infty e^{-(\omega^2/2)\varsigma_t^2} \cos[\omega u(1 - e^{-t})] dt$. Acting once more with the free resolvent $(I - L_{\text{free}})^{-1} = \int_0^\infty e^{-s} e^{sL_{\text{free}}} ds$ and changing variables to $\lambda = \omega(1 - e^{-t})e^{-s}$ gives

$$\beta(u) = - \int_0^\omega \Pi(\lambda) \cos(\lambda u) d\lambda, \quad \Pi(\lambda) = \frac{e^{\lambda^2/2}}{\omega} \int_{\lambda/\omega}^1 (1-y)^{\omega^2-1} \frac{e^{\omega^2 y}}{y} dy. \quad (45)$$

The density Π is positive on $(0, \omega)$, decreasing and convex on a dense sample, and has the logarithmic endpoint behaviour

$$\Pi(\lambda) = \frac{1}{\omega} \ln \frac{\omega}{\lambda} - \frac{1.6394386}{\omega} + o(1) \quad (\lambda \downarrow 0), \quad (46)$$

with $\Pi(\omega) = 0$ to infinite order. Two consequences are exact. First, the large- u tail follows from (46) by Watson’s lemma:

$$\beta(u) = -\frac{\pi}{2\omega u} + O(u^{-2}), \quad \beta'(u) = \frac{\pi}{2\omega u^2} + O(u^{-3}), \quad (47)$$

so $\beta' > 0$ at infinity in a strict asymptotic sense. Second, the endpoint value is $\beta(0) = - \int_0^\omega \Pi = -0.2577726643$ for $k = 1$, and β increases from that value to 0^- , as Figure 2 shows together with the strictly positive derivative. The representation was checked independently in two ways: by inverting the Hermite resolvent (32), and by direct quadrature of (45). The two agree to the digits quoted, and both agree with the high-resolution sample of the archived certificate to 2.6×10^{-6} in $\beta(0)$. The monotonicity identities (38)–(39) were verified numerically on the same footing: the relative deviation of the difference identity is below 10^{-7} for every even j tested, and $\int u^{j+1} \beta' d\mu > 0$ for all of them.

5.5 A three-region certificate for $\beta' > 0$

The positivity of β' on the whole half-line is the only ingredient of the refutation that is not algebraic. It follows from

$$\beta'(u) = \int_0^\omega \lambda \Pi(\lambda) \sin(\lambda u) d\lambda > 0 \quad (u > 0), \quad (48)$$

which is a sine-transform positivity statement, and it is established in three regions.

In region (i), $0 < u \leq \pi/\omega$, every $\lambda \in [0, \omega]$ satisfies $\lambda u \in [0, \pi]$, so $\sin(\lambda u) \geq 0$ and (48) is positive because $\Pi > 0$ on $(0, \omega)$. This region is exact and needs no numerics.

In region (ii), $u \geq U_0$, write the density as $\Pi'' = b - 1/(\omega\lambda)$ and integrate by parts twice; the logarithmic endpoint at $\lambda = 0$ is handled as an $\varepsilon \rightarrow 0$ limit and Π, Π' vanish to infinite order at ω . The result is the exact identity

$$\beta'(u) = \frac{1}{u^2} \left[\frac{1}{\omega} \operatorname{Si}(\omega u) - \int_0^\omega b(\lambda) \sin(\lambda u) d\lambda \right], \quad (49)$$

where b is continuous on $(0, \omega)$ with $b(0+) = -2/\omega^2$, $b(\omega) = 1/\omega^2$ and bounded variation $\operatorname{TV}(b) = \int_0^\omega |b'| d\lambda$. Stieltjes integration by parts bounds the oscillatory term by C_b/u with $C_b = 3/\omega^2 + \operatorname{TV}(b)$, and $\operatorname{Si}(t) \geq \pi/2 - 2/t$ bounds the first term, so

$$\beta'(u) \geq \frac{1}{u^2} \left[\frac{\pi}{2\omega} - \frac{2/\omega^2 + C_b}{u} \right] > 0 \quad \text{for every } u > U_0, \quad (50)$$

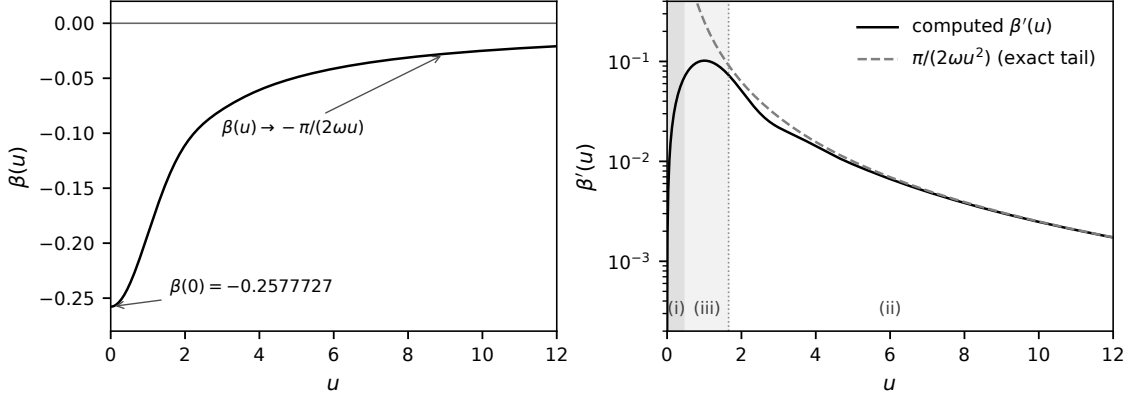


Figure 2: The boundary function β of (37) for $k = 1$, computed from the exact representation (45). Left: $\beta(u)$, increasing from $\beta(0) = -0.25777$ towards 0^- and matching the tail $-\pi/(2\omega u)$. Right: $\beta'(u)$, positive on the whole half-line, with the minimum over a grid of $u \leq 120$ equal to 1.7×10^{-5} , consistent with the exact asymptotics $\pi/(2\omega u^2)$. The shaded bands and the dotted vertical line mark the three regions of the certificate of Section 5.5: region (i), $0 < u \leq \pi/\omega$, where the sign follows from the representation without numerics; region (iii), $\pi/\omega \leq u \leq U_0 = 1.6539$, certified by a sampled lower bound; and region (ii), $u \geq U_0$, covered by the analytic tail bound of Appendix B. The plotted minimum is the smallest sampled value of β' , not a bound on its infimum.

where

$$U_0 = \frac{4}{\pi\omega} + \frac{2\omega C_b}{\pi} \quad (51)$$

is the value at which the bracket vanishes. This region is analytic once the two constants $\text{TV}(b)$ and C_b are known; they are evaluated by quadrature and are listed in Table 3, and the derivation of the bound, the rounding convention that fixes where the region starts and an independent reproduction of all six constants are in Appendix B.

