

Finite-size and boundary-condition corrections to full counting statistics in the open XY chain

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Abstract

Does the way a finite quantum chain is closed change what a counting experiment on it measures, and by how much? We answer this for the fermion number of the anisotropic XY chain in a transverse field, treating the open chain exactly at every finite length instead of solving it with the periodic Fourier reduction that the counting literature adopts for convenience. Every protocol we study — the total number, the number in a subregion, the charge transferred across a cut, the same quantities at finite temperature, the joint outcome of two projective number measurements, and the counts registered by a detector of finite efficiency — is reduced to a determinant built from the Bogoliubov covariance matrix of the chain, and each reduction is checked against exact diagonalisation of the full many-body Hilbert space on small chains. Open and periodic boundaries agree in the bulk and differ by a boundary term that grows without bound at the critical field: over chain lengths from 64 to 1024 the open-minus-periodic mean and variance fall as logarithms of the length whose fitted coefficients reproduce the values expected from the critical mode sums to within a few parts in a hundred thousand, while any fixed nonzero temperature rounds both logarithms into terms that stay bounded with length. Two further results concern measurement rather than boundaries: an imperfect detector described by independent clicks and the same detector described by a rescaled counting operator differ in variance by an exactly predictable binomial term, and the distribution of transferred charge obtained from two projective measurements differs from the operator-level current statistics because the first measurement changes the state it subsequently probes. We do not prove the logarithmic coefficients analytically, do not treat interactions beyond the quadratic Hamiltonian, and do not claim the finite-temperature partial-number projective protocol; each boundary is stated where it arises.

1 Introduction

Counting particles is the oldest measurement in quantum many-body physics and, for a correlated ground state, one of the most informative: the distribution of the number found in a region, not merely its mean, distinguishes a product state from a critical one. Cold-atom experiments made the question concrete. A quantum-non-demolition readout of an optical lattice registers how many atoms sit on the illuminated sites, with a known probability per atom, and the resulting histogram is the counting distribution of the many-body state [1]. For the family of models that are quadratic in fermion operators — the anisotropic XY chain in a transverse field, equivalently the transverse-field Ising model through the Jordan–Wigner map — the whole distribution collapses to a determinant

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of a matrix built from the Bogoliubov covariance of the state [2]. That collapse is exact at every finite chain length, which makes this family the place where a counting question can be answered without approximation.

The counting literature nevertheless solves the model with periodic boundary conditions. The reason is technical and is stated openly in the paper that established the fermion-counting results for this family: the Jordan–Wigner transformation is a map for open chains, but the Fourier and Bogoliubov transformations that diagonalise it require a closed chain, so periodic boundary conditions are imposed and the procedure is justified by its leading behaviour at large length [1]. Nothing in that argument is wrong, and for extensive quantities it is the right thing to do. It leaves open the question this paper asks: at the finite lengths an experiment or a numerical study actually uses, what does the boundary condition cost, and does the cost ever compete with the signal?

The answer turns out to be quantitative and, at criticality, unbounded. We evaluate the counting statistics of the *open* chain exactly, at every length from a few sites to a thousand, and compare them term by term with the periodic reference of [1]. The bulk densities agree, as they must. The boundary corrections do not vanish: at the critical field the open-minus-periodic difference of the mean number and of its variance both grow logarithmically with the chain length, and the fitted logarithmic coefficients match the values $-1/(2\pi)$ and $-1/\pi^2$ to a few parts in a hundred thousand over the range $N = 64$ to 1024 . Any fixed nonzero temperature rounds both logarithms into terms that stay bounded in N . The mechanism is visible in the spectrum: the periodic chain at criticality carries an exact zero mode, so its many-body gap is zero at every length, while the open chain’s gap closes only as $\pi/(2N)$. A finite-temperature state cannot resolve a splitting that small once β^{-1} exceeds it, which is why the logarithm disappears at any fixed temperature.

Three further questions belong to the same finite- N story and are treated here with the same tool. First, *counting in time*: the joint distribution of two projective number measurements separated by an interval t is a different object from the statistics of the operator $\hat{N}(t) - \hat{N}(0)$, because the first measurement changes the state the second one probes. We compute both, and the transferred charge they imply, and show where and why they differ. Second, *imperfect detection*: a detector that clicks independently on each atom with probability κ thins the counting distribution binomially, whereas rescaling the counting operator by κ does not; the two conventions differ in variance by exactly $\kappa(1 - \kappa)\langle\hat{N}\rangle$, a term that is extensive at criticality and therefore not a small correction. Third, *cutting the chain*: the variance of the number in one half is smaller than half the variance of the total, and at criticality the deficit grows logarithmically with N while at any fixed temperature it is bounded.

Every number in this paper comes from one of two routes, and the distinction is kept visible throughout. The *determinant route* evaluates the Gaussian formulas of [2] for the open chain; it is the method, and it scales to the lengths of interest. The *exact route* diagonalises the full 2^N many-body Hilbert space and evaluates the same quantity as a trace; in this work it reaches $N = 8$ and makes no assumption about the Gaussian reduction. Each formula we use is checked by the exact route before it is used, and each recorded table is reproduced by an independent implementation of the covariance algebra. Section 4 states the coverage and the tolerances; Appendix A tabulates them. The checks found four places where the numerics fail or leave a choice undetermined, and all four are reported rather than hidden: the extraction of a total-number projective distribution from a determinant square root loses its sign resolution at $N = 8$, so the recorded grids violate fermion-parity conservation — 66% and 99% of their mass sits on forbidden pairs, they differ from exact Kraus traces by up to 0.38 per cell, and they are not even reproducible from the code shipped with them — and we regenerate those figures with a branch-tracked quadrature that is exact to machine precision for the ground state; one recorded row of the projective-versus-operator comparison carries a discretisation error of 3.6×10^{-2} in its third cumulant and is replaced by

exact values; the site-resolved mean profile at a subcritical field is ambiguous in a ground-state doublet split by less than 10^{-12} ; and at the periodic critical point the determinant route returns the counting data of no state at all. None of them affects any of the headline results; each is quantified where it arises.

The paper is organised as follows. Section 2 fixes the model, the counting observables and the detector conventions. Section 3 derives the determinant evaluation for each protocol, including the finite temperature, subregion, two-time and projective cases, and states the numerical protocol. Section 4 reports the independent validation. Sections 5–7 contain the results: boundary conditions, finite size and temperature; counting in time; imperfect detection. Section 8 collects the interpretation, the limits of the evidence and the open directions, and Section 9 restates the claims with their quantifiers. Appendices A–C carry the validation coverage, the derivations whose results the body quotes, and the analysis of zero modes, parity and branch choices.

The implementation, the recorded evidence tables and the parameters behind every figure are those of the source repository [3]; the independent checks and the regenerated figures were produced for this manuscript.

2 Model and counting observables

2.1 The chain, its boundary conditions and its phases

The system is a chain of N sites, $j = 1, \dots, N$, carrying spinless fermions $c_j, c_j^\dagger$ with $\{c_j, c_l^\dagger\} = \delta_{jl}$ and $\hat{n}_j = c_j^\dagger c_j$. The Hamiltonian is the anisotropic XY chain in a longitudinal fermion field, written in the convention of [1],

$$H = -\frac{J}{2} \sum_{\langle j, j+1 \rangle} \left[c_j^\dagger c_{j+1} + \gamma c_j^\dagger c_{j+1}^\dagger + \text{h.c.} \right] + Jg \sum_{j=1}^N \hat{n}_j + \frac{JgN}{2}, \quad (1)$$

where $J > 0$ is the tunnelling energy, $\gamma \in [0, 1]$ the pairing anisotropy and g the field in units of J . The sum $\langle j, j+1 \rangle$ runs over the $N-1$ nearest-neighbour bonds of the *open* chain; the *periodic* chain adds the wrap bond $(N, 1)$. The additive constant $JgN/2$ commutes with everything, including the counting observables below, so it drops out of every spectrum, every Gibbs state and every Heisenberg-evolved operator; it is kept only because it makes the Jordan–Wigner image of (1) a spin chain with no residual field term,

$$H = -J \sum_{\langle j, j+1 \rangle} \left[(1+\gamma) S_j^x S_{j+1}^x + (1-\gamma) S_j^y S_{j+1}^y \right] + Jg \sum_j S_j^z + JgN, \quad S_j^z = \hat{n}_j - \frac{1}{2}. \quad (2)$$

Throughout, $J = 1$ sets the unit of energy, so times are in units of $\hbar/J$ and inverse temperatures β in units of $\hbar/J$ as well.

The only structural difference between the two boundary conditions that matters here is the following. On the periodic chain the translation operator commutes with H , the single-particle problem is diagonalised by the Fourier modes $\Phi_k = 2\pi k/N$, and the Bogoliubov quasiparticle energy is

$$E_k = \sqrt{\xi_k^2 + \gamma^2 \sin^2 \Phi_k}, \quad \xi_k = g - \cos \Phi_k, \quad (3)$$

which vanishes at $\Phi_k = 0$ exactly when $g = 1$ (for $\gamma > 0$). On the open chain translation is broken, no mode label survives, and the same quantity is obtained only as the spectrum of a $2N \times 2N$ matrix (Section 3.1). The critical field is therefore $g_c = 1$ for both boundary conditions, but the way

the gap closes with N is different, and that difference is the subject of Section 5. The line $\gamma = 0$ is special: the pairing vanishes, the number operator commutes with H , and for $|g| \leq 1$ the periodic ground manifold is degenerate. That case is treated separately in Appendix C and excluded from the critical statements, which all assume $\gamma > 0$.

2.2 What is counted

Five observables are counted in this paper. Each is diagonal in the occupation basis, which is what makes a projective measurement of it well defined and what makes its statistics accessible from a covariance matrix.

Definition 2.1 (counted observables). Let ρ be either the ground state of (1) or the Gibbs state $\rho_\beta = e^{-\beta H} / \text{Tr } e^{-\beta H}$, and let $U(t) = e^{-iHt}$.

- (i) The *total number* $\hat{N} = \sum_{j=1}^N \hat{n}_j$, with spectrum $\{0, \dots, N\}$ and projectors P_n onto $\hat{N} = n$. Counting $\hat{N}$ is the same experiment as counting the total z -magnetisation $\sum_j S_j^z = \hat{N} - N/2$, shifted by a known constant.
- (ii) The *partial number* $\hat{N}_A = \sum_{j \in A} \hat{n}_j$ in a subregion $A \subseteq \{1, \dots, N\}$, taken to be the left half $A = \{1, \dots, \lfloor N/2 \rfloor\}$ wherever a specific region is needed.
- (iii) The *weighted number* $\hat{W} = \sum_j w_j \hat{n}_j$ with real weights w_j , of which (i) is $w_j \equiv 1$ and (ii) is $w_j = \nu_j$, the indicator of A . A standing-wave readout of the kind discussed in [1] corresponds to $w_j \propto \cos(k_L r_j)$.
- (iv) The *transferred charge* across the boundary of A after a time t ,

$$\hat{Q}(t) = \hat{N}_A(t) - \hat{N}_A(0), \quad \hat{N}_A(t) = U^\dagger(t) \hat{N}_A U(t), \quad (4)$$

with spectrum $\{-|A|, \dots, |A|\}$. This is the integrated current through the cut, defined as an operator difference rather than as a time integral of a current operator; the two agree because $\dot{\hat{N}}_A(t) = i[H, \hat{N}_A(t)]$ integrates to (4).

- (v) The *detected count* m registered by a detector of efficiency $\kappa \in [0, 1]$, defined in Section 2.3.

For any of these, with eigenvalues q and projectors Π_q , the object computed is the full counting statistics: the probability distribution $p(q) = \text{Tr}[\rho \Pi_q]$, its generating function and cumulant generator

$$\chi(z) = \langle z^{\hat{q}} \rangle = \sum_q p(q) z^q, \quad \mathcal{K}(\lambda) = \ln \chi(e^\lambda) = \sum_{j \geq 1} \kappa_j \frac{\lambda^j}{j!}, \quad (5)$$

and the cumulants $\kappa_j = \partial_\lambda^j \mathcal{K}(0)$. Writing κ for the detector efficiency and κ_j for cumulants is an unavoidable collision; the two never appear in the same formula, and where both appear in the same section the cumulant carries the subscript j and the efficiency never does. The counting field λ is real for the moment expansion and $\lambda = i\theta$ on the unit circle for the distribution itself, $z = e^{i\theta}$; all our generators are evaluated for real λ and continued to the unit circle only in the inversion step, where the square-root branch has to be tracked (Section 3.4).

Two-time counting needs one more piece of structure. A projective measurement of $\hat{N}$ at time 0, followed by unitary evolution and a second projective measurement at time t , has the joint distribution

$$p(n_1, n_2) = \text{Tr} \left[P_{n_1} U(t) P_{n_2} \rho P_{n_2} U^\dagger(t) \right], \quad \sum_{n_1, n_2} p(n_1, n_2) = 1, \quad (6)$$

where n_2 is the *earlier* outcome and n_1 the later one; this ordering is used consistently in every table and figure. Its characteristic function is $F(\theta_1, \theta_2) = \sum_{n_1, n_2} p(n_1, n_2) e^{i(\theta_1 n_1 + \theta_2 n_2)}$. Distinct from (6) is the *operator-level* two-time generator, which keeps the state intact,

$$Z(\lambda_1, \lambda_2; t_1, t_2) = \left\langle e^{\lambda_1 \hat{N}(t_1)} e^{\lambda_2 \hat{N}(t_2)} \right\rangle, \quad (7)$$

and the connected number correlator obtained from it,

$$C(t_1, t_2) = \langle \hat{N}(t_1) \hat{N}(t_2) \rangle - \langle \hat{N}(t_1) \rangle \langle \hat{N}(t_2) \rangle = \partial_{\lambda_1} \partial_{\lambda_2} \ln Z|_{\lambda_1 = \lambda_2 = 0}. \quad (8)$$

Because the exponentials in (7) do not commute with the projectors in (6), the two are different measurements, and Section 6 quantifies the difference. Both reduce correctly at equal times: $Z(\lambda, 0; t, t) = \chi(e^\lambda)$ for the state evolved to t , and $C(t, t) = \text{Var}(\hat{N})$, which is one of the checks in Section 4.

2.3 Two detector conventions and the efficiency κ

A real detector does not register every particle. The model used here is the one of [1]: the readout is a quantum-non-demolition scattering process, and the operator coupled to the detector is the rescaled number operator $I = \kappa \sum_j c_j^\dagger c_j$, with

$$\kappa = 1 - e^{-\zeta t}, \quad (9)$$

where ζ is a scattering rate per occupied site and t the exposure time. Equation (9) is the saturation of a two-level scatterer driven far from resonance: in the limit of large detuning Δ compared with both the natural linewidth Γ and the Rabi frequency Ω , the scattering rate is $r_{\text{sc}} = \Gamma \Omega^2 / (4\Delta^2) = \Gamma^3 I_{\text{probe}} / (8I_{\text{sat}} \Delta^2)$, so $\zeta = r_{\text{sc}}$ and κ is the probability that at least one photon was scattered during the exposure. We use (9) as the definition of the efficiency and do not re-derive the scattering rate; what the paper establishes is the counting statistics at fixed κ .

Two conventions turn κ into a prediction for the measured variance, and they are not the same.

Independent clicks. Each of the n present particles is detected with probability κ , independently of the others, so the detected count is a binomial thinning of the true count,

$$p_{\text{miss}}(m) = \sum_{n \geq m} p(n) \binom{n}{m} \kappa^m (1 - \kappa)^{n-m}, \quad \chi_{\text{miss}}(z) = \chi(1 - \kappa + \kappa z). \quad (10)$$

Rescaled operator. The counting operator itself is replaced by $\kappa \hat{N}$ and its full statistics are taken, which is the map $\chi_{\text{scale}}(z) = \chi(z^\kappa)$ and rescales the j th cumulant by κ^j .

Both give the same mean, $\kappa \langle \hat{N} \rangle$, and differ in the variance by a term that is fixed by (10) alone:

$$\text{Var}_{\text{miss}} = \kappa^2 \text{Var}(\hat{N}) + \kappa(1 - \kappa) \langle \hat{N} \rangle, \quad \text{Var}_{\text{miss}} - \text{Var}_{\text{scale}} = \kappa(1 - \kappa) \langle \hat{N} \rangle. \quad (11)$$

The second term in Var_{miss} is the shot noise of the detection process itself. It is not a small correction at criticality, where $\langle \hat{N} \rangle \propto N$: at $\kappa = 0.9$ and $N = 128$ it amounts to 2.01 out of a variance of 27.37 (Table 5). Which convention applies is a property of the apparatus, not a matter of taste, and Section 7 reports both.

3 Gaussian evaluation of the counting statistics

Every observable in Definition 2.1 is a quadratic form in the Nambu spinor, and every state we use is Gaussian. The whole of the computation is therefore an application of the trace formula for products of exponentials of fermion bilinears [2], specialised to this model. This section states the resulting formulas in a single basis and notation, gives the reduction that makes them cheap, and isolates the one step — the square root of a determinant — where a numerical implementation can go wrong. Proofs of the algebraic identities quoted here are in Appendix B.

3.1 Nambu representation and covariance matrices

Write $\alpha = (c_1, \dots, c_N, c_1^\dagger, \dots, c_N^\dagger)^T$, so that $\{\alpha_a, \alpha_b^\dagger\} = \delta_{ab}$. The Hamiltonian (1) is $H = \frac{1}{2}\alpha^\dagger W \alpha + \frac{1}{2} \text{Tr } M + JgN/2$ with the real symmetric $2N \times 2N$ BdG matrix

$$W = \begin{pmatrix} M & d \\ -d & -M \end{pmatrix}, \quad M_{jl} = g\delta_{jl} - \frac{1}{2}[\delta_{l,j+1} + \delta_{j,l+1}]_{\text{bonds}}, \quad d_{j,j+1} = -\frac{\gamma}{2}, \quad d_{j+1,j} = +\frac{\gamma}{2}, \quad (12)$$

where the bond terms are present for the $N - 1$ bonds of the open chain and additionally for $(N, 1)$ on the periodic chain. Particle-hole symmetry makes the spectrum of W symmetric, $\{E_k, -E_k\}$, with $E_k \geq 0$.

Let U_- be the $2N \times N$ matrix whose columns are eigenvectors of W for its N negative eigenvalues, ties at $E_k = 0$ broken by the prescription of Appendix C, and define the *covariance matrix*

$$\Gamma = U_- U_-^\dagger \quad (\text{ground state}), \quad \Gamma_\beta = (I + e^{\beta W})^{-1} \quad (\text{Gibbs state at inverse temperature } \beta). \quad (13)$$

Both are Hermitian contractions with $0 \preceq \Gamma \preceq I$; the ground-state one is a rank- N projector and the thermal one is a function of W . The physical contractions are recovered from the two off-diagonal blocks,

$$\langle c_i^\dagger c_j \rangle = \delta_{ij} - \Gamma_{N+i, N+j}, \quad \langle c_i c_j \rangle = -\Gamma_{i, N+j}, \quad (14)$$

and all higher correlations follow from Wick's theorem. The three matrices that represent the counting observables in Nambu space are

$$\sigma_z = \begin{pmatrix} I_N & 0 \\ 0 & -I_N \end{pmatrix}, \quad \sigma_z^A = \begin{pmatrix} \nu & 0 \\ 0 & -\nu \end{pmatrix}, \quad X_w = \begin{pmatrix} w & 0 \\ 0 & -w \end{pmatrix}, \quad (15)$$

with $\nu = \text{diag}(\nu_1, \dots, \nu_N)$ the indicator of the subregion A and $w = \text{diag}(w_1, \dots, w_N)$ arbitrary real site weights. They correspond to the operator identities

$$\hat{N} = \frac{N}{2} + \frac{1}{2}\alpha^\dagger \sigma_z \alpha, \quad \hat{N}_A = \frac{|A|}{2} + \frac{1}{2}\alpha^\dagger \sigma_z^A \alpha, \quad \hat{W} = \frac{1}{2} \sum_j w_j + \frac{1}{2}\alpha^\dagger X_w \alpha. \quad (16)$$

Heisenberg evolution of a one-body matrix is generated by

$$R(t) = e^{-iWt}, \quad S(t) = R^\dagger(t) \sigma_z R(t), \quad S_A(t) = R^\dagger(t) \sigma_z^A R(t), \quad (17)$$

which is the single-particle image of $\hat{N}(t) = U^\dagger(t) \hat{N} U(t)$ and $\hat{N}_A(t)$; the constants $N/2$ and $|A|/2$ are unchanged because R is unitary.

3.2 One-time generators

Proposition 3.1 (ground-state and thermal number generator). *For real λ and Γ either of (13),*

$$\chi(\lambda) \equiv \langle e^{\lambda \hat{N}} \rangle = e^{\lambda N/2} \det^{1/2} [I - \Gamma + \Gamma e^{\lambda \sigma_z}], \quad \mathcal{K}(\lambda) = \frac{\lambda N}{2} + \frac{1}{2} \ln \det[\cdots], \quad (18)$$

with the positive square root, i.e. the branch that equals 1 at $\lambda = 0$. The same formula with $\sigma_z \rightarrow X_w$ and $N/2 \rightarrow \frac{1}{2} \sum_j w_j$ gives the generator of the weighted number $\hat{W}$; with $w = \nu$ it gives the partial number $\hat{N}_A$.

For real λ the determinant in (18) is positive, so the branch is unambiguous [2]; this is the case that produces all cumulants. The branch matters only when λ is continued to the unit circle, which is what the distribution requires (Section 3.6).

The determinant is $2N \times 2N$. For the ground state it collapses to $N \times N$.

Lemma 3.2 ($N \times N$ reduction). *With $\Gamma = U_- U_-^\dagger$ and $D = e^{\lambda X_w}$,*

$$\det_{2N} [I - \Gamma + \Gamma D] = \det_N [I_N - U_-^\dagger (I - D) U_-] = \det_N \left[I_N - \sum_{j=1}^N u_j^\dagger (I_2 - D_j) u_j \right], \quad (19)$$

where u_j is the $2 \times N$ matrix formed of rows j and $N + j$ of U_- and D_j is the corresponding 2×2 diagonal block of D . For $w = \nu$ or $w \equiv 1$, $D_j = \text{diag}(e^{\lambda w_j}, e^{-\lambda w_j})$.

The first equality is Sylvester's determinant identity; the second uses the fact that $I - D$ is block diagonal in the site pairs $(j, N + j)$. Lemma 3.2 halves the dimension of every determinant and makes the $N = 1024$ evaluations of Section 5 cheap. The thermal covariance is not a projector, so (18) stays $2N \times 2N$ there: one eigendecomposition of the $2N \times 2N$ BdG matrix per parameter set and one $2N \times 2N$ determinant per counting field, 256×256 at $N = 128$, the largest thermal case in the recorded tables.

Cumulants come from $\mathcal{K}$ by differentiation. Two routes are used and both are reported where they differ in accuracy: analytic Wick reduction for the first two cumulants,

$$\langle \hat{N} \rangle = \text{Tr } \hat{G}, \quad \text{Var}(\hat{N}) = \text{Tr } \hat{G} - \text{Tr } \hat{G}^2 + \|F\|_F^2, \quad \hat{G}_{ij} = \delta_{ij} - \Gamma_{N+i, N+j}, \quad F_{ij} = -\Gamma_{i, N+j}, \quad (20)$$

with $\hat{G}$ and F restricted to $A \times A$ blocks for the partial number; and centred fourth-order finite differences of $\mathcal{K}$ in λ with step $h = 0.01$ for the third and fourth cumulants. The two agree to the accuracy of the finite-difference step, which is the origin of the 10^{-7} offsets quoted in Appendix A.

3.3 Operator-level two-time generator and integrated current

Proposition 3.3 (two-time generator and correlator). *For real λ_1, λ_2 ,*

$$Z(\lambda_1, \lambda_2; t_1, t_2) = e^{(\lambda_1 + \lambda_2)N/2} \det^{1/2} [I - \Gamma + \Gamma e^{\lambda_1 S(t_1) + \lambda_2 S(t_2)}], \quad (21)$$

with $S(t)$ from (17) and Γ either covariance of (13). The connected correlator $C(t_1, t_2)$ of (8) is the mixed second derivative of $\ln Z$ at $\lambda_1 = \lambda_2 = 0$.

Two limits are immediate and are used as checks: $Z(\lambda, 0; t, t) = \chi_t(\lambda)$, the one-time generator of the state evolved to t (equal to the stationary one for the ground and Gibbs states), and $C(t, t) = \text{Var}(\hat{N})$. Because $e^{\lambda_1 S(t_1) + \lambda_2 S(t_2)}$ is the exponential of a sum rather than a product, (21)

is *not* the characteristic function of a sequential measurement; it keeps the state intact and is the operator-level object. It is symmetric under $(\lambda_1, t_1) \leftrightarrow (\lambda_2, t_2)$ only after the trace is taken, and C is symmetric in its two arguments for any state with real covariance, which the recorded tables verify to machine precision.

Proposition 3.4 (integrated current). *For the transferred charge $\hat{Q}(t)$ of (4) and real λ ,*

$$\chi_{\hat{Q}}(\lambda) = \det^{1/2}[I - \Gamma + \Gamma e^{\lambda \Delta_{\hat{Q}}}], \quad \Delta_{\hat{Q}} = S_A(t) - \sigma_z^A, \quad (22)$$

with no exponential prefactor, because the constants $|A|/2$ cancel in the difference.

For a stationary state $\langle \hat{Q}(t) \rangle = 0$ identically: the thermal and ground states commute with H , so $\langle \hat{N}_A(t) \rangle = \langle \hat{N}_A(0) \rangle$. The variance of $\hat{Q}$ is therefore its second cumulant and grows from zero as t^2 at short times.

3.4 Projective two-measurement distributions

The joint distribution (6) is not of the form (21), and reducing it to a determinant needs one extra step.

Proposition 3.5 (twirl reduction). *For any state ρ and any $U = U(t)$,*

$$\begin{aligned} F(\theta_1, \theta_2) &= \int_0^{2\pi} \frac{d\phi}{2\pi} \left\langle e^{-i\phi \hat{N}} U^\dagger e^{i\theta_1 \hat{N}} U e^{i(\phi+\theta_2) \hat{N}} \right\rangle \\ &= \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i(\theta_1+\theta_2)N/2} \det^{1/2} \left[I - \Gamma + \Gamma e^{-i\phi \sigma_z} R^\dagger e^{i\theta_1 \sigma_z} R e^{i(\phi+\theta_2) \sigma_z} \right], \end{aligned} \quad (23)$$

and $p(n_1, n_2)$ is the two-dimensional Fourier coefficient of F .

The first equality is the identity $\sum_n P_n X P_n = \int (d\phi/2\pi) e^{i\phi \hat{N}} X e^{-i\phi \hat{N}}$ applied to the dephasing that the intermediate measurement performs on ρ ; it holds for any ρ , not only for number-diagonal ones, and this is the step that makes the projective protocol computable at all. The second equality is Proposition 3.1 applied to a product of four exponentials of quadratic forms [2].

Two facts about (23) organise everything that follows.

First, *parity*. Since H conserves fermion parity $\Pi = (-1)^{\hat{N}}$, the unitary U commutes with Π , and $p(n_1, n_2) = 0$ unless $n_1 - n_2$ is even. This is an exact selection rule for the *total* number and it is the sharpest available diagnostic of a numerically extracted distribution: any weight on $n_1 - n_2$ odd is error, not signal. For a *partial* number $\hat{N}_A$ no such rule holds, because $\hat{N}_A$ does not commute with Π ; the subregion distribution has support on every transferred charge q , and that is a physical difference between counting the whole chain and counting part of it (Section 6.2).

Second, *the square root*. The determinant in (23) is the square of an analytic function of $(e^{i\phi}, e^{i\theta_1}, e^{i\theta_2})$, but a numerical square root chooses a sign at every grid point independently. Unlike the real-axis case of Proposition 3.1, positivity no longer pins the branch: F is complex. The sign ambiguity of determinant formulas for fermionic traces and the Pfaffian alternative that avoids it are exactly the subject of [2], and they are the only place in this paper where the numerical method can fail while the formula is right.

Two extraction routes are used, and their accuracy is measured in Section 4.

Protocol 3.6 (branch-tracked twirl quadrature). Evaluate the integrand determinant on a grid of L values of θ_1 , L of θ_2 and M_ϕ values of ϕ . For each (θ_1, θ_2) walk the ϕ circle from $\phi = 0$, taking

at each step the square root of the ratio to the previous determinant that lies in the right half plane, so that the branch is continued along the circle; average the tracked values and multiply by $\sqrt{D(\phi=0)}$ and by the prefactor. The remaining global sign of each (θ_1, θ_2) entry is fixed by continuity from its neighbours and, where needed, against the exactly known row $\theta_1 = 0$, whose characteristic function is that of the one-time distribution. Extract $p(n_1, n_2)$ by a two-dimensional discrete Fourier transform on the $L \times L$ grid, keep the $(N+1) \times (N+1)$ physical block, clip negative values and renormalise.

The alternative route, used for the recorded total-number grids, expands the integrand determinant as a Laurent polynomial $D(z) = f(z)^2$ in $z = e^{i\theta}$, recovers the coefficients of f by the recursion implied by squaring, and fixes its overall sign from the known parity of N . That recursion is exact in principle and loses precision in practice: it involves differences of large coefficients, and at $N = 8$ the resulting distribution is wrong by up to 10^{-2} in a single entry. Protocol 3.6, which never subtracts large numbers, is exact to machine precision for the ground state at the same N (Appendix A).