In region (iii), the finite middle interval $\pi/\omega \leq u \leq U$, where $U \geq U_0$ is the outer start rounded up to two decimals, the second derivative is bounded exactly by

$$|\beta''(u)| \leq M_2 := \int_0^\omega \lambda^2 \Pi(\lambda) d\lambda, \quad (52)$$

and sampling β' on a mesh of width h with linear interpolation gives $\beta'(u) \geq \min_j \beta'(u_j) - M_2 h^2/8$. The sampled minimum is positive by several orders of magnitude for every mode considered, so the middle region is certified.

Proposition 6 (Positivity of β' , computer-assisted). *For every mode $k \in \{1, 2, 3, 4, 5, 6\}$, that is $\omega = 2\pi k$ with $k \leq 6$, $\beta'(u) > 0$ for all $u > 0$.*

Proof. Regions (i) and (ii) are the exact arguments above; region (ii) needs the constants $\text{TV}(b)$, C_b and U_0 of Table 3, for which the lower bound (50) is positive by construction at $u = U_0$ and increasing beyond it. Region (iii) is the interpolation bound with the recorded M_2 , mesh width and sampled minimum; the certified margin in Table 3 is the sampled minimum minus $M_2 h^2/8$, and it is positive for all six modes, the smallest being 5.98×10^{-3} at $k = 6$. $\square$

Remark 7 (Status of the certificate). *The inequalities in the three regions are exact; what is numerical is the evaluation of the four constants M_2 , $\text{TV}(b)$, C_b and the sampled minimum. For $k \leq 4$ these are computed in double precision and for $k = 5, 6$ the single inner integral of (45), which contains $e^{\omega^2 y}$, is computed in 50-digit arithmetic because e^{ω^2} overflows double precision at*

Table 3: Constants of the three-region certificate for $\beta'(u) > 0$ (Proposition 6), for the modes $k = 1, \dots, 6$. M_2 bounds $|\beta''|$ in the middle region, $\text{TV}(b)$ and $C_b = 3/\omega^2 + \text{TV}(b)$ control the oscillatory tail term in (49), and U_0 is the analytic start of region (ii). The certified margin is the sampled minimum of β' on the middle mesh minus the interpolation error $M_2 h^2/8$; the tail bound is the value of the right-hand side of (50) at $u = U_0$.

k	M_2	$\text{TV}(b)$	C_b	U_0	$\min_j \beta'(u_j)$	certified margin	tail bound
1	0.16721736	0.28682251	0.36281340	1.65389597	0.07274540	7.27×10^{-2}	3.34×10^{-4}
2	0.12415577	0.14821925	0.16721697	1.43905696	0.02944914	2.94×10^{-2}	3.95×10^{-5}
3	0.10271950	0.10120941	0.10965284	1.38338155	0.01659497	1.66×10^{-2}	2.05×10^{-4}
4	0.08926030	0.07676665	0.08151608	1.35491792	0.01092196	1.09×10^{-2}	1.26×10^{-4}
5	0.07979867	0.06181660	0.06485623	1.33765317	0.00785422	7.85×10^{-3}	4.88×10^{-5}
6	0.07268042	0.05173476	0.05384562	1.32606864	0.00598181	5.98×10^{-3}	6.96×10^{-5}

$\omega = 10\pi$ and 12π . The quadratures were validated by grid refinement, but the certificate is not interval-validated: it is a computer-assisted proof in the sense of Tucker [43, 42] with margins of order 10^{-2} to 10^{-5} , not a machine-checked enclosure. Upgrading it to interval arithmetic would make the refutation fully rigorous; nothing in the argument would change.

Remark 8 (Modes beyond six). The argument depends on k only through the explicit constants of Table 3, so it can be repeated for any further finite mode. A finite enumeration does not establish every $k \geq 7$, which would need a positivity argument uniform in k ; no such argument is claimed here. The exact recurrence (33), by contrast, runs for any k and gives $\kappa_7 < 0$ for every mode tested.

5.6 The refutation

Theorem 9 (Failure of the infinitesimal bound for $n \geq 7$). Let $k \in \{1, \dots, 6\}$ and let n be odd with $7 \leq n \leq 401$. For the one-mode family (27) the coefficient of (28) satisfies $\kappa_n < 0$.

Proof. The recurrence (33)–(35) evaluated at 110-digit precision gives $\kappa_7 < 0$ exactly for each of the six modes, the smallest in magnitude being $\kappa_7 = -0.3263205946\dots$ at $k = 1$. By Proposition 6 and Theorem 5, a_n is strictly decreasing over the odd integers, so (36) gives $\kappa_n < \kappa_7 < 0$ for every odd n with $7 \leq n \leq 401$. $\square$

Corollary 10 (Refutation of the conjectured bound). Assume that $V_0 \mapsto R_{0,n}(V_0)$ is analytic at $V_0 = 0$ for the one-mode family, as the expansion (28) asserts. Let $k \in \{1, \dots, 6\}$ and let n be odd with $7 \leq n \leq 401$. Then there is $V_* > 0$ such that for every $0 < V_0 < V_*$ there is $\tau_n > 0$ with

$$Q_n(\tau; 1) < Q_n^0(\tau) \quad \text{for all } 0 < \tau < \tau_n. \quad (53)$$

The conjectured bound (15) is therefore false for every odd order $n \geq 7$ in a weak-potential, short-time regime.