3.5 Detector thinning as a composition law

Thinning is a linear map on distributions and therefore a substitution on generators. For the one-time case, (10) gives

$$\mathcal{K}_{\text{miss}}(\lambda) = \mathcal{K}(\ln(1 - \kappa + \kappa e^\lambda)), \quad (24)$$

and the cumulants follow by composition; the first two are the identity (11). For the two-time projective distribution, thinning at the two measurement times is a convolution with independent binomials,

$$p_{\text{miss}}(m_1, m_2) = \sum_{n_1 \geq m_1} \sum_{n_2 \geq m_2} p(n_1, n_2) \binom{n_1}{m_1} \kappa_1^{m_1} (1 - \kappa_1)^{n_1 - m_1} \binom{n_2}{m_2} \kappa_2^{m_2} (1 - \kappa_2)^{n_2 - m_2}, \quad (25)$$

i.e. $p_{\text{miss}} = B_1^T p B_2$ with B_a the binomial matrix of κ_a . The same map applies to the partial-number distribution and hence to the measured transferred charge. Equation (25) is exact — no approximation is involved — and the thinning of the recorded $N = 64$ distributions was verified against it entrywise.

3.6 Numerical protocol

The parameters behind every number in this paper are collected in Table 7 of Appendix A; the essentials are these. Eigendecompositions of W are dense symmetric ones, and U_- is taken as the N lowest eigenvalues, which is well defined even at a zero mode because exactly N eigenvalues are non-positive. Distributions are obtained by a discrete Fourier inversion on L points of the unit circle with $L \geq 2N + 2$ for two-time quantities and $L \geq 4N + 4$ (rounded up to a power of two) for one-time ones, with the analytic branch tracked as in Protocol 3.6; the twirl integral uses $M_\phi = 2048$ points for the total number and $M_\phi = 1024$ for the partial one. Cumulants beyond the second use centred fourth-order finite differences with $h = 0.01$, and the resulting truncation error, not the determinant, dominates their accuracy.

4 Independent validation

The determinant route of Section 3 is exact mathematics, but it is only worth as much as its implementation. Two independent routes are therefore run in parallel throughout, and every formula is checked before it is used to produce a number that appears in this paper.

The *exact route* builds the full 2^N many-body Hamiltonian from the Jordan–Wigner fermions as explicit matrices, diagonalises it, forms the ground or Gibbs state as a density matrix, and evaluates the counting quantity as a trace: $\chi = \text{Tr}[\rho e^{\lambda \hat{N}}]$, $p(n_1, n_2) = \text{Tr}[P_{n_1} U P_{n_2} \rho P_{n_2} U^\dagger]$, $\chi_{\hat{Q}} = \text{Tr}[\rho e^{\lambda \hat{Q}(t)}]$ with $\hat{Q}(t)$ formed from the exact Heisenberg-evolved occupation operators. It makes no use of Gaussianity, of the BdG matrix or of any determinant, and it is limited to $N \leq 8$ by memory. The *covariance route* recomputes means and variances from the Wick reduction (20) using an independently assembled BdG matrix, and is used to check the recorded tables at large N where the exact route cannot reach. Where the periodic chain is involved, a third route is added: the BCS mode product $\prod_k [(1 - v_k^2) + v_k^2 z^2]$ of [1], evaluated from (3) with the self-paired modes handled explicitly.

Table 1 summarises the outcome; Appendix A lists the coverage of each recorded table row by row. Three statements cover the whole manuscript.

- (V1) *Every formula of Section 3 reproduces the exact trace.* Checked separately against the implementation of [3] and against the formulas as written here: the ground-state generator agrees with exact 2^N traces to 8.0×10^{-15} for the open chain ($N = 2$ to 6) and 5.3×10^{-15} for the periodic chain away from the critical zero mode ($N = 3$ to 6, $g \in \{0.8, 1.2\}$); the thermal generator to 4.4×10^{-15} ($N = 2$ to 5, $\beta \in \{0.6, 2\}$); the two-time generator to 1.7×10^{-15} ($N = 2$ to 4); the integrated-current generator to 4.3×10^{-16} ($N = 3, 4$); and the projective joint distribution to 1.5×10^{-9} in absolute value ($N = 2$ to 4). The formulas in the Nambu form used in this paper — one-time at zero and finite temperature over 96 evaluations with both boundary conditions, $\gamma \in \{0.6, 1\}$, $g \in \{0.8, 1, 1.2\}$, $\beta \in \{1.3, \infty\}$ and counting fields of both signs; the weighted generator; the $N \times N$ reduction; the two-time generator; the current generator; and the twirl integrand of Proposition 3.5, the last up to the sign of its square root — all agree with exact traces to 3.5×10^{-15} or better. The recorded two-time generator table reproduces the exact traces to 9.7×10^{-15} over 24 of its 32 rows, the remaining 8 being $N = 8$ rows whose exact evaluation is the most expensive and which were spot-checked instead.
- (V2) *Every recorded table quoted in this paper is reproduced by the independent covariance route.* The critical open-versus-periodic table ($N = 64$ to 1024) to 2.3×10^{-7} , the thermal table (48 rows) to 2.2×10^{-8} , the gap table (14 rows) to 1.6×10^{-15} , the half-chain table (21 rows) to 5.1×10^{-8} , the three $N = 64$ distribution files in normalisation, parity support, mean and variance to 10^{-12} and in the thinning relation to 4×10^{-17} , the detector conventions to 2×10^{-6} , and the integrated-current rows to 2×10^{-9} . All of these offsets are at the level of the fourth-order finite-difference step $h = 0.01$ used to produce the recorded cumulants, not at the level of the determinants.
- (V3) *The structural identities hold exactly.* The thinning identity (11) and its two-time version (25) hold to 5×10^{-15} ; the detector-convention gap $\text{Var}_{\text{miss}} - \text{Var}_{\text{scale}} = \kappa(1 - \kappa)\langle \hat{N} \rangle$ holds to 0.0 in double precision on all seven recorded rows of the κ sweep; $C(t, t) = \text{Var}(\hat{N})$ holds to 2.9×10^{-10} at $N = 40$; and the projective distribution of the total number is supported on $n_1 - n_2$ even, with parity-forbidden mass 8.5×10^{-15} .

4.1 Where the numerics fail, and what was done

Four numerical problems were found. None of them changes a physical conclusion, and each is reported here rather than repaired silently, because two of them affect figures.

Table 1: Validation summary. “Exact” compares a formula of Section 3 with an exact 2^N Fock trace; “recorded” compares an independent recomputation with the evidence tables of the source repository [3]. Worst deviations over the coverage listed in Appendix A.

quantity	route	N range	worst deviation
ground-state generator (18)	exact	2–6	3.5×10^{-15} (rel.)
thermal generator (18)	exact	2–6	4.4×10^{-15} (rel.)
weighted generator	exact	4–6	9.3×10^{-16} (rel.)
$N \times N$ reduction (19)	exact	4–8	3.7×10^{-16} (rel.)
two-time generator (21)	exact	2–8	9.7×10^{-15} (rel.)
integrated current (22)	exact	3–8	2.0×10^{-9} (cumulants)
twirl integrand (23)	exact	4–6	2.5×10^{-15} (rel., up to sign)
projective distribution, branch-tracked	exact	4–8	3.1×10^{-15} (abs.)
projective distribution, coefficient recursion	exact	8	9.8×10^{-3} (abs.)
partial projective distribution	exact	8	1.3×10^{-15} (abs.)
critical boundary table	recorded	64–1024	2.3×10^{-7} (abs.)
thermal table	recorded	16–128	2.2×10^{-8} (abs.)
gap table	recorded	4–256	1.6×10^{-15} (abs.)
half-chain table	recorded	16–512	5.1×10^{-8} (abs.)
distribution files	recorded	64	10^{-12} (moments)
detector conventions	recorded	128	2×10^{-6} (abs.)
κ -sweep identity (11)	recorded	64	0.0

The recorded total-number projective grids are wrong. The two recorded grids for $p(n_1, n_2)$ of the total number at $N = 8$, $t = 0.4$, in the ground state and at $\beta = 0.7$, place 0.661 and 0.989 of their total probability on pairs with $n_1 - n_2$ odd, where the exact selection rule gives zero. They differ from exact 2^8 Kraus traces by up to 0.378 and 0.237 in a single entry. The failure is not in the determinant reduction: evaluating the same twirl integrand on the same grid and comparing with the exact characteristic function gives agreement to 2×10^{-14} *up to a sign*, so what is lost is the branch of the square root, exactly as Section 3.4 warns. Rerunning the source routine at the parameters documented by the analysis script — `projective_two_time_distribution_fft(N=8, t=0.4, L=32)` at $\gamma = g = 1$, open chain — is deterministic, returning bit-identical grids under one, two and four BLAS threads, but it does not return the recorded grid: the two differ by up to 0.372 in a single cell and the rerun still carries 2.6×10^{-3} of parity-forbidden mass. The recorded file is thus not reproducible from the code supplied with it. Recomputing the grids with Protocol 3.6 at $L = 18$, $M_\phi = 2048$ gives the ground state to 3.1×10^{-15} and parity-forbidden mass 8.5×10^{-15} , and the thermal case to 2.3×10^{-5} with parity-forbidden mass 4.2×10^{-5} . Figures 13 and 14 are drawn from the recomputed grids, not from the recorded ones; the residual thermal error is a property of the sign resolution and is stated in the captions.

The site-resolved mean at a subcritical field is branch-ambiguous. At $g = 0.8$ the open chain has an exponentially small many-body splitting: the smallest positive BdG eigenvalue is 2.9×10^{-4} at $N = 32$, 2.3×10^{-7} at $N = 64$, 1.4×10^{-13} at $N = 128$ and 2.6×10^{-15} at $N = 256$, against 2.4×10^{-2} , 1.2×10^{-2} and 6.1×10^{-3} at criticality and ≈ 0.2 at $g = 1.2$. Which linear combination the eigensolver returns is then an arbitrary choice inside a numerically degenerate doublet, and the site-resolved density profile $\langle \hat{n}_j \rangle$ is not continuous across that choice: it loses the exact reflection symmetry $\langle \hat{n}_j \rangle = \langle \hat{n}_{N+1-j} \rangle$ of the true finite-chain ground state by up to 4.4×10^{-4} at $N = 128$

and 3.7×10^{-2} at $N = 256$, and the raw half-chain mean misses the recorded one by 6.1×10^{-4} and 5.1×10^{-2} at those two lengths. The physically meaningful profile is the reflection-symmetric one, obtained by averaging the two members of the doublet, for which the half-chain mean is exactly half the total mean; this reproduces every recorded half-chain mean to 1.3×10^{-8} . Variances and the cut deficit of Section 5.5 are insensitive to the choice, agreeing to 5.1×10^{-8} either way, because they are traces over the doublet. The mean profile at criticality, where the splitting is a power law rather than exponential, has no such ambiguity.

One recorded cumulant row carries a discretisation error. The projective cumulants of the transferred charge at $t = 0.8$ differ from the exact 2^8 values by up to 3.6×10^{-2} in the third cumulant (0.4786 recorded, 0.5145 exact); a rerun on a refined grid converges to the exact value, so the recorded row reflects the coarse grid used for it. Table 4 therefore quotes the exact 2^8 values at that time, and the discrepancy is noted there.

At the periodic critical point the determinant route has no branch, and worse. On the periodic chain the mode $\Phi_k = 0$ has $E_k = 0$ exactly whenever $g = 1$, for every $\gamma > 0$, so the BdG matrix has a two-dimensional null space, the many-body ground space is two-dimensional, and U_- must contain one arbitrary unit vector of that null space. The consequence is not merely that an arbitrary branch is returned. The assembled $\Gamma = U_- U_-^\dagger$ is still a projector with $\text{Tr } \Gamma = N$ and $\langle cc^\dagger \rangle = I - \langle c^\dagger c \rangle^T$ to 10^{-15} , but its pairing block $B = \langle c^\dagger c^\dagger \rangle$ fails to be antisymmetric by up to 8.8×10^{-2} (measured for $N = 3$ to 8 and $\gamma \in \{0.4, 0.7, 1\}$, whereas every non-degenerate control case satisfies it to 10^{-15}), and a covariance with a non-antisymmetric pairing block is the covariance of no fermionic state. The two exact parity branches differ in mean by exactly 1 and have identical variance, so any state $a|\text{even}\rangle + b|\text{odd}\rangle$ of the ground manifold obeys $\text{Var} = v + x(1 - x)$ with $x = |b|^2$; in 16 of the 21 degenerate cases examined the determinant route returns a pair outside that range, for example $(\langle \tilde{N} \rangle, \text{Var}) = (1.728404, 1.398798)$ at $N = 7$, $\gamma = g = 1$, where the manifold allows at most 1.324399 at that mean. The implementation supplied with the data behaves the same way and differently from ours — at $\gamma = g = 1$ it returns the occupied $k = 0$ branch at $N = 3$, the empty one at $N = 4$ and unbalanced values at $N = 5$ and 6, differing from the nearer parity branch by up to 2.6×10^{-2} in relative terms — which is what one expects of a choice that nothing determines. At $N = 5$ its weight on the occupied branch is 0.061 at $\lambda = +0.35$ and 0.040 at $\lambda = -0.60$, so the returned generating function is not even that of a single fixed state of the manifold: for a state $a|\text{even}\rangle + b|\text{odd}\rangle$ the weight would be λ -independent, because $\tilde{N}$ is diagonal in the branch basis. All periodic critical numbers in this paper therefore come from the analytic mode product of [1] with the branch stated explicitly, not from the determinant; Appendix C gives the three branches, how they differ, and the measurement above.

Two things are not verified and are marked as such wherever they appear. The $\gamma = 0$ parity protocol is taken from the source’s recorded table, which lists a worst cumulant error of 1.2×10^{-14} over 54 non-degenerate rows and a two-fold ground-manifold degeneracy for the two degenerate rows; it is consistent with our own checks but was not re-derived. Running the repository’s own `test_parity_protocol` in a sandbox reproduces that table row by row, including the degeneracy flags, with per-row differences of at most 8.0×10^{-15} , i.e. BLAS round-off. And the logarithmic coefficients of Section 5 are fitted, not proved: the evidence for them is the quality of the fit over $N = 64$ to 1024, nothing more.

Finally, the repository ships its own test suite, eleven scripts covering the formula reductions, the thermal, spatial, current and two-time routines, the two projective two-time protocols, the periodic $k = 0$ branch, the $g = 1$ zero mode and the parity protocol. All eleven pass when run against the supplied code in a sandbox copy, in 406 s in total. Ten of them report errors no larger than 6.0×10^{-14} ; the eleventh, `test_thermal_projective_two_time`, reports 2.2×10^{-5} at $N = 4$ and passes with it, which is the thermal sign caveat of Section 3.4 and which our own check reproduces

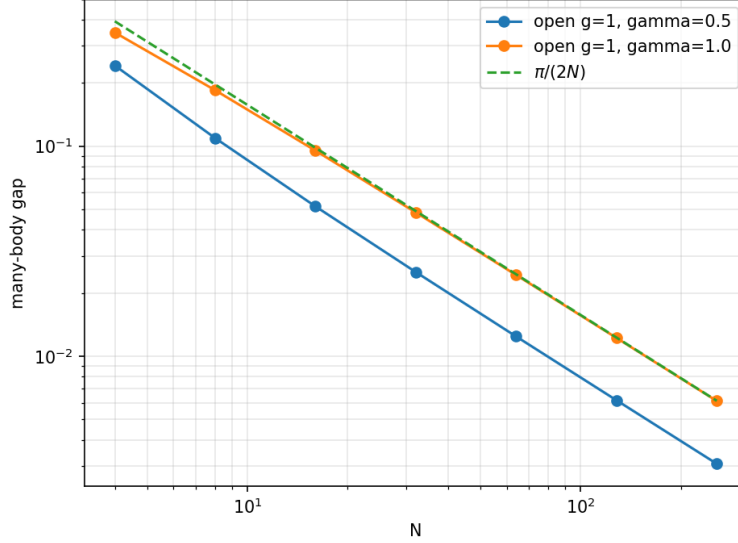


Figure 1: Many-body gap of the open chain at $\gamma = g = 1$ (circles, both $\gamma = 0.5$ and $\gamma = 1$ series) against N on logarithmic axes, with the reference line $\pi/(2N)$ (dashed). The open-chain gap closes as a power of N ; the periodic chain at the same parameters has an exact zero mode and is not plotted on these axes because its gap is zero for every N . At $\gamma = 1$ the ratio of the computed gap to $\pi/(2N)$ is 0.884, 0.940, 0.969, 0.985, 0.9922, 0.9961 and 0.9980 for $N = 4, 8, \dots, 256$; at $\gamma = 0.5$ the same ratio against $\gamma\pi/(2N)$ approaches unity from above, reaching 1.0030 at $N = 256$. The reference line is a numerical observation, not a derived quantisation condition.

independently as 2.244×10^{-5} . `test_periodic_k0` in particular confirms that the empty- $k = 0$ branch and the $g \neq 1$ determinant agree to 4.7×10^{-14} at $N = 400$, which is the convention used throughout Section 5. The suite validates the code against small- N exact traces; it does not test the recorded evidence tables against the code, which is what the checks summarised above do.

5 Boundary conditions, finite size and temperature

5.1 The two spectra close differently

The counting statistics of a quadratic fermion model are controlled by the spectrum of its BdG matrix W of (12), and the boundary condition changes that spectrum qualitatively at the critical point. On the periodic chain the mode $\Phi_0 = 0$ always exists, and at $g = 1$ it is an exact zero mode: two eigenvalues of W vanish to within 10^{-10} for every N from 4 to 256 and every $\gamma > 0$. The many-body ground state is then two-fold degenerate and the periodic gap is exactly zero at every length. On the open chain no mode is forced to zero; the gap closes only as a power of N .

The recorded gap table (Figure 1) is reproduced from an independently assembled BdG matrix to 1.6×10^{-15} over all 14 of its rows. Two features of it matter later. First, the open gap at criticality is

$$\Delta_{\text{open}}(N) \simeq \frac{\gamma\pi}{2N}, \quad \frac{\Delta_{\text{open}}}{\gamma\pi/(2N)} = \begin{cases} 0.9922, 0.9980 & (\gamma = 1, N = 64, 256) \\ 1.0121, 1.0030 & (\gamma = 0.5, N = 64, 256) \end{cases} \quad (26)$$

with the ratio approaching unity from opposite sides for the two anisotropies; the factor γ is the

Table 2: Total-number mean and variance at $\gamma = g = 1$, $T = 0$, open chain (determinant route) and periodic chain (analytic mode product with the empty $k = 0$ branch), with their differences $d_{\text{mean}} = \langle \hat{N} \rangle_{\text{open}} - \langle \hat{N} \rangle_{\text{per}}$ and $d_{\text{var}} = \text{Var}_{\text{open}} - \text{Var}_{\text{per}}$. The last two rows are the least-squares fits of the differences to $a \ln N + b + c/N$ over the five lengths: for d_{mean} the fitted slope is -0.1591523 against $-1/(2\pi) = -0.1591549$, a relative difference of $+1.6 \times 10^{-5}$, and for d_{var} it is -0.1013244 against $-1/\pi^2 = -0.1013212$, relative -3.1×10^{-5} .

N	$\langle \hat{N} \rangle_{\text{open}}$	Var_{open}	$\langle \hat{N} \rangle_{\text{per}}$	Var_{per}	d_{mean}	d_{var}
64	10.863569	15.373957	11.132258	15.500000	-0.268689	-0.126043
128	22.382287	31.304135	22.758380	31.500000	-0.376093	-0.195865
256	45.528740	63.234106	46.013692	63.500000	-0.484951	-0.265894
512	91.931311	127.163975	92.525850	127.500000	-0.594539	-0.336025
1024	184.846441	255.093795	185.550932	255.500000	-0.704492	-0.406205
fit of d_{mean} : a, b, c , max residual			-0.1591523	0.399032	-0.372830	5.9×10^{-7}
fit of d_{var} : a, b, c , max residual			-0.1013244	0.296172	-0.052435	6.2×10^{-7}

critical group velocity, since (3) at $g = 1$ gives $E_k \rightarrow \gamma|k|$ as $k \rightarrow 0$. Second, on the periodic chain slightly off criticality the gap is *exactly* $J|g - 1|$ at every length: at $g = 1.05$ the computed value is 0.050000000000 for all N and both anisotropies, to twelve decimal places. That is not a numerical coincidence but an identity, because the $\Phi_0 = 0$ mode is unpaired, $\Delta_0 = \gamma \sin 0 = 0$, so its BdG eigenvalues are $\pm|\xi_0| = \pm|g - 1|$ and no other level lies below them for g close to 1 from above. The periodic smallest *positive* level at criticality, where the zero mode is excluded, approaches $2\pi J\gamma/N$: the ratio is 0.9996 at $N = 64$ and 0.99997 at $N = 256$ for $\gamma = 1$, from below, and 1.0032 and 1.0002 at the same lengths for $\gamma = 0.5$, from above.

The consequence for counting is immediate. A thermal state at inverse temperature β cannot resolve the open chain's small gap once $\beta^{-1} \gg \gamma\pi/(2N)$, which for $\gamma = 1$ and $N = 128$ means $\beta^{-1} \gg 0.012$, i.e. any temperature an experiment would call low. The periodic chain, whose gap is exactly zero, has no such scale at all. Section 5.4 shows that this is precisely the regime in which the boundary correction to the counting statistics stops growing.

5.2 Critical boundary corrections to the mean and the variance

At $\gamma = g = 1$ and $T = 0$ the open and periodic chains have the same bulk density and the same bulk variance density, and differ by a term that grows without bound. Table 2 collects the two moments for $N = 64, \dots, 1024$; every entry is reproduced by the independent covariance route of Section 4 to 2.3×10^{-7} , and the periodic entries agree with the closed forms derived in Appendix B to 1.4×10^{-14} .

The periodic column is not a fit. Appendix B derives, from $v_k^2 = (1 - \sin(\pi k/N))/2$ at criticality and the pair product of [1] with the $k = 0$ mode empty,

$$\text{Var}_{\text{per}} = \frac{N}{4} - \frac{1}{2} \quad \text{exactly, for every even } N, \quad (27)$$

and

$$\langle \hat{N} \rangle_{\text{per}} = \sum_{k=1}^{N/2-1} \left(1 - \sin \frac{\pi k}{N} \right) = \frac{N}{2} - 1 - \frac{\sin(\frac{\pi}{4} - \frac{\pi}{2N})}{\sqrt{2} \sin \frac{\pi}{2N}} = N \left(\frac{1}{2} - \frac{1}{\pi} \right) - \frac{1}{2} + O\left(\frac{1}{N}\right), \quad (28)$$

both of which the table confirms to fourteen digits. The large- N form (28) has a remainder that Appendix B.9 computes exactly, $\pi/12N + O(N^{-3})$: the measured residual is $+1.023 \times 10^{-3}$ at

$N = 256$ against $\pi/12N = 1.0230 \times 10^{-3}$, and $+2.557 \times 10^{-4}$ at $N = 1024$ against 2.5574×10^{-4} . The bulk density at criticality is therefore $\langle \hat{n}_j \rangle = \frac{1}{2} - \frac{1}{\pi} = 0.18169$ and the bulk variance density is $\frac{1}{4}$, and the variance identity (27) is independent of the $k = 0$ branch, while the mean is not: the occupied branch is z times the empty one and shifts the mean by exactly $+1$ (Appendix C).

The open column has no such closed form, and the statement of this paper about it is numerical. Fitting the differences to $a \ln N + b + c/N$ over the five lengths $N = 64$ to 1024 gives

$$d_{\text{mean}}(N) = -\frac{1}{2\pi} \ln N + 0.399032 - \frac{0.372830}{N} + r_N, \quad \max_N |r_N| = 5.9 \times 10^{-7}, \quad (29)$$

$$d_{\text{var}}(N) = -\frac{1}{\pi^2} \ln N + 0.296172 - \frac{0.052435}{N} + r_N, \quad \max_N |r_N| = 6.2 \times 10^{-7}, \quad (30)$$

where in each case the logarithmic coefficient has been *fixed* to the quoted rational multiple of π and only b and c fitted; a free three-parameter fit returns $a = -0.1591523$ and $a = -0.1013244$, differing from $-1/(2\pi)$ and $-1/\pi^2$ by $+1.6 \times 10^{-5}$ and -3.1×10^{-5} in relative terms. Fitting only the four lengths up to $N = 512$ changes a in the eighth decimal place. Over one decade in N the residuals of a two-term logarithmic form are at the 10^{-7} level, which is four orders of magnitude below the differences themselves.

What (29) and (30) do *not* come with is a derivation. That the coefficients are $-1/(2\pi)$ and $-1/\pi^2$ to five decimal places is a numerical fact about the range $64 \leq N \leq 1024$; the paper does not prove that they are the asymptotic coefficients, and does not exclude a $(\ln \ln N)/\ln N$ or $1/\ln N$ correction that this range cannot resolve. The magnitude is worth stating plainly: at $N = 1024$ the boundary term is 0.70 in the mean, 0.38% of it, and 0.41 in the variance, 0.16% of it. Relative to the bulk it is small and shrinking; relative to any fixed subregion's fluctuations — the half-chain deficit of Section 5.5 is 0.16 at $N = 512$ — it is not, and it grows.

Figures 2 and 3 show the same comparison away from the critical point, on a 61-point grid of g from 0.5 to 1.5 at $N = 64, 128, 256$.

5.3 Away from criticality: the self-paired mode and the parity of the distribution

Off criticality the boundary correction is bounded, and the periodic chain acquires a feature the open chain does not have: a mode that pairs with itself. On the periodic chain the Fourier modes k and $N - k$ form the pairs that the Bogoliubov transformation mixes, and the product representation of the counting function runs over those pairs [1],

$$\chi_{\text{per}}(z) = z^{[g < 1]} \prod_{k=1}^{[(N-1)/2]} \left[(1 - v_k^2) + v_k^2 z^2 \right], \quad v_k^2 = \frac{1}{2} \left(1 - \frac{\xi_k}{E_k} \right), \quad (31)$$

where each factor contributes 0 or 2 fermions — the paired modes are occupied in pairs — and the two self-paired modes contribute single-occupancy factors. The mode $k = N/2$, present for even N , has $\xi = g + 1 > 0$ and is always empty. The mode $k = 0$ has $\xi_0 = g - 1$, so it is occupied for $g < 1$, empty for $g > 1$, and degenerate at $g = 1$; the factor $z^{[g < 1]}$ in (31) is that occupancy.

The counting distribution therefore has a parity that depends on which side of g_c the periodic chain is on, while the open chain's does not. At $\gamma = 1$, $g = 0.95$, $N = 64$ (Figure 4) the open distribution is supported entirely on even n , with odd- n mass 2×10^{-15} , and the periodic one entirely on odd n , with odd- n mass 1.0: the occupied self-paired mode contributes one unpaired

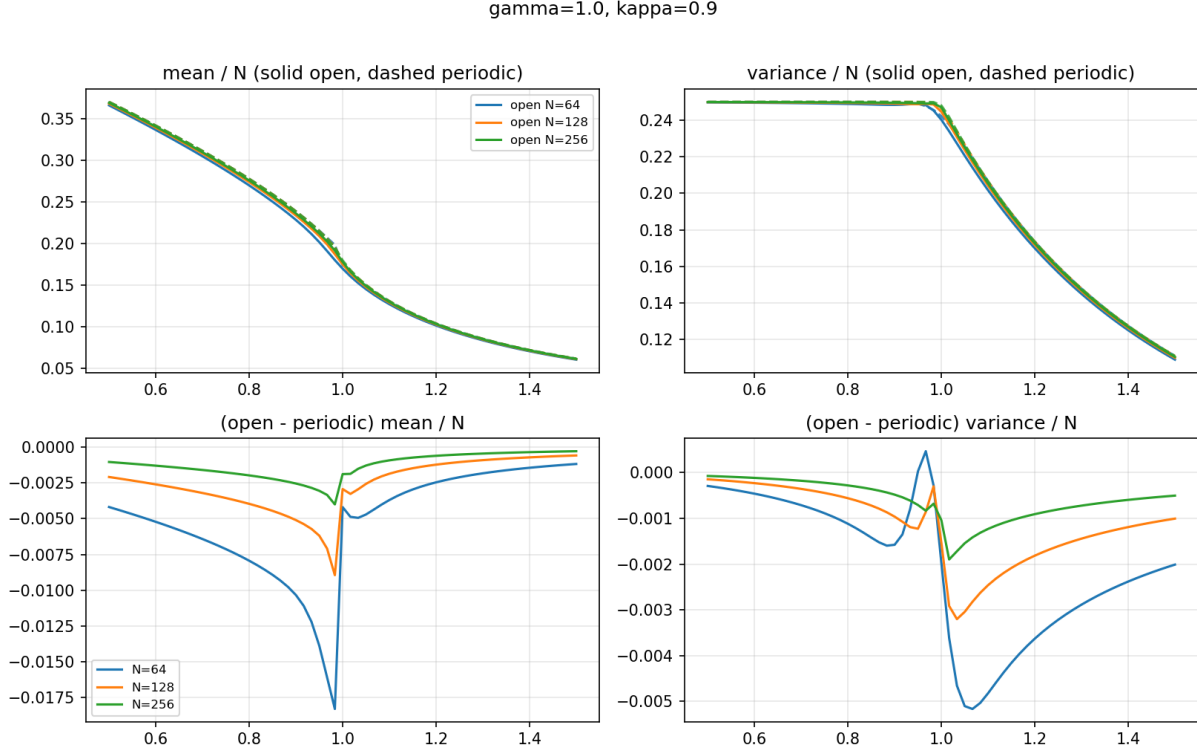


Figure 2: Total-number cumulants at $\gamma = 1$, $T = 0$, as functions of the field g for $N = 64, 128, 256$ (three colours, one per length). Top left: mean per site, solid for the open chain and dashed for the periodic reference. Top right: variance per site, same line convention. Bottom left and bottom right: the open-minus-periodic differences of mean and variance per site. The differences change sign at $g = g_c = 1$ and are the only place where the boundary condition shows up at the scale of the plot; at criticality they are the logarithmic corrections of Table 2 divided by N , which is why they are plotted per site here and unnormalised there.

fermion. At $g = 1.05$ both are supported on even n . The two moments at $N = 64$ are

		$\langle \hat{N} \rangle$	$\text{Var}(\hat{N})$	support	
$\gamma = 1, g = 0.95 :$	open	12.9086	15.9345	n even	
	periodic	13.7963	15.9326	n odd	(32)
$\gamma = 1, g = 1.05 :$	open	9.2805	14.1174	n even	
	periodic	9.5821	14.4439	n even	

and at $\gamma = 0.5, g = 1, N = 64$ (Figure 5) the open and periodic moments are 6.5932, 9.9600 and 6.8746, 10.1667, both distributions even. Below g_c the mean difference is dominated by the single occupied self-paired mode, which accounts for exactly one fermion: subtracting it leaves the empty-branch periodic mean at 12.796284 and a residual $+0.112273$ from the remaining $N - 1$ modes, positive here in contrast with the critical value -0.268689 of Table 2.