Proof. $\kappa_n < 0$ by Theorem 9 and analyticity give $R_{0,n}(V_0) = 1 + \kappa_n V_0^2 + O(V_0^4) < 1$ for all sufficiently small $V_0 > 0$. By (19) this implies $R_n(\tau; 1) < 1$ for all sufficiently small $\tau > 0$ at that amplitude. $\square$

Corollary 10 is an existence statement in two limits, and the $O(V_0^4)$ remainder is not bounded rigorously. The finite-amplitude version is numerical, and Section 6 supplies it: at $V_0 = 0.05$ and $V_0 = 0.2$ with $f = 3 \times 10^{-4}$ the seventh-order ratio is below one on the whole computed window $\tau \in [0.02, 50]$, and the three independent routes to the infinitesimal value agree there, $R(0.02) = 0.999199$ against $A_7/c_7 = 0.999173$ against $1 + \kappa_7 V_0^2 = 0.999184$ at $V_0 = 0.05$. The conjecture of Fischer et al. [8] is stated for all odd $n \geq 1$; the orders $n = 1, 3, 5$ pass the infinitesimal test, $n = 1$ by proof and $n = 3, 5$ by computation, and $n = 7$ is the first order at which it fails.

6 Violations at finite observation times

The infinitesimal result says nothing about the size of the window. To resolve finite times we evaluate (10) with the full propagator: the generator (3) is discretised in complex Fourier modes $e^{2\pi i k \theta}$ with $|k| \leq K$ and in Hermite functions with index $\ell < N_h$, the stationary density is the right null vector of the forward Fokker–Planck matrix, and

$$\text{Var}(Y_n(\tau; w)) = 2 \left\langle w u^n, L^{-2} (e^{\tau L} - I - \tau L) (w u^n - \langle w u^n \rangle \rho) \right\rangle \quad (54)$$

is evaluated with the Moore–Penrose inverse of L and a matrix exponential. Appendix C records the discretisation, the null-vector convention and the controls.

6.1 The homogeneous seventh-order window

At the weak-coupling probe $k = 1$, $f = 3 \times 10^{-4}$, $w \equiv 1$, $n = 7$ and $V_0 \in \{0.05, 0.2\}$, the ratio $R(\tau) = Q_7(\tau; 1)/Q_7^0(\tau)$ is below one on the entire computed window $\tau \in [0.02, 50]$ for both amplitudes. At $V_0 = 0.05$ it deepens slowly from 0.9992 to 0.9984; at $V_0 = 0.2$ it runs from 0.985 to 0.975 and decreases monotonically over the last four decades of τ (Table 4, left column group, and Figure 3). No crossing is visible, and the earlier estimate of a crossing near $\tau \approx 0.8$, obtained by truncating the short-time expansion at $B\tau^2$, is wrong: for $n = 7$ that term is already comparable to the omitted orders at $\tau \sim 0.1$.

Three controls pin these rows. The $V \equiv 0$ self-check reproduces the exact reference (12) to 7×10^{-6} relative over $\tau \leq 5$, to 2×10^{-6} over $\tau \in [0.5, 10]$ and to 5.4×10^{-7} over $\tau \in [10, 50]$, so the propagator machinery does not degrade with τ . The $\tau \rightarrow 0$ rows converge to the propagator-free stationary slope ratio A_7/c_7 and to the exact quadratic law $1 + \kappa_7 V_0^2$; at $V_0 = 0.2$ the three values are $R(0.02) = 0.985062$, $A_7/c_7 = 0.985127$ and $1 + \kappa_7 V_0^2 = 0.986947$, and the visible offset is the negative $O(V_0^4)$ term. Raising the truncation from $(K, N_h) = (16, 28)$ to $(20, 36)$ changes the sign-defining rows by about 10^{-4} in absolute value, well inside the margin. At the stationary endpoint of the $V_0 = 0.05$ window the resolved ratio is $A_7/c_7 = 0.99917331$ against $1 + \kappa_7 V_0^2 = 0.99918420$, an absolute difference of 1.1×10^{-5} at the weakest amplitude, where the quadratic law is most nearly exact.

6.2 A weighted violation that the infinitesimal test misses

Two positive, lower-harmonic spatial weights were added at the same probe: $w = 1 + 0.5 \cos(2\pi\theta)$, which is even under $\theta \mapsto -\theta$, and $w = 1 + 0.5 \sin(2\pi\theta)$, which is odd. At $V_0 = 0.05$ the three weights separate cleanly and stay where they start: the homogeneous ratio is below one throughout, the cosine ratio is above one and decays towards it, and the sine ratio is above one and approaches it from above, remaining at $R - 1 = +8.3 \times 10^{-4}$ even at $\tau = 50$.

At $V_0 = 0.2$ the sine weight crosses. Its infinitesimal ratio is $R_{0,7} = A_7/c_7 = 1.1075 > 1$, so the short-time test passes, and indeed $R(0.05) = 1.0938$ and $R(0.2) = 1.0084$; but R falls below one between $\tau = 0.2$ and $\tau = 0.25$ and stays below it out to $\tau = 10$, reaching 0.9786. Bisection on the default truncation locates the crossing at

$$\tau_* = 0.2321, \quad \text{bracket half-width } 9.8 \times 10^{-6}, \quad (55)$$

and the raised truncation (20, 36) gives $R(0.23) = 1.000197$ and $R(0.24) = 0.998285$, moving τ_* by less than 2×10^{-3} ; no digit beyond the truncation sensitivity is quoted. The cosine weight stays above one on the same window, and the homogeneous weight stays below one, so the crossing is specific to the parity-odd modulation.

An amplitude scan at $V_0 = 0.2$ with $w = 1 + a \sin(2\pi\theta)$, $a \in \{0, 0.1, 0.25, 0.5, 0.75, 1\}$, gives $R < 1$ at $\tau = 1$ and $\tau = 10$ for every a tested; at $\tau = 1$ the deficit $1 - R$ decreases monotonically from 2.03×10^{-2} at $a = 0$ to 1.58×10^{-3} at $a = 1$. A phase scan at $a = 0.5$, $\tau = 1$ gives $R < 1$ for

Table 4: Finite-time ratios $R(\tau) = Q_7(\tau; w)/Q_7^0(\tau)$ for the one-mode potential at $k = 1$, $f = 3 \times 10^{-4}$, $n = 7$, from the propagator formula (54) at $(K, N_h) = (16, 28)$. The weight columns are the homogeneous weight $w \equiv 1$, the parity-odd weight $w_s = 1 + 0.5 \sin 2\pi\theta$ and the parity-even weight $w_c = 1 + 0.5 \cos 2\pi\theta$. Values below one violate the conjectured bound. The $V_0 = 0.2$ homogeneous column and the $V_0 = 0.2$ odd-weight column are both below one, from $\tau = 0.05$ and $\tau = 0.25$ respectively, out to $\tau = 50$; the odd-weight column crosses at $\tau_* = 0.2321$ although its infinitesimal ratio is 1.1075. The $V_0 = 0.05$ even-weight column stays above one throughout. Dashes mark combinations that were not computed.