5.4 Temperature rounds the logarithm into a bounded term

At any fixed nonzero temperature the boundary correction stops growing. The open and periodic variances at $\gamma = g = 1$ are compared in Table 3 and Figure 6 over $N = 16$ to 128 and $\beta = 0.5$ to 5;

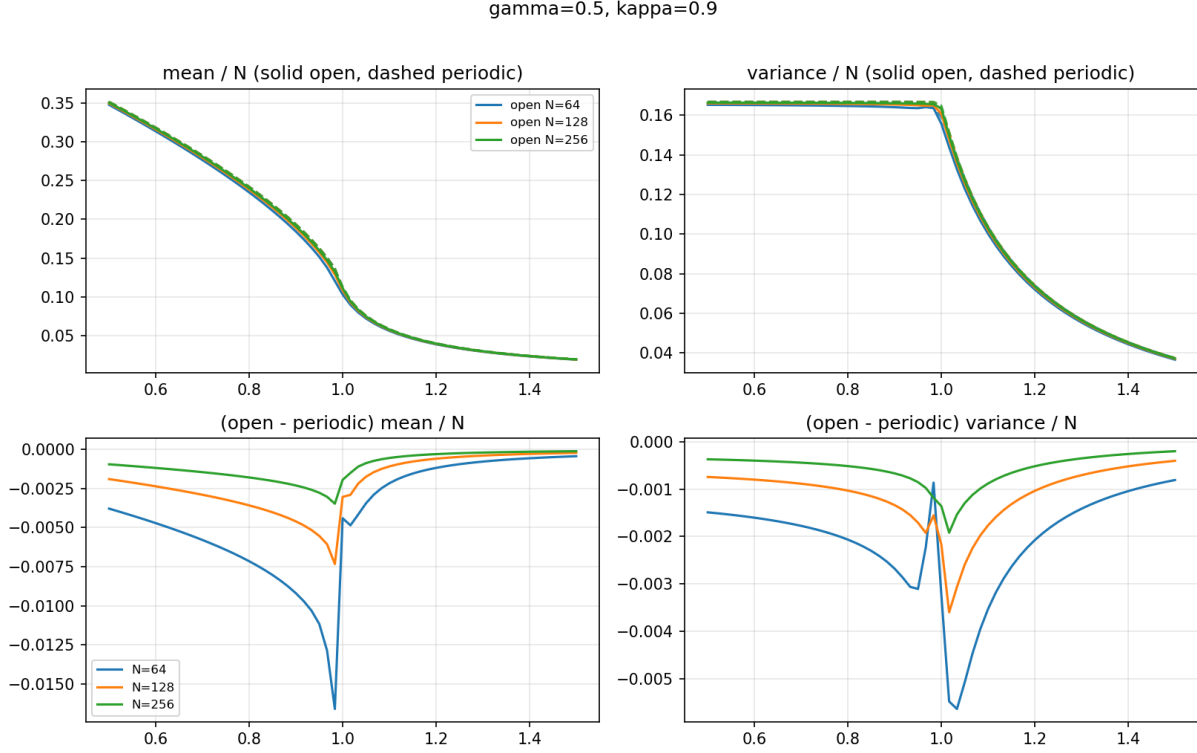


Figure 3: The same comparison at $\gamma = 0.5$, $T = 0$, on the same grid of g and the same three lengths. Reducing the anisotropy reduces both moments; the open-minus-periodic differences keep the same sign structure across g_c and remain subextensive.

the values come from the recorded thermal table, whose 48 rows the independent covariance route reproduces to 2.2×10^{-8} .

Three readings of the table. The variance density itself is smooth in β and approaches its $T = 0$ value $\frac{1}{4}$ from below as β grows; the $\beta = 5$ rows are visibly not yet there, so $\beta = 5$ is not the zero-temperature limit at these lengths. The open-minus-periodic difference is *constant in N to the last printed digit* for $\beta = 0.5, 1$ and 2 , and constant to three digits for $\beta = 5$; the logarithm of Table 2 has been rounded into an $O(1)$ term whose value is set by β alone. And the ordering in β is monotone: $|d_{\text{var}}|$ grows from 0.0016 at $\beta = 0.5$ through 0.0162 and 0.0783 to 0.203 at $\beta = 5$, approaching the zero-temperature value 0.1959 at $N = 128$ and crossing it, as it must, since the $T = 0$ correction continues to grow with N while the thermal one does not.

The mechanism is the one identified in Section 5.1. The boundary correction at $T = 0$ accumulates over modes down to the lowest one, at energy $\gamma\pi/(2N)$; a thermal weight cuts that accumulation off at β^{-1} , and once $\beta^{-1} \gg \gamma\pi/(2N)$ the correction saturates at a value controlled by $\beta\gamma\pi/(2N)$ rather than by N . The table is consistent with this and does not test it: at $N = 128$ and $\beta = 5$ the thermal energy $\beta^{-1} = 0.2$ is sixteen times the open-chain gap 0.0122, so every entry in Table 3 sits deep in the saturated regime, and the crossover from logarithmic to constant behaviour as β is lowered at fixed N is not resolved by these lengths.

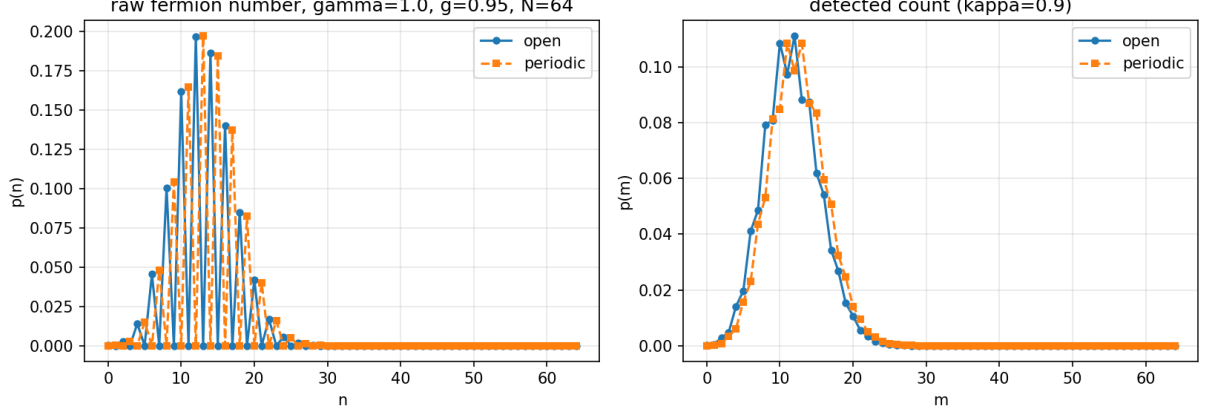


Figure 4: Number distributions at $\gamma = 1$, $g = 0.95$, $N = 64$, $T = 0$. Left: the raw distribution $p(n)$, open chain (circles, solid) and periodic reference (squares, dashed). Right: the same distributions after binomial thinning at $\kappa = 0.9$. The raw open distribution is supported on even n and the raw periodic one on odd n , the signature of the occupied self-paired $k = 0$ mode; the whole periodic profile is displaced by one particle relative to the open one. Thinning destroys the parity structure — the detected-count distributions carry 54% and 47% even mass for the open and periodic chains — while preserving the displacement.

5.5 Cutting the chain in half

A different boundary effect appears when the counted observable is the number in one half of the chain, $A = \{1, \dots, N/2\}$, evaluated with the weighted generator of Proposition 3.1. For a translation-invariant bulk one would expect $\text{Var}(\hat{N}_A) = \frac{1}{2} \text{Var}(\hat{N})$; the difference

$$c_N = \frac{1}{2} \text{Var}(\hat{N}) - \text{Var}(\hat{N}_A) \quad (33)$$

is the cost of the interior cut, and it is positive: the two halves fluctuate less independently than the sum rule assumes. All 21 recorded rows of the half-chain table are reproduced to 5.1×10^{-8} by the independent covariance route.

At $T = 0$ and $g = 1$ the deficit grows logarithmically (Figure 7, left panel),

$$c_N = 0.013672 + 0.023651 \ln N, \quad \max_{32 \leq N \leq 512} |r_N| = 6.5 \times 10^{-4}, \quad (34)$$

with the values 0.096287, 0.111622, 0.127850, 0.144634 and 0.161750 at $N = 32, 64, 128, 256, 512$. Adding the natural next term does not change the slope but improves the residual by an order of magnitude, $c_N = 0.005697 + 0.024970 \ln N + 0.130044/N$ with residual 4.5×10^{-5} , so the coefficient of $\ln N$ is 0.0237 to within 5% under either fit. Unlike (29) and (30), this coefficient is not a clean multiple of a power of π ; we report it as a fitted number and do not conjecture its exact value. Off criticality the deficit saturates: $c_N = 0.031758$ at $g = 0.8$ for $N \geq 128$ and 0.051862 at $g = 1.2$ for $N \geq 64$, with the $N = 32$ values 0.034394 and 0.051843 already within 8% and 0.04% of the large- N numbers.

At finite temperature the deficit is bounded in N and smaller (Figure 7, right panel): 0.002783 at $\beta = 1$, constant to eight decimal places over $N = 16$ to 128, and 0.041001 at $\beta = 5$, constant to six places for $N \geq 32$ and 0.041011 at $N = 16$. The same rounding that removes the logarithm from the boundary correction removes it from the cut deficit, and the ordering in β is the same: $0.0028 < 0.0410 < c_N(T=0)$.

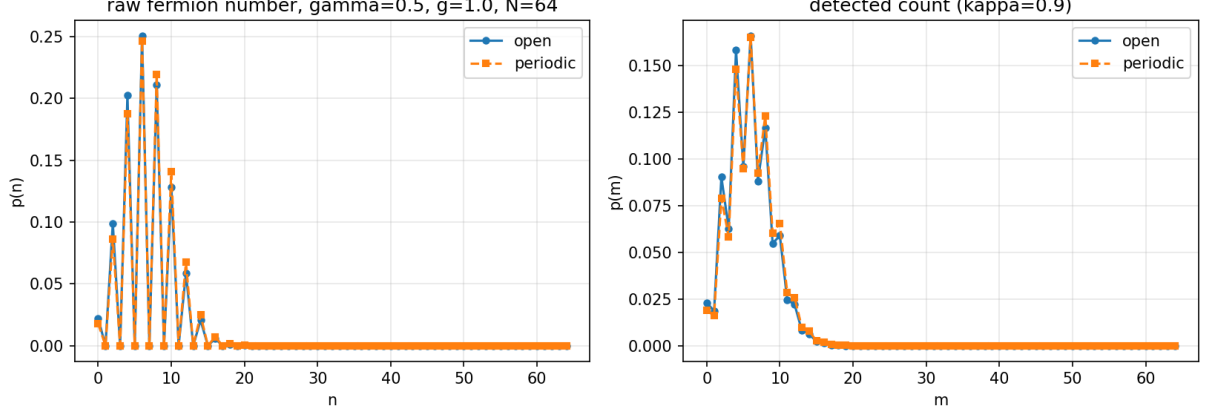


Figure 5: Number distributions at $\gamma = 0.5$, $g = 1$, $N = 64$, $T = 0$, raw (left) and thinned at $\kappa = 0.9$ (right), open against periodic. Both raw distributions are supported on even n : at criticality the self-paired $k = 0$ mode is degenerate rather than occupied, and the empty-branch convention of (31) is used. The periodic profile is broader (Var = 10.1667 against 9.9600) and displaced to larger n (6.8746 against 6.5932).

One caveat belongs to this subsection rather than to the appendix. At $g = 0.8$ the open chain's BdG spectrum contains a doublet split by 2.3×10^{-7} at $N = 64$, 1.4×10^{-13} at $N = 128$ and less than 10^{-14} at $N = 256$, so the ground state returned by a numerical eigensolver is an arbitrary member of a numerically degenerate pair. The site-resolved density $\langle \hat{n}_j \rangle$, and therefore the half-chain *mean*, is not continuous across that choice; the half-chain *variance* and the deficit (33) are, agreeing to 3×10^{-9} whichever member is selected. The recorded half-chain means are the reflection-symmetric ones, for which the half-chain mean is exactly half the total mean, and the symmetrised profile reproduces them to 1.3×10^{-8} while a raw eigenvector selection misses them by 6.1×10^{-4} at $N = 128$ and 5.1×10^{-2} at $N = 256$. Every number quoted above for $g = 0.8$ is a variance and is unaffected.

6 Counting in time

Three different objects share the phrase “two-time counting statistics”, and they are not equal. This section keeps them apart, states which identities link them, and reports where the recorded data supports each one.

- (i) The *operator-level* (Weyl) generator

$$Z(\lambda_1, \lambda_2; t_1, t_2) = \left\langle \exp[\lambda_1 \hat{N}(t_1) + \lambda_2 \hat{N}(t_2)] \right\rangle, \quad \hat{N}(t) = e^{iHt} \hat{N} e^{-iHt}, \quad (35)$$

a single exponential of a sum of Heisenberg operators. Its mixed derivative at the origin is the symmetrised connected correlator $C(t_1, t_2) = \frac{1}{2} \langle \{\Delta \hat{N}(t_1), \Delta \hat{N}(t_2)\} \rangle$.

- (ii) The *projective two-measurement* distribution

$$p(n_1, n_2; t) = \text{Tr}[P_{n_1} U(t) P_{n_2} \rho P_{n_2} U^\dagger(t)], \quad U(t) = e^{-iHt}, \quad (36)$$

where n_2 is the outcome at time 0 and n_1 the outcome at time t . This is the ordering used by the recorded data and by the figures below.

Table 3: Thermal rounding of the boundary correction at $\gamma = g = 1$. Var/N for the open and periodic chains and their difference d_{var} , at two lengths spanning the table’s range. The difference is constant in N to six decimal places for $\beta \leq 2$ and to three for $\beta = 5$, whereas at $T = 0$ the same quantity grows logarithmically, reaching -0.406 at $N = 1024$ (Table 2).