τ	$V_0 = 0.2$			$V_0 = 0.05$		
	1	w_s	w_c	1	w_s	w_c
0.05	0.984587	1.093809	1.088506	0.999151	1.110347	1.110275
0.2	0.980663	1.008353	1.031722	0.998892	1.027221	1.034016
0.25	0.979949	0.996914	1.024788	—	—	—
0.3	0.979604	0.992212	1.022007	—	—	—
0.4	0.979455	0.989324	1.019792	—	—	—
0.5	0.979476	0.987564	1.018028	0.998814	1.007128	1.015635
1	0.979704	0.984397	1.014090	0.998828	1.003628	1.011946
2	0.979819	0.983195	1.011351	0.998836	1.002282	1.010209
5	0.977919	0.980621	1.007206	0.998697	1.001448	1.008954
10	0.976147	0.978649	1.004659	0.998554	1.001100	1.008448
15	0.975535	—	—	0.998503	1.000984	1.008282
20	0.975235	—	—	0.998478	1.000928	1.008201
30	0.974941	—	—	0.998453	1.000872	1.008122
50	0.974708	—	—	0.998434	1.000829	1.008059

the odd phases $\phi = 0, \pi$ (0.9844 and 0.9843) and $R > 1$ for $\phi = \pi/4, \pi/2, 3\pi/4$ (1.0051, 1.0141, 1.0050). The finite-time violation is therefore carried by the parity-odd component of the weight, not by its parity-even part. These are floating-point spectral observations, not certified bounds.

6.3 Monte Carlo consistency

An independent Monte Carlo check of the violating rows themselves is not feasible at fixed precision: at $f = 3 \times 10^{-4}$ the signal is $\langle u^7 \rangle \approx 0.03$ against an equilibrium fluctuation scale $\sqrt{\text{Var}(u^7)} = \sqrt{13!!} \approx 368$, and the uncertainty product scales as $\langle u^7 \rangle^{-2}$. The propagator was therefore checked on a large-signal row, $f = 0.5$, $V_0 = 0.2$, $k = 1$, $\tau = 1$, where $\langle u^7 \rangle \approx 62$. Euler–Maruyama integration with step 10^{-3} , a burn-in of 20 relaxation times and 1.5×10^5 blocks of length $\tau = 1$ over five seeds gives $m \in [0.4351, 0.4446]$ against the spectral value 0.4404, and $Q \in [8.99, 12.98]$ against the spectral value 11.7495; the three sine-weight seeds give $Q \in [9.89, 12.90]$ against 11.8314. The spectral value lies inside the seed scatter, which is dominated by the heavy tail of u^7 . This is a coarse consistency check, not a tight cross-validation; the tight controls are the $V \equiv 0$ self-check and the truncation stability reported above.

7 The stationary sweep and the rows that decide nothing

7.1 Sweep statistics

A stationary sweep covers the one-mode family at $k = 1, 2$, amplitudes $V_0 \in \{0.5, 1.5, 4\}$, drives $f \in \{0.2, 0.7, 1.5, 3\}$, weights $w \in \{1, \cos 2\pi\theta\}$, orders $n \in \{1, 3, 5\}$ and observation times $\tau \in \{0.05, 0.1, 0.2, 0.5, 1, 2, 5\}$: 1008 rows. A row is marked resolved when the current mean is not near zero, since Q_n diverges or changes sign as $\langle g \rangle \rightarrow 0$; near-zero-mean rows are excluded rather than reported as violations. Table 5 summarises the outcome.

At $n = 1$ every one of the 175 resolved rows satisfies $R \geq 1.1314$, consistent with Corollary 4,

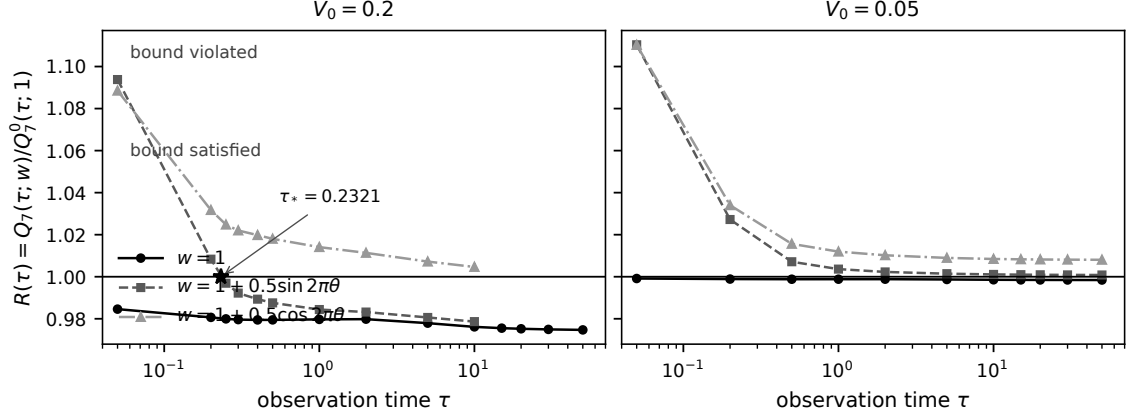


Figure 3: Finite-time ratios for the seventh-order current at $k = 1$, $f = 3 \times 10^{-4}$, for the homogeneous weight and for the two parity weights of Section 6.2. Left: $V_0 = 0.05$. Right: $V_0 = 0.2$. The horizontal line is the conjectured bound. The homogeneous ratio stays below it at both amplitudes; the sine weight starts above the bound at $V_0 = 0.2$ and crosses it at $\tau_* \approx 0.232$, which is the finite-time failure mode that the infinitesimal test cannot detect.

Table 5: Stationary sweep of 1008 rows over the one-mode family ($k = 1, 2$; $V_0 \in \{0.5, 1.5, 4\}$; $f \in \{0.2, 0.7, 1.5, 3\}$; $w \in \{1, \cos 2\pi\theta\}$; $\tau \in \{0.05, \dots, 5\}$). “Resolved” counts the rows whose current mean is not near zero. “min(Q/Q^0)” is the minimum over all resolved rows and “min physical” the minimum over those with a positive resolved variance. At $n = 1$ and $n = 3$ every resolved row obeys the bound, and the single $n = 3$ row closest to it is truncation-limited rather than physical (Section 7.1). The fifth-order minima are not violations of the bound: they are default-truncation artifacts, as Section 7.2 shows, and no converged fifth-order violation was found.

n	rows	resolved	min(Q/Q^0)	min physical	below 1	negative variance
1	336	175	1.1314	1.1314	0	0
3	336	252	1.0082	1.0082	0	0
5	336	252	-1.8028	0.2629	26	5

which proves the infinitesimal version. At $n = 3$ every one of the 252 resolved rows satisfies $R \geq 1.0082$ at the default truncation; the row closest to the bound is at $V_0 = 4$, where truncation sensitivity is strongest, and a convergence check moves it to between 1.19 and 1.28. At $n = 5$ the picture is different: of the 252 resolved rows, 26 fall below one and 5 have a negative resolved variance, which is not a physical value at all. A short-time sweep over the same family with three weights (216 rows, $w \in \{1, \cos, \sin\}$) gives $R_0 \geq 1.1559$ on every resolved row, and at $V_0 = 0.05$ the infinitesimal ratios are $R_{0,5} - 1 = +1.0086 \times 10^{-3}$, $R_{0,7} - 1 = -8.2703 \times 10^{-4}$ and $R_{0,9} - 1 = -1.9768 \times 10^{-3}$, changing only in the fifth digit when the truncation is raised from (16, 40) to (24, 48).

7.2 Retiring the fifth-order artifacts

The sub-boundary fifth-order rows behave like truncation error, not like physics. A retirement check on six representative rows at $V_0 = 4$ shows every uniform-weight row moving above the bound as the truncation is raised: 0.2629 to 4.1827 between $(K, N_h) = (12, 20)$ and $(16, 28)$ for one row, and -1.1222 to 8.8631 to 22.4147 across $(12, 20)$, $(16, 28)$ and $(20, 36)$ for the worst row, which is still growing at the deepest truncation and is therefore not converged there either. The check covers representative rows, not all 26.

The exception is a weighted row at $f = 0.2$, $k = 1$, $V_0 = 4$, $w = \cos(2\pi\theta)$, $n = 5$, $\tau = 5$. Its

Table 6: Convergence ladder for the single unresolved fifth-order sweep row ($f = 0.2$, $k = 1$, $V_0 = 4$, $w = \cos 2\pi\theta$, $n = 5$, $\tau = 5$). The ratio stays below one at every rung but does not converge; the last column is the change from the previous rung. This row is excluded from the evidence.

truncation (K, N_h)	Q/Q^0	change from previous rung
(16, 28)	0.988106	–
(20, 36)	0.946539	–0.041567
(24, 44)	0.906087	–0.040452
(28, 48)	0.909995	+0.003908
(32, 56)	0.940079	+0.030084

convergence ladder over five truncations does not settle: the ratio moves $0.9881 \rightarrow 0.9465 \rightarrow 0.9061 \rightarrow 0.9100 \rightarrow 0.9401$ as (K, N_h) runs through (16, 28), (20, 36), (24, 44), (28, 48) and (32, 56) (Table 6). It stays below one at every rung but oscillates rather than converging, so the row is reported as an unresolved floating-point candidate and is used as no evidence whatsoever, in either direction. A converged statement about it would need certified truncation error control, which the present spectral method does not provide.

7.3 A ratchet candidate that disappears

A low-resolution search over random Fourier potentials flagged an asymmetric ratchet with an apparent m/f of about 28, which by (20) would have been an infinitesimal first-order counterexample. Refining the truncation collapses the ratio: the flagged state is unconverged and has a negative kinetic energy, and $|m/f|$ falls to 0.16 or below as (K, N_h) increase, with the stationary residuals of (6) dropping to 10^{-4} – 10^{-5} . An independent Euler–Maruyama estimate for the same potential gives $m/f = 1.1 \times 10^{-5}$: the particle is essentially trapped. No violation of Theorem 3 is supported by either route, as the theorem says it cannot be.

7.4 A search for stationary violations

Beyond the cosine family, 300 random smooth asymmetric periodic potentials were probed, with Fourier amplitudes drawn as $a_j \sim U(0, 2)/j$ and random phases, together with drives $f \sim U(0.05, 2.5)$ and a set of curated asymmetric two- and three-mode cases. Every refined row has $m/f < 1$. The largest refined values, between 0.9914 and 0.9992, occur for weak single-mode potentials, where the ratio approaches one from below exactly as the equality case of Theorem 3 predicts, with stationary residuals of 10^{-5} – 10^{-7} . Strong asymmetric potentials give m/f well below one. The search is finite and non-certified; it widens the empirical support for $m \leq \chi f \leq f$ beyond the cosine family and finds no counterexample, which is what the proof already excludes.

8 A weight-variational form of the infinitesimal bound

For a general smooth positive weight the infinitesimal coefficient of (18) is

$$A_n(w) = \sigma\left(\frac{\langle w^2 u^{2n} \rangle}{\langle w u^n \rangle^2} - 1\right), \quad R_{0,n}(w) = \frac{A_n(w)}{c_n}. \quad (56)$$

Introduce the stationary θ -profiles

$$G_n(\theta) = \int_{\mathbb{R}} u^n \rho(\theta, u) du, \quad H_n(\theta) = \int_{\mathbb{R}} u^{2n} \rho(\theta, u) du, \quad (57)$$

so that $\langle w u^n \rangle = \int_0^1 w G_n d\theta$ and $\langle w^2 u^{2n} \rangle = \int_0^1 w^2 H_n d\theta$.