β	N	$\text{Var}_{\text{open}}/N$	$\text{Var}_{\text{per}}/N$	d_{var}	$\langle \hat{N} \rangle_{\text{open}}$
0.5	16	0.236570	0.236671	-0.001619	6.110806
0.5	128	0.236658	0.236671	-0.001619	48.918807
1.0	16	0.213349	0.214359	-0.016158	4.725982
1.0	128	0.214232	0.214359	-0.016158	37.999063
2.0	16	0.194442	0.199335	-0.078290	3.428509
2.0	128	0.198724	0.199335	-0.078290	28.126138
5.0	16	0.207145	0.219793	-0.202379	2.749002
5.0	128	0.218272	0.219862	-0.203476	23.709870
$T = 0$	128	0.244564	0.246094	-0.195865	22.382287

- (iii) The *integrated current* through a cut, $\hat{Q}(t) = \hat{N}_A(t) - \hat{N}_A$, counted either at the operator level through (35) with $\hat{N} \rightarrow \hat{N}_A$ or projectively through (36) with the projectors of $\hat{N}_A$.

6.1 Operator-level correlators and their reductions

Three reductions of (35) are identities, and all three hold in the recorded data to the accuracy of the finite differences that produced it. Setting $\lambda_2 = 0$ and using $[\rho, H] = 0$,

$$Z(\lambda, 0; t, t) = \chi(e^\lambda) \quad (37)$$

independently of t ; the recorded values at $N = 20$, $\gamma = g = 1$, $\lambda \in \{-0.6, -0.3, 0, 0.3, 0.6\}$ and $t \in \{0, 0.4, 0.8\}$ are constant in t to 1.8×10^{-15} and equal both the independent Nambu-basis generator and the one-time $\ln \chi$ to 4.9×10^{-15} (Figure 8). Setting $t_1 = t_2$ in the correlator,

$$C(t, t) = \text{Var}(\hat{N}), \quad (38)$$

and the diagonal of the recorded 6×6 maps is constant to 2.9×10^{-10} at the value $\text{Var}(\hat{N})$ computed independently. And $C(t_1, t_2) = C(t_2, t_1)$ holds *exactly* on the recorded grid, the deviation being 0.0 in double precision.

The correlator itself is shown in Figure 9. At $N = 40$, $\gamma = 1$, $T = 0$ the equal-time value $C(0, 0) = \text{Var}(\hat{N})$ is 9.931997 at $g = 0.8$, 9.421082 at $g = 1.0$ and 6.712258 at $g = 1.2$; the critical value sits 5.8% below the bulk expectation $N/4 = 10$, which is the negative logarithmic boundary correction of Table 2 at $N = 40$. Scanning the second time at $g = 1$, $C(0, t)$ falls through 8.662603 at $t = 0.2$, 6.618503 at 0.4, 3.895033 at 0.6 and 1.247309 at 0.8 to -0.690297 at $t = 1.0$: the correlator changes sign inside the window, and the whole map ranges from -0.690297 to 9.421082. Off criticality the sign change comes sooner and the swing is larger — $C(0, 1) = -1.360940$ at $g = 0.8$ and -2.051109 at $g = 1.2$, the latter crossing zero between $t = 0.6$ and $t = 0.8$. An independent recomputation of the recorded scan reproduces it to 4.7×10^{-10} , which is the resolution of the fourth-order finite differences ($h = 0.004$) that produced it.

Temperature enters in two places. The equal-time value is *non-monotonic* in β : $C(0, 0) = 9.465220$ at $\beta = 0.5$, 8.558182 at $\beta = 1$, 7.895124 at $\beta = 2$, against 9.421082 at $T = 0$. The variance dips at intermediate temperature and rises towards both limits, the infinite-temperature binomial

Finite-temperature number variance of the open and periodic XY chain, $\gamma = 1$ (redrawn from the recorded table)

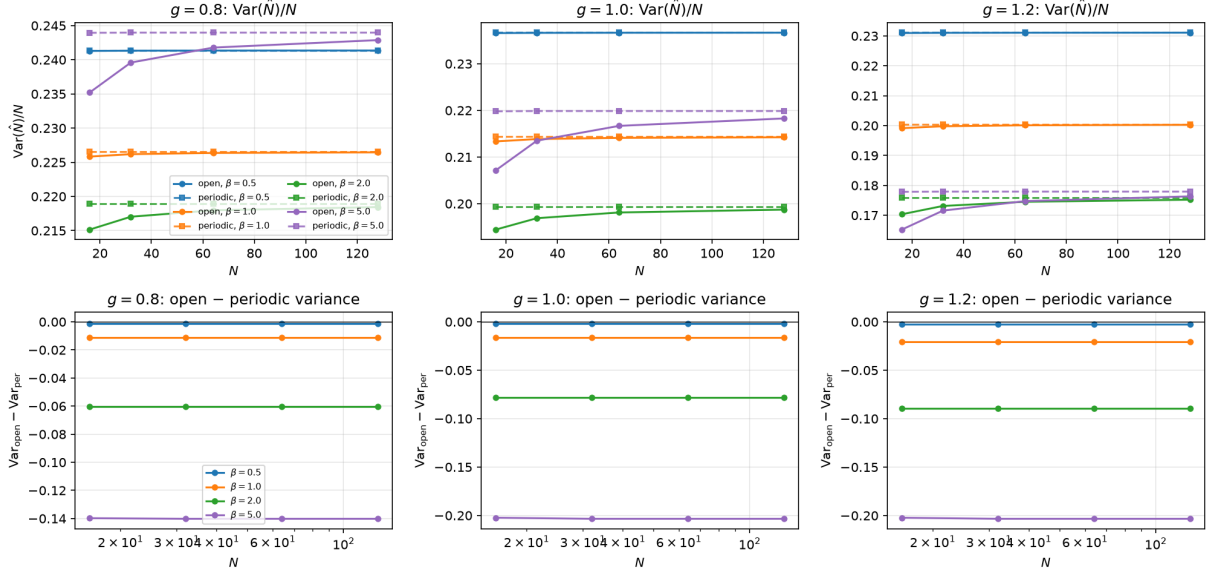


Figure 6: Finite-temperature number variance of the open (circles, solid) and periodic (squares, dashed) chain at $\gamma = 1$, for $g = 0.8, 1.0$ and 1.2 left to right and $\beta = 0.5, 1, 2, 5$ in four colours, over $N = 16$ to 128 . Top row: $\text{Var}(\hat{N})/N$, which converges in N at every temperature. Bottom row: the open-minus-periodic difference of the variance, unnormalised and plotted against N on a logarithmic axis; the curves are flat, in contrast with the $T = 0$ case where the same quantity grows logarithmically. Redrawn from the recorded thermal table with the values unscaled, so no axis factor has to be interpreted.

value $N/4 = 10$ and the zero-temperature critical value; the same dip is visible per site in Table 3, where $\text{Var}/N = 0.1987$ at $\beta = 2$ against 0.2446 at $T = 0$ and 0.2367 at $\beta = 0.5$. Away from the diagonal, temperature suppresses the oscillation: over $t \in [0, 2.5]$ the $\beta = 0.5$ scan has minimum 3.454416 and ends at 4.301023 , the $\beta = 1$ scan 1.547888 and 2.775497 , and only the $\beta = 2$ scan goes negative, minimum -1.001147 and end value 0.912121 . None of the three has settled by $t = 2.5$; each has passed a minimum and is rising again, and the window is too short to call that a plateau.

6.2 Projective two-measurement counting

The protocol (36) is what a detector that resolves individual fermions actually records, and it differs from (35) because the first measurement destroys the coherences of ρ between different eigenspaces of the counted observable. Two consequences are exact and worth stating before the numbers.

First, the marginal of the *first* measurement is the equilibrium distribution: summing (36) over n_1 gives $\text{Tr}[P_{n_2}\rho]$, because $\sum_{n_1} P_{n_1} = 1$ and the evolution is unitary. This is a strong internal consistency test that needs no exact diagonalisation, and it is the test that the recorded total-number grids fail (Section 6.3).

Second, the mean of the second outcome is the evolved observable in the *dephased* state. Writing $\rho_d = \sum_n P_n \rho P_n$,

$$\langle n_1 \rangle_{\text{proj}} = \text{Tr}[\hat{N}(t) \rho_d], \quad \langle n_2 \rangle_{\text{proj}} = \text{Tr}[\hat{N} \rho_d] = \text{Tr}[\hat{N} \rho], \quad (39)$$

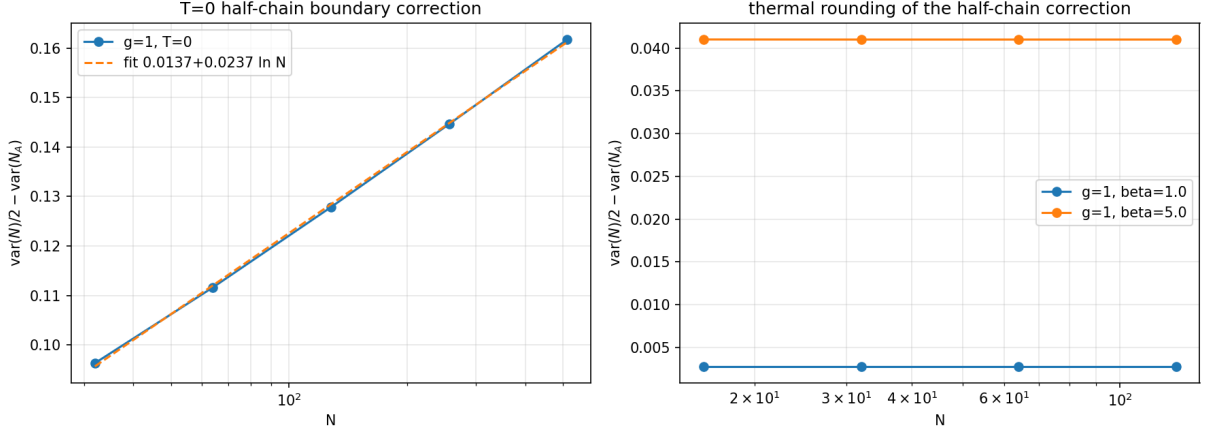


Figure 7: Half-chain cut deficit $\frac{1}{2} \text{Var}(\hat{N}) - \text{Var}(\hat{N}_A)$ against N on a logarithmic axis, at $\gamma = 1$. Left: $T = 0$, $g = 1$, for $N = 32$ to 512 , with the least-squares line $0.0137 + 0.0237 \ln N$ of (34) (dashed); the growth is logarithmic and the fit residual is at most 6.5×10^{-4} . Right: the same quantity at $g = 1$ for $\beta = 1$ and $\beta = 5$, both flat in N . The two panels are the interior-cut analogue of Table 2 and Table 3.

and the same holds for the second moments of the difference, so that for the transferred charge $\hat{Q}(t) = \hat{N}_A(t) - \hat{N}_A$,

Proposition 6.1 (dephasing identity). *For a projective two-measurement protocol on a region A ,*

$$\langle q \rangle_{\text{proj}} = \text{Tr}[\hat{Q}(t)\rho_d], \quad \text{Var}_{\text{proj}}(q) = \text{Tr}[\hat{Q}(t)^2\rho_d] - \text{Tr}[\hat{Q}(t)\rho_d]^2. \quad (40)$$

No such identity holds for the third and fourth cumulants, because $\hat{N}_A(t)$ and $\hat{N}_A$ do not commute and the ordering of the operators inside $\hat{Q}^3$ is not the ordering the projection produces. Numerically, at $N = 8$, $A = \{0, 1, 2, 3\}$, $\gamma = g = 1$, $T = 0$, $t = 0.1$: the identity reproduces the recorded projective mean and variance to 1.9×10^{-17} and 1.4×10^{-15} , while its third and fourth cumulants miss by 4.4×10^{-2} and 1.2×10^{-1} ; at $t = 0.4$ the first two agree to 4.7×10^{-16} and 1.1×10^{-16} and the last two miss by 0.58 and 1.33 .

The identity is the whole story of the measured current. Since ρ is stationary, $\text{Tr}[\hat{Q}(t)\rho] = 0$ identically, and the operator-level mean of the transferred charge vanishes — the recorded current table has $|\langle \hat{Q} \rangle| \leq 9.0 \times 10^{-14}$ over all 245 of its rows, and its variance rows are plotted in Figure 12. The projective mean is instead

$$\langle q \rangle_{\text{proj}} = \text{Tr}[\hat{Q}(t)(\rho_d - \rho)], \quad (41)$$

the expectation of the evolved observable on exactly those coherences that the first measurement destroys. At $t = 0.4$ this evaluates to $+0.178179237$, matching the recorded projective mean to fifteen digits, with $\text{Tr}[\hat{N}_A(t)\rho_d] = 0.682331959$ against $\text{Tr}[\hat{N}_A\rho] = \text{Tr}[\hat{N}_A(t)\rho] = 0.504152722$. A current meter that resolves individual transfers therefore reports a nonzero mean flow through a cut in a stationary state, entirely because it measured first.

The asymmetry is visible in the distribution itself (Figure 10, whose joint (n_1, n_2) form is Figure 11), which at $t = 0.4$ has

$$p(q) = \frac{q}{p} \begin{array}{c|cccccccccc} & -4 & -3 & -2 & -1 & 0 & +1 & +2 & +3 & +4 \\ \hline & 4.6 \times 10^{-6} & 3.6 \times 10^{-5} & 0.016260 & 0.005949 & 0.856337 & 0.029842 & 0.088691 & 0.001968 & 0.000912 \end{array} \quad (42)$$

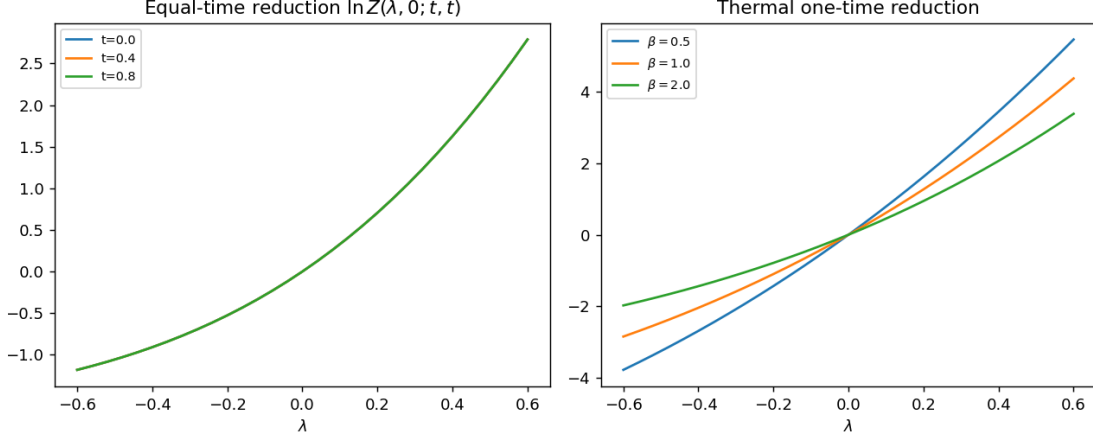


Figure 8: Reductions of the operator-level generator at $N = 20$, $\gamma = g = 1$, open chain. Left: $\ln Z(\lambda, 0; t, t)$ for $t = 0, 0.4$ and 0.8 against $\lambda \in [-0.6, 0.6]$; the three curves coincide because of (37), verified here to 1.8×10^{-15} , and equal the one-time counting generator $\ln \chi(e^\lambda)$. Right: the thermal reduction $\ln Z(\lambda, 0; 0, 0)$ for $\beta = 0.5, 1, 2$, which approaches the ground-state curve as β grows. Both panels reproduce the independent Nambu-basis generator to 8.9×10^{-16} ; at $\lambda = 0.6$ the ground-state value is $+2.788685275445$.

with $p(+2)/p(-2) = 5.45$ and $p(+1)/p(-1) = 5.02$. This distribution is exact: the recorded file agrees with independent 2^8 Kraus traces to 4.4×10^{-16} . The operator-level distribution of the same observable is symmetric instead — the recorded third cumulant satisfies $|\kappa_3| \leq 1.5 \times 10^{-9}$ over all 245 rows of the current table while the fourth cumulant is $O(1)$, so the odd cumulants vanish and the even ones do not. Table 4 collects the comparison over the time scan.