Proposition 11 (Weight-variational lower bound). *For every smooth positive 1-periodic weight w ,*

$$R_{0,n}(w) \geq \frac{\sigma(1/S_n - 1)}{c_n} =: R_{0,n}^{\text{inf}}, \quad S_n = \int_0^1 \frac{G_n(\theta)^2}{H_n(\theta)} d\theta, \quad (58)$$

with equality if and only if $w \propto G_n/H_n$. The right-hand side is a purely stationary quantity: no weight family has to be guessed, and the worst weight is known explicitly.

Proof. Cauchy–Schwarz in the measure $H_n d\theta$ gives

$$\left(\int_0^1 w G_n d\theta \right)^2 = \left(\int_0^1 (w H_n^{1/2})(G_n/H_n^{1/2}) d\theta \right)^2 \leq \left(\int_0^1 w^2 H_n d\theta \right) S_n,$$

with equality exactly when $w H_n^{1/2} \propto G_n/H_n^{1/2}$, that is, $w \propto G_n/H_n$. Substituting into (56) gives (58). $\square$

At $n = 1$ the bound becomes weight-independent. The stationary θ -current identity $\partial_\theta \int u \rho du = 0$ makes $G_1(\theta) = m$ constant, so $S_1 = m^2 \int_0^1 H_1^{-1} d\theta$ and the right-hand side of (58) is the same for every weight. This is the weight-variational counterpart of Corollary 4: the worst-case weight cannot improve on the homogeneous one at first order. The constancy of G_1 was verified numerically on the archived profiles; the printed columns vary by at most 8.4×10^{-8} in absolute value and 3.4×10^{-3} in relative value, which is the discretisation and printing tolerance of the solve rather than the 10^{-10} level quoted in the exploratory notes.

For $n = 3$ the optimizer does attain the bound. At $V_0 = 0.2$, $k = 1$, $f = 10^{-4}$, direct short-time evaluation of the optimal weight $w = G_3/H_3$ gives $R_{0,3}(w_{\text{opt}}) = 1.054209138$ against $R_{0,3}^{\text{inf}} = 1.054209138$, a difference of 6.8×10^{-14} , while the homogeneous weight gives $R_{0,3}(1) = 1.058810295$. The same evaluation on the one-mode cosine family gives $R_{0,1}^{\text{inf}} = 1.008974702$, 1.131533439 and 2.491589474 at $V_0 = 0.05$, 0.2 and 1.0; the value at $V_0 = 0.05$ is within 10^{-5} of the seventh-order homogeneous endpoint A_7/c_7 discussed in Section 6.1 only in the sense that both are close to one, and the two are different quantities.

Proposition 11 bounds the infinitesimal ratio from below for every weight, and it is sharp. It does not bound the finite-time ratio, and the crossing of Section 6.2 shows why that matters: at $V_0 = 0.2$ the sine weight has $R_{0,7} = 1.1075$, comfortably above the bound, and $R(\tau) < 1$ for $\tau > \tau_*$.

9 Discussion and limitations

The three results of this paper have different strengths, and it is worth stating what each one is.

The refutation for $n \geq 7$ is the strongest and the most complete. The sign of κ_7 is an exact arithmetic fact about the equilibrium resolvent, evaluated independently by two routes and at six modes; the propagation of that sign to every odd order up to 401 rests on a monotonicity theorem whose only numerical content is the positivity of β' , established by a three-region argument with explicit margins. What is missing is interval validation of four constants, and a uniform-in- k argument for modes beyond six. Neither gap affects the statement that the conjecture as written is false: a single mode and a single order suffice for that, and the $k = 1$, $n = 7$ case is the best-resolved one.

The first-order theorem is unconditional but infinitesimal. It gives $m \leq \chi f \leq f$ for every smooth periodic potential, identifies the equality case, and turns (15) at $n = 1$ into a statement that holds in a right-neighbourhood of $\tau = 0$. It says nothing about finite τ , and the finite-time first-order bound remains open. The intermediate quantity χ is the sharper object here: $m \leq \chi f$ is strictly stronger than $m \leq f$ whenever the position marginal is non-uniform, and it is saturated by the local-mean part of the entropy production alone.

The finite-time results are numerical. The homogeneous seventh-order violation out to $\tau = 50$ and the odd-weight crossing at $\tau_* = 0.2321$ are floating-point spectral evaluations, controlled by a self-check against the exact reference, by truncation stability at the 10^{-4} level, and by a coarse Monte Carlo consistency check on a large-signal row. They are not certified bounds. The mechanism they reveal is nevertheless sharp and does not depend on the numerics: an infinitesimal test that passes can still be followed by a finite-time crossing, because the sign of A_n says nothing about B_n or the higher short-time coefficients, and the parity scan shows that the crossing is carried by the parity-odd part of the weight.

Three negative results are part of the record. The orders $n = 3$ and $n = 5$ are open: $\kappa_3 > 0$ and $\kappa_5 > 0$ at every mode examined, and no converged violation was found, so the conjecture is neither proved nor refuted there. The sub-boundary fifth-order sweep rows move above the bound under refinement wherever they were checked, which is what a truncation artifact does, and the one row that does not retire oscillates and is unresolved; the honest summary of the fifth-order sweep is that it decides nothing. The ratchet candidate with $m/f \approx 28$ was an unconverged state with negative kinetic energy, and an independent Monte Carlo estimate puts the same potential at $m/f \approx 10^{-5}$.

Two methodological limitations run through all of the numerics. The spectral method reports resolved quantities without a rigorous bound on the remaining Fourier and Hermite truncation error, so convergence is checked empirically by raising (K, N_h) and comparing. And rows whose current mean is near zero are excluded, because Q_n diverges there and carries no information about the bound; this exclusion removes 161 of the 336 first-order rows and 84 of the 336 rows at each of $n = 3$ and $n = 5$, which is a substantial fraction of the sweep and a reason to treat the sweep minima as indicative rather than exhaustive.