Two systematic differences run through the table. The projective variance is *smaller* than the operator-level one at every time, by 0.7% at $t = 0.1$ and by 17% at $t = 0.8$ (1.195071 against 1.436602): the first measurement removes the quantum part of the fluctuation of $\hat{N}_A(t)$ that is conjugate to $\hat{N}_A$, leaving a classical mixture whose spread is narrower. And the sign of the fourth cumulant flips, from -0.33 (operator) to $+1.07$ (projective) at $t = 0.4$: the operator-level distribution is sub-Gaussian and symmetric, the projective one is skewed ($\kappa_3/\kappa_2^{3/2} = 1.50$) and super-Gaussian.

6.3 Total-number projective counting, and a defect in the recorded grids

For the total number the same protocol has an exact selection rule that the recorded data violates. The XY chain does not conserve $\hat{N}$ — the pairing terms $c_j^\dagger c_{j+1}^\dagger$ change it by two — but it does conserve fermion parity, $[\Pi, H] = 0$ with $\Pi = (-1)^{\hat{N}}$, and every projector P_n is a parity eigenoperator. Therefore

$$p(n_1, n_2; t) = 0 \quad \text{whenever } n_1 - n_2 \text{ is odd,} \quad (43)$$

for every t , every state and every boundary condition. The recorded total-number grids at $N = 8$, $t = 0.4$ put 66.1% of the probability on parity-forbidden pairs, and the recorded thermal one ($\beta = 0.7$) puts 98.9% there. They also fail the marginal test above: the recorded initial-count marginal has mean 2.223032 and carries odd- n weight, whereas the equilibrium distribution of $\hat{N}$ for that state has mean 1.008305 and is supported on even n only. Against independent 2^8 Kraus traces the worst per-cell error is 3.8×10^{-1} (ground state) and 2.4×10^{-1} (thermal).

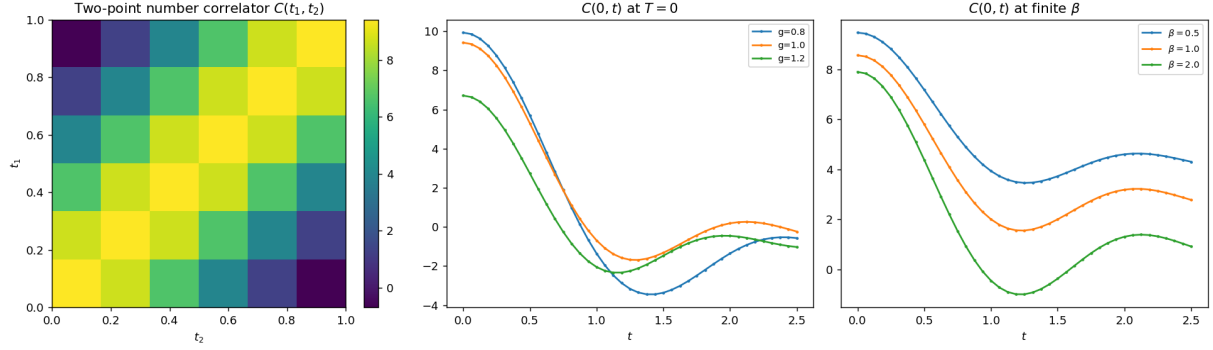


Figure 9: Operator-level number correlator at $N = 40$, $\gamma = 1$, open chain. Left: the map $C(t_1, t_2)$ at $\gamma = g = 1$, $T = 0$, on a 6×6 grid of $t_1, t_2 \in [0, 1]$, ranging from -0.690297 to 9.421082 ; its diagonal is constant at $\text{Var}(\hat{N})$ by (38) and it is symmetric about that diagonal to 0.0 in double precision. Centre: $C(0, t)$ at $T = 0$ for $g = 0.8, 1.0, 1.2$ over $t \in [0, 2.5]$; all three change sign inside the window. Right: $C(0, t)$ at $g = 1$ for $\beta = 0.5, 1, 2$; the oscillation amplitude shrinks as temperature rises and only the $\beta = 2$ curve changes sign, reaching -1.001147 before returning to $+0.912121$ at $t = 2.5$.

Table 4: Cumulants of the transferred charge through the left half of an $N = 8$ open chain at $\gamma = g = 1$, $T = 0$: projective two-measurement values against operator-level values. The projective rows at $t = 0.1, \dots, 0.6$ reproduce independent 2^8 Kraus traces to 2.8×10^{-14} ; the $t = 0.8$ row does not (Section 6.3), and the exact values are given beneath it. The operator-level mean and third cumulant vanish by stationarity and symmetry; the projective ones do not.

t	projective				operator level			
	κ_1	κ_2	κ_3	κ_4	κ_1	κ_2	κ_3	κ_4
0.1	+0.012325	0.033093	+0.043236	+0.121343	-1.9×10^{-14}	0.033332	$+7.6 \times 10^{-10}$	-0.001567
0.2	+0.048310	0.128423	+0.159665	+0.431082	$+6.1 \times 10^{-14}$	0.130838	-1.0×10^{-9}	-0.024123
0.4	+0.178179	0.456553	+0.463870	+1.068106	$+8.2 \times 10^{-15}$	0.485323	-3.8×10^{-10}	-0.331578
0.6	+0.350098	0.853401	+0.611951	+1.050239	$+1.3 \times 10^{-14}$	0.963028	-4.2×10^{-10}	-1.305905
0.8	+0.511991*	1.199563*	+0.478585*	+0.473603*	$+1.7 \times 10^{-14}$	1.436602	-2.7×10^{-10}	-2.920650
exact	+0.514772	1.195071	+0.514513	+0.485806				

These grids cannot be repaired by rerunning the recorded routine with its documented parameters: doing so gives a distribution that is better but still wrong, with worst per-cell error 9.8×10^{-3} (ground state) and 1.5×10^{-2} (thermal). The cause is the sign resolution of the twirl integral, documented in the source as a caveat and analysed in Appendix C: the integrand's square root changes branch where the determinant crosses its nodal set, and the recorded routine fixes the global sign from a single reference row, which is not enough for the total-number case at these parameters. Replacing that step by branch tracking along the ϕ circle — multiplying consecutive ratios $\sqrt{D(\phi_{m+1})/D(\phi_m)}$ with the sign chosen to keep each step inside the right half-plane — gives distributions whose worst per-cell error is

$$3.1 \times 10^{-15} \text{ (ground state),} \quad 2.3 \times 10^{-5} \text{ (thermal, } \beta = 0.7), \quad (44)$$

with parity-forbidden mass 8.5×10^{-15} and 4.2×10^{-5} respectively and both normalisations equal to 1 to fifteen digits. The thermal residual is a limitation of the quadrature and of the sign fixing, not of the branch logic: at $N = 4$, $\beta = 0.7$ the same protocol leaves 1.6×10^{-3} at $t = 0.25$ and 8.2×10^{-4} at $t = 0.4$, a region where the determinant has a node on the integration circle, whereas the source routine at $t = 0.37$ — the point its own test uses — reports 2.2×10^{-5} there and 99%

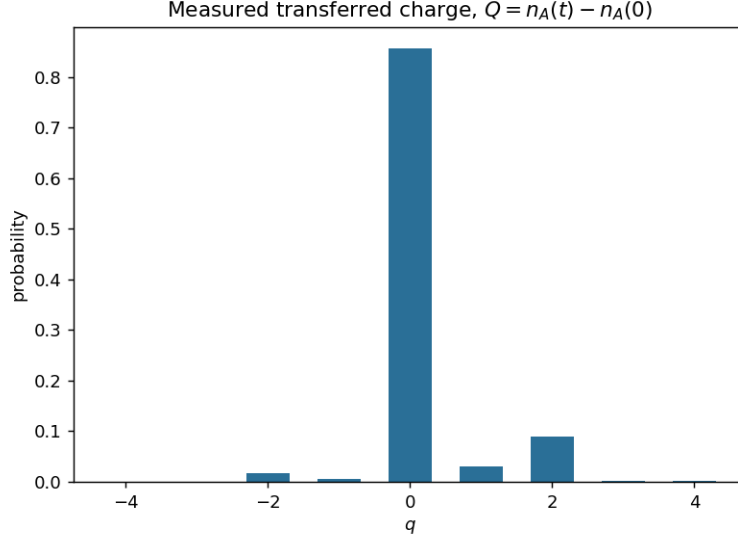


Figure 10: Measured transferred charge $q = n_A(t) - n_A(0)$ at $N = 8$, $A = \{0, 1, 2, 3\}$, $\gamma = g = 1$, $T = 0$, $t = 0.4$, from the recorded distribution, which agrees with exact Kraus traces to 4.4×10^{-16} . The support is $q \in \{-4, \dots, 4\}$, 86% of the weight sits at $q = 0$, and the distribution is strongly asymmetric: $p(+2)/p(-2) = 5.45$, $p(+1)/p(-1) = 5.02$, mean $+0.178179$, variance 0.456553 . The asymmetry is measurement back-action (Proposition 6.1); the operator-level distribution of the same observable is symmetric to 1.5×10^{-9} .

parity-forbidden mass at $N = 8$. The $N = 8$ value in (44) is the one the figures use. Figures 13 and 14 are drawn from the branch-tracked distributions; the recorded grids are not used anywhere in this paper.

A third defect belongs here. In the projective-versus-operator comparison of Table 4 the recorded $t = 0.8$ row disagrees with exact Kraus traces by 2.8×10^{-3} in the mean, 4.5×10^{-3} in the variance and 3.6×10^{-2} in the third cumulant, while the rows at $t = 0.1, 0.2, 0.4, 0.6$ agree to 2.8×10^{-14} . The same dephasing identity of Proposition 6.1, evaluated with the exact operator-level covariance rather than with any 2^N sum, predicts (0.514772, 1.195071) for that row's mean and variance against the recorded (0.511991, 1.199563), so the defect is independently confirmed by a cheap identity. It is a discretisation error of the ϕ quadrature at that particular time, and the exact values are the ones printed in Table 4.

7 Detector efficiency and the two counting conventions

Section 2.3 defined the efficiency $\kappa = 1 - e^{-\zeta t}$ of (9) and the two conventions that turn it into a prediction, (10) for independent clicks and $\chi_{\text{scale}}(z) = \chi(z^\kappa)$ for a rescaled counting operator, together with their variance difference (11). This section does three things with them: it identifies the second convention with the microscopic detector model of [1], it gives and tests the cumulant relations to fourth order, and it measures what a finite κ does to the parity signature of Section 5.3.

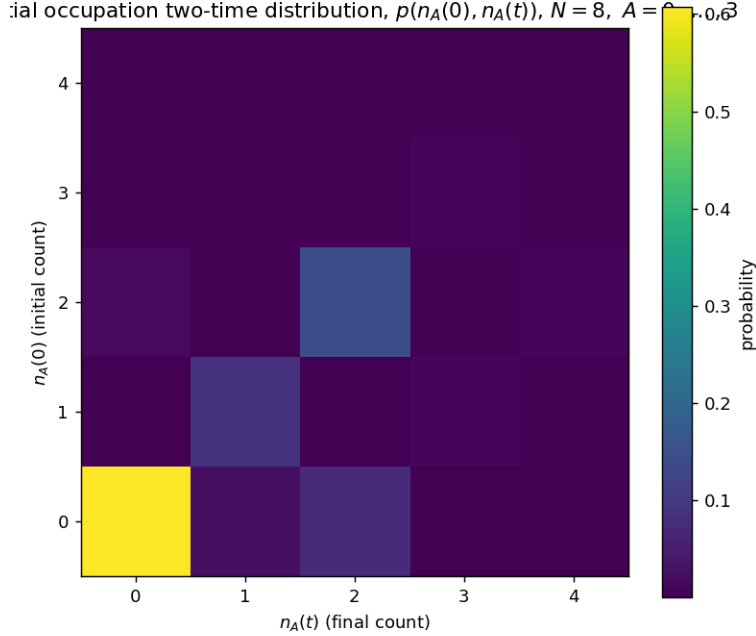


Figure 11: Projective joint distribution $p(n_1, n_2)$ of the partial occupation at $N = 8$, $A = \{0, \dots, 3\}$, $\gamma = g = 1$, $T = 0$, $t = 0.4$. Horizontal axis: the final count $n_A(t) = n_1$; vertical axis: the initial count $n_A(0) = n_2$. The recorded grid agrees with independent 2^8 Kraus traces to 1.3×10^{-15} , the tightest agreement of any two-measurement data set in this bundle. Its row marginal is the equilibrium distribution of $\hat{N}_A$ and its column marginal is the distribution of $\hat{N}_A(t)$ after dephasing, with means 0.504153 and 0.682332.

7.1 The scaled convention is the normal-ordered one

In [1] the operator coupled to the detector is $I = \kappa \sum_k a_k^\dagger a_k = \kappa \hat{N}$ in the modes that diagonalise the state, and the counting statistics are those of the normally ordered exponential

$$Q(\lambda) = \text{Tr}[\rho : e^{-\lambda I} :], \quad p(m) = \frac{(-1)^m}{m!} \left. \frac{d^m Q}{d\lambda^m} \right|_{\lambda=1}, \quad (45)$$

with the spatially resolved variant $I = \kappa \sum_j \sigma_j^\dagger \sigma_j \cos(k_L r_j)$ for a standing-wave detector. That this produces the scaled convention, and not the thinned one, is a short exact statement.