What would settle the remaining questions is a finite-time argument. The infinitesimal analysis reduces to stationary expectations and closes cleanly in both directions; the finite-time problem needs control of the propagator, and the present evidence is that the seventh-order homogeneous violation persists over four decades of τ , while the first-order bound has not been seen to fail anywhere. A proof or a counterexample at finite τ for $n = 1$, and any statement at all for $n = 3$ and $n = 5$, remain open.

10 Conclusions

The conjectured free-diffusion bound on the precision of odd underdamped currents is false. For the one-mode potential family at modes $k = 1, \dots, 6$, the weak-potential coefficient κ_n of the short-time ratio is negative for every odd order from 7 to 401, which reverses the bound in a short-time, weak-potential window at each of those orders. The sign is exact arithmetic; the one numerical ingredient is a monotonicity property of an explicitly known boundary function, certified in three regions with recorded margins. At finite times the seventh-order homogeneous violation persists out to fifty relaxation times, and a positive parity-odd weight crosses the bound at a finite observation time despite passing the infinitesimal test.

At first order the bound holds near zero time, and this is proved: the stationary mean velocity of a driven periodic diffusion satisfies $m \leq \chi f \leq f$, with equality only for a constant potential. The third and fifth orders remain open, and the fifth-order sweep rows that appear to violate the bound are truncation artifacts, one of which is reported as unresolved.

A Exact coefficients of the free-diffusion reference

For $V \equiv 0$ and $w \equiv 1$ the velocity is an Ornstein–Uhlenbeck process with stationary law $\mathcal{N}(f, 1)$ and correlation $\langle \delta u(t) \delta u(0) \rangle = e^{-|t|}$, where $\delta u = u - f$. Expanding u^n in probabilists’ Hermite

polynomials of δu ,

$$u^n = \sum_{\substack{k=0 \\ n-k \text{ even}}}^n c_{n,k} \text{He}_k(\delta u), \quad c_{n,k} = \frac{n!}{2^{(n-k)/2} \left(\frac{n-k}{2}\right)! k!} f^{n-k}, \quad (59)$$

and using $\langle \text{He}_k(\delta u_t) \text{He}_k(\delta u_0) \rangle = k! e^{-k|t|}$, the autocorrelation of (10) becomes a finite sum of exponentials. With $A_k = c_{n,k}^2 k!$ evaluated at the odd-order scaling that survives the ratio, the double time integral gives

$$\text{Var}(Y_n) = 2 \sum_k A_k \left(\frac{\tau}{k} - \frac{1 - e^{-k\tau}}{k^2} \right), \quad \langle Y_n \rangle^2 = \tau^2 \langle u^n \rangle^2, \quad (60)$$

and since $\sigma = f^2$ and $\langle u^n \rangle = n!! f + O(f^3)$ for odd n ,

$$Q_n^0(\tau) = \frac{2}{(n!!)^2} \sum_k A_k \left(\frac{1}{k} - \frac{1 - e^{-k\tau}}{k^2 \tau} \right) = q_n - \frac{s_n}{\tau} + \frac{1}{\tau} \sum_k d_{n,k} e^{-k\tau}, \quad (61)$$

with

$$d_{n,k} = \frac{2A_k}{(n!!)^2 k^2}, \quad s_n = \sum_k d_{n,k}, \quad q_n = \sum_k k d_{n,k}, \quad c_n = \frac{1}{2} \sum_k k^2 d_{n,k} = \frac{(2n-1)!!}{n^2 ((n-2)!!)^2}. \quad (62)$$

Every coefficient is a positive rational number, and the last identity in (62) is exact. The Laplace inversion that turns the resolvent poles at $-k$ into the exponentials of (61) is the standard one [47].

Three identities follow from (62) and hold exactly in rational arithmetic for every odd $n \leq 401$:

$$\sum_k d_{n,k} = s_n, \quad q_n = \sum_k k d_{n,k}, \quad \frac{1}{2} \sum_k k^2 d_{n,k} = c_n. \quad (63)$$

They are the consistency conditions for the three limits of Q_n^0 : the value at $\tau \rightarrow \infty$, the absence of a $1/\tau$ divergence as $\tau \rightarrow 0$, and the short-time slope. The first three orders reproduce the published closed forms,

$$Q_1^0(\tau) = \frac{2}{\tau} (\tau - 1 + e^{-\tau}), \quad (64)$$

$$Q_3^0(\tau) = \frac{2}{\tau} \left(\frac{11}{9} \tau - \frac{29}{27} + e^{-\tau} + \frac{2}{27} e^{-3\tau} \right), \quad (65)$$

$$Q_5^0(\tau) = \frac{2}{\tau} \left(\frac{449}{225} \tau - \frac{4447}{3375} + e^{-\tau} + \frac{8}{27} e^{-3\tau} + \frac{8}{375} e^{-5\tau} \right), \quad (66)$$

and the $n = 7$ curve agrees with an independent integration of the moment hierarchy at $f = 10^{-4}$ to 3.6×10^{-7} in absolute value over 133 grid points, the residual being the expected $O(f^2)$ offset of a finite-drive computation against an equilibrium reference. Table 7 lists the exact coefficients.

B The tail region of the certificate

Region (ii) of Section 5.5 deserves its constants written out. From (45),

$$\beta'(u) = \int_0^\omega \lambda \Pi(\lambda) \sin(\lambda u) d\lambda, \quad (67)$$

and the logarithmic singularity of Π at $\lambda = 0$ prevents a direct integration by parts. Isolating it through the identity $\Pi'' = b - 1/(\omega\lambda)$, where b is continuous on $(0, \omega)$ with $b(0+) = -2/\omega^2$ and

Table 7: Exact rational coefficients of the free-diffusion reference (12) for odd $n \leq 13$, computed in exact rational arithmetic from (62). The three identities (63) were verified exactly for every odd $n \leq 401$; a dash marks an exponential that does not occur at that order.

n	c_n	$q_n = Q_n^0(\infty)$	s_n	d_1	d_3	d_5	d_7	d_9
1	$\frac{1}{5}$	$\frac{2}{22}$	$\frac{2}{58}$	$\frac{2}{2}$	—	—	—	—
3	$\frac{3}{21}$	$\frac{9}{898}$	$\frac{27}{8894}$	$\frac{1}{2}$	$\frac{4}{16}$	—	—	—
5	$\frac{5}{429}$	$\frac{225}{9862}$	$\frac{3375}{480542}$	$\frac{1}{2}$	$\frac{27}{4}$	$\frac{375}{48}$	—	—
7	$\frac{35}{2431}$	$\frac{1225}{1882418}$	$\frac{128625}{194253574}$	$\frac{1}{2}$	$\frac{3}{64}$	$\frac{125}{192}$	$\frac{32}{512}$	—
9	$\frac{63}{4199}$	$\frac{99225}{23930966}$	$\frac{31255875}{20123841326}$	$\frac{1}{2}$	$\frac{27}{100}$	$\frac{125}{64}$	$\frac{1715}{640}$	$\frac{25515}{1280}$
11	$\frac{33}{185725}$	$\frac{480249}{1271834738}$	$\frac{1664062785}{10918374667898}$	$\frac{1}{2}$	$\frac{27}{16}$	$\frac{15}{48}$	$\frac{343}{2560}$	$\frac{5103}{1280}$
13	$\frac{429}{429}$	$\frac{9018009}{9018009}$	$\frac{406216215405}{406216215405}$	$\frac{1}{1}$	$\frac{3}{3}$	$\frac{5}{5}$	$\frac{343}{343}$	$\frac{567}{567}$

$b(\omega) = 1/\omega^2$, and integrating by parts twice with the $\varepsilon \rightarrow 0$ limit at the origin, produces (49). The two terms are then bounded separately. For the oscillatory term, Stieltjes integration by parts against $\sin(\lambda u)$ gives

$$\left| \int_0^\omega b(\lambda) \sin(\lambda u) d\lambda \right| \leq \frac{1}{u} \left(\frac{2}{\omega^2} + \frac{1}{\omega^2} + \text{TV}(b) \right) = \frac{C'_b}{u}, \quad (68)$$

where the endpoint values of b contribute $3/\omega^2$ and the recorded bound is written with $C_b = 3/\omega^2 + \text{TV}(b)$. For the first term, $\text{Si}(t) \geq \pi/2 - 2/t$ for $t > 0$, so $\omega^{-1}\text{Si}(\omega u) \geq \pi/(2\omega) - 2/(\omega^2 u)$. Combining,

$$\beta'(u) \geq \frac{1}{u^2} \left[\frac{\pi}{2\omega} - \frac{2/\omega^2 + C_b}{u} \right], \quad (69)$$

The bracket vanishes exactly at

$$U_0 = \frac{4}{\pi\omega} + \frac{2\omega C_b}{\pi} \quad (70)$$

and increases with u beyond it, so the outer region is covered once the bound is positive at its start. The computations take that start to be U , the value of U_0 rounded up to two decimals (1.66, 1.44, 1.39, 1.36, 1.34 and 1.33 for $k = 1, \dots, 6$), and the middle region is then $[\pi/\omega, U]$. The tail bounds recorded in Table 3 are the values of the right-hand side at $u = U$; they are positive for all six modes, the smallest being 3.95×10^{-5} at $k = 2$. They were reproduced independently from the formula, together with the identity $C_b = 3/\omega^2 + \text{TV}(b)$, the endpoint π/ω of region (i) and the six values of U_0 , to within the printing tolerance of the archived constants (10^{-7} for U_0 and 5×10^{-9} for the tail bounds).

C Numerical protocol

Discretisation. The generator (3) is discretised in complex Fourier modes $e^{2\pi i k \theta}$ with $|k| \leq K$ and in orthonormal Hermite functions with index $0 \leq \ell < N_h$. The stationary density is the right null vector of the forward Fokker–Planck matrix, obtained from a singular value decomposition. For a complex operator the null vector is the conjugate of the last row of the right singular factor; taking that row without conjugation returns the density reflected by $\theta \mapsto -\theta$, which leaves velocity moments unchanged but flips θ -odd expectations and corrupts propagator-resolved variances. The convention was validated on a representative resolved case, $V = 0.5 \cos 2\pi\theta$, $f = 0.7$, $(K, N_h) = (16, 24)$, where the residuals of the two stationary identities (6) are 6×10^{-4} for $\langle u^2 \rangle = 1 + fm$ and 10^{-6} for $\langle V' \rangle = f - m$, against a force residual of 0.56 for the unconjugated row.

Variances. Time-integrated variances come from the resolvent identity (54), evaluated with the Moore–Penrose inverse of the discretised generator and a matrix exponential. Short-time ratios $R_{0,n} = A_n/c_n$ are evaluated separately from stationary expectations alone, without any propagator, which is why they can be compared against the propagator rows as an independent check.

Truncation controls. Sign-defining rows were recomputed at raised truncation. The finite-time seventh-order rows move by about 10^{-4} in absolute value between (16, 28) and (20, 36); the crossing rows move by less than 2×10^{-3} in τ_* ; the large- τ homogeneous rows agree to 10^{-4} ; and the $V_0 = 0.05$ sine-weight rows at $\tau = 20, 50$ agree to 2×10^{-7} . The $V \equiv 0$ self-check against the exact reference (12) is the tightest control available and is reported at 5×10^{-7} to 7×10^{-6} relative depending on the window.

High-precision arithmetic. The exact recurrence of Section 5.2 is run at 110-digit working precision with the minimal-solution boundary constant taken from a backward continued fraction. The rational reference coefficients of Appendix A are computed in exact integer arithmetic. The certificate constants of Table 3 are computed in double precision for $k \leq 4$ and in 50-digit arithmetic for $k = 5, 6$, where the factor $e^{\omega^2 y}$ in (45) overflows double precision; all quadratures were validated by grid refinement.

Monte Carlo. The independent trajectory check uses Euler–Maruyama integration of (2) with step 10^{-3} , a burn-in of 20 relaxation times and 1.5×10^5 blocks of length $\tau = 1$ over five seeds, on a large-signal row where $\langle u^7 \rangle \approx 62$. Its role is consistency, not validation: at the small drives where the violation lives the signal-to-noise ratio of a seventh-order current makes a fixed-precision comparison infeasible.

Data and code availability

The archived computations and result tables underlying every number quoted in this paper, including the spectral solver, the exact reference and recurrence evaluations, the certificate quadratures and the sweep and finite-time tables, are available in the source repository [11].

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