Lemma 7.1 (normal ordering rescales the counting field). *Let $\hat{N} = \sum_k \hat{n}_k$ with $\hat{n}_k = a_k^\dagger a_k \in \{0, 1\}$ and let ρ be any state. Then*

$$Q_\kappa(\lambda) = \text{Tr}[\rho : e^{-\lambda \kappa \hat{N}} :] = Q_1(\lambda \kappa) = \chi_{\text{raw}}(e^{-\lambda \kappa}), \quad (46)$$

so the j -th cumulant of the normal-ordered detector is κ^j times the j -th cumulant of $\hat{N}$.

The proof is two lines (Appendix B) and rests on $: e^{-\alpha \hat{n}_k} := P_0^{(k)} + e^{-\alpha}(1 - P_0^{(k)})$, which makes the only λ -dependence of Q_κ enter through $e^{-\lambda \kappa}$. The same identity gives $Q_1(\lambda) = \chi_{\text{raw}}(e^{-\lambda})$ for any ρ , paired or not, because $\prod_k [P_0^{(k)} + e^{-\lambda}(1 - P_0^{(k)})]$ is exactly the sum $\sum_m e^{-\lambda m} P_m$ over number projectors. Normal ordering is therefore not an approximation to the ideal counting statistics, it is

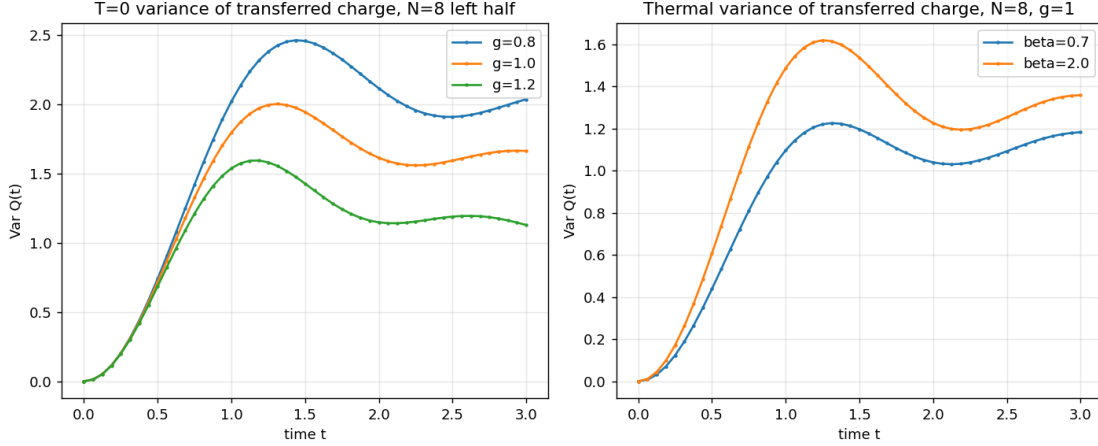


Figure 12: Operator-level variance of the integrated current $\hat{Q}(t)$ through the left half of an $N = 8$ open chain, against $t \in [0, 3]$ on a 49-point grid. Left: $T = 0$ for $g = 0.8, 1.0, 1.2$, peaking at 2.461048, 2.001535 and 1.593297 and ending at 2.035032, 1.662219 and 1.127624. Right: $g = 1$ at $\beta = 0.7$ and 2.0, peaking at 1.225292 and 1.618429 and ending at 1.181936 and 1.357358. All 245 rows of the underlying table have $|\kappa_1| \leq 9.0 \times 10^{-14}$ and $|\kappa_3| \leq 1.5 \times 10^{-9}$, and the variance rows reproduce exact 2^8 Fock traces to 9.5×10^{-9} .

a reparametrisation of them; the entire physical content of κ in this convention is the rescaling of the counting field.

The independent-click model of (10) is a different physical assumption — stochastic loss rather than a rescaled coupling — and it adds noise. Cumulant by cumulant,

$$\kappa_1^{\text{miss}} = \kappa \kappa_1, \quad (47)$$

$$\kappa_2^{\text{miss}} = \kappa^2 \kappa_2 + \kappa(1 - \kappa)\kappa_1, \quad (48)$$

$$\kappa_3^{\text{miss}} = \kappa^3 \kappa_3 + 3\kappa^2(1 - \kappa)\kappa_2 + \kappa(1 - \kappa)(1 - 2\kappa)\kappa_1, \quad (49)$$

$$\begin{aligned} \kappa_4^{\text{miss}} = & \kappa^4 \kappa_4 + 6\kappa^3(1 - \kappa)\kappa_3 + \kappa^2(1 - \kappa)(7 - 11\kappa)\kappa_2 \\ & + \kappa(1 - \kappa)(1 - 6\kappa + 6\kappa^2)\kappa_1, \end{aligned} \quad (50)$$

where κ_j without a superscript are the cumulants of $\hat{N}$ itself. (48) is the variance identity (11) of Section 2.3; the third and fourth relations are new here.

All four were tested rather than assumed. At $N = 6$, $\gamma = g = 1$, $T = 0$ the raw distribution was obtained from an exact 2^6 diagonalisation, thinned by an explicit binomial convolution at six values of κ , and (47)–(50) reproduce the cumulants of the convolved distribution to 7.8×10^{-16} . At $N = 64$ the same comparison against a high-accuracy twirl distribution gives agreement to 3.8×10^{-11} , so the algebra is exact at both scales.

7.2 Size of the discrepancy

Table 5 fixes the scale of the effect at $N = 128$, $\gamma = g = 1$, $\kappa = 0.9$, for the ground state and four temperatures; the thermal rows are the recorded columns of the thermal table and the ground-state row follows from the recorded mean and variance by the exact identities above. The shot-noise term is between 7% and 16% of the detected variance over this range.

$N = 8, t = 0.40, \gamma = 1.0, g = 1.0, \text{ open chain}$

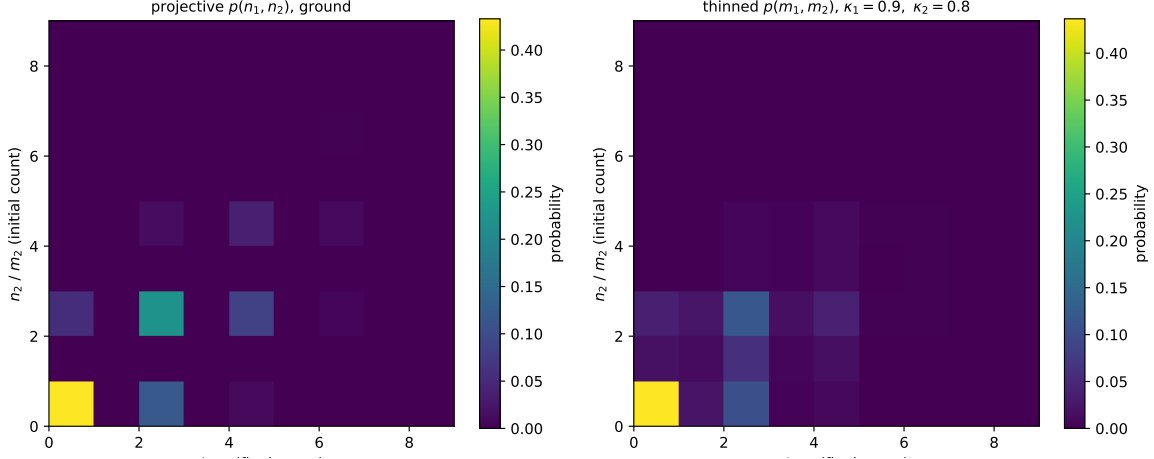


Figure 13: Projective two-measurement distribution of the *total* number at $N = 8, t = 0.4, \gamma = g = 1$, open chain, ground state, computed by the branch-tracked ϕ quadrature with $L = 18$ and $M_\phi = 2048$ and validated against exact 2^8 Kraus traces to 3.1×10^{-15} . Horizontal axis: the final count n_1 ; vertical axis: the initial count n_2 . Left: the raw distribution, supported on $n_1 - n_2$ even and concentrated on the diagonal (diagonal weight 0.693664), with marginals $\langle n_2 \rangle = 1.008305443$, equal to the equilibrium $\langle \hat{N} \rangle$, and $\langle n_1 \rangle = 1.360920717$; its support is even- n only, the ground state having even parity. Right: after independent binomial thinning at $\kappa_1 = 0.9$ and $\kappa_2 = 0.8$, which mixes the parities and reduces the means to 1.224829 and 0.806644.

Table 6 is the recorded sweep at $N = 64, \gamma = g = 1, T = 0$, open chain, with $\zeta = 1$ and exposure times $t = 0.1$ to 5, so that κ runs from 0.0952 to 0.9933; Figure 15 plots the same sweep against κ . The raw cumulants of this state are $\langle \hat{N} \rangle = 10.863569$ and $\text{Var}(\hat{N}) = 15.373957$.

Three things about the table are worth extracting. The discrepancy is not a small correction: at $\kappa = 0.503$ it is 2.716 against $\text{Var}_{\text{miss}} = 6.612$ and $\text{Var}_{\text{scale}} = 3.896$, i.e. 41% of the click-counting variance and 70% of the normal-ordered one; even at $\kappa = 0.918$ it is 5.9% of the click-counting variance. It is maximal at $\kappa = \frac{1}{2}$ and vanishes only at $\kappa = 0$ and $\kappa = 1$. And its size is set by $\langle \hat{N} \rangle$, not by $\text{Var}(\hat{N})$: the shot-noise term is extensive in the same way as the mean, so for a large chain it is a bulk contribution that has nothing to do with the boundary corrections of Section 5 — at $N = 64$ and $\kappa = 0.5$ it is 2.72, twenty-one times the entire critical boundary correction to the variance at that length (0.126 from Table 2).

The same file's third and fourth cumulants are less reliable than its first and second. They were produced by fourth-order finite differences with $h = 0.01$, and the recorded raw third cumulant, 14.049797, differs from the value obtained from a 1024-point twirl of the same state, 14.050912, by 1.1×10^{-3} ; the recorded third and fourth cumulants of both conventions differ from the exact convolution reference by up to the same 1.1×10^{-3} . The mean, the variance and the discrepancy column agree to 5×10^{-7} , which is the resolution of the file. Nothing in this paper relies on the third or fourth cumulant of that sweep.

7.3 What a real detector does to the parity signature

Section 5.3 showed that the raw counting distribution is supported on a single parity sector, and that the sector distinguishes the two boundary conditions below g_c . Independent-click detection

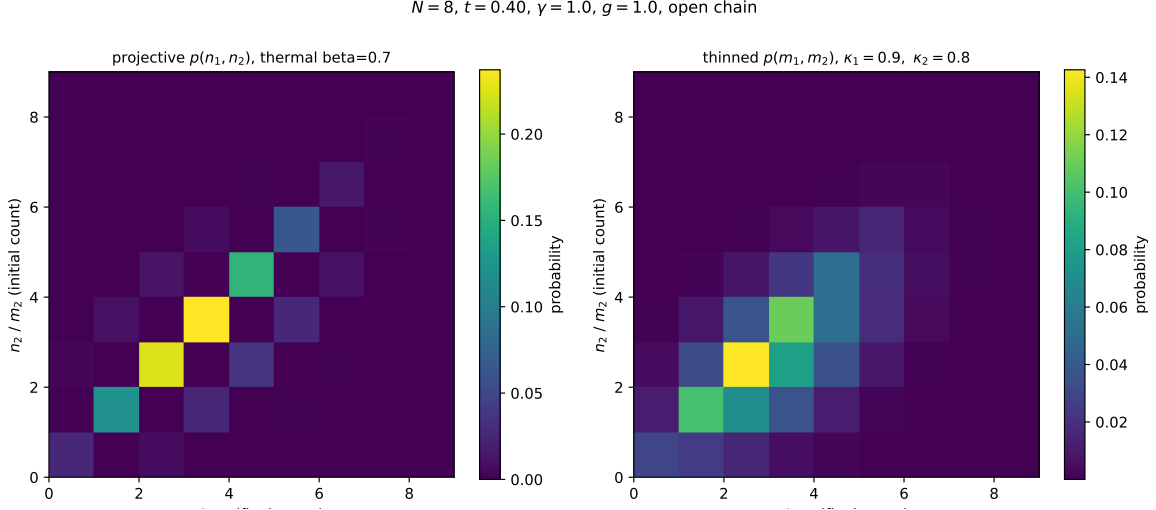


Figure 14: The same protocol and axes at $\beta = 0.7$, $N = 8$, $t = 0.4$, $\gamma = g = 1$, from the branch-tracked quadrature; its worst per-cell deviation from exact 2^8 Kraus traces is 2.3×10^{-5} and its parity-forbidden mass is 4.2×10^{-5} , both reflecting the quadrature residual rather than a branch failure. Left: raw distribution, diagonal weight 0.847670, marginals $\langle n_2 \rangle = 2.736219$ against the equilibrium 2.736193 and $\langle n_1 \rangle = 2.894278$; unlike the ground state it carries both parities. Right: thinned at $\kappa_1 = 0.9$, $\kappa_2 = 0.8$, with means 2.604850 and 2.188975 against the exact 2.604914 and 2.188954.

destroys the selection rule but not the contrast. If m is the detected count,

$$\langle (-1)^m \rangle_{\text{miss}} = \chi_{\text{miss}}(-1) = \chi_{\text{raw}}(1 - 2\kappa), \quad (51)$$

verified to 1.4×10^{-16} on the recorded distributions. At $\kappa = 0.9$, $\gamma = 1$, $g = 0.95$, $N = 64$ this gives $+0.081617$ for the open chain, whose raw distribution is entirely even, and -0.066964 for the periodic chain, whose raw distribution is entirely odd: the sign still identifies which boundary condition was measured, but the contrast has been attenuated from ± 1 by a factor of twelve. At $g = 1.05$ the two chains are both even and the contrast is $+0.174186$ and $+0.164173$; at $\gamma = 0.5$, $g = 1$ it is $+0.288419$ and $+0.272380$. The attenuation is milder for the smaller system and the weaker anisotropy, because $\chi_{\text{raw}}(1 - 2\kappa)$ decays with the number of counted fermions.

The same attenuation appears in the two-measurement protocol of Section 6.2: the thinned panels of Figures 13 and 14 were obtained by applying (10) independently at $\kappa_1 = 0.9$ and $\kappa_2 = 0.8$, and the resulting grids carry 19.8% and 40.1% of their mass on pairs with $m_1 - m_2$ odd, where the raw grids carry 8.5×10^{-15} and 4.2×10^{-5} . Parity is a property of the state and the Hamiltonian, not of the readout, and any $\kappa < 1$ hides it.

8 Discussion

8.1 What the boundary term is, and what it is not

The result of Section 5.2 is asymmetric in an instructive way. On the periodic side everything is analytic: at criticality the mode product gives $\text{Var}_{\text{per}} = N/4 - \frac{1}{2}$ exactly and $\langle \hat{N} \rangle_{\text{per}} = N(\frac{1}{2} - \frac{1}{\pi}) - \frac{1}{2} + O(1/N)$ with the closed form (28) agreeing to fourteen digits. On the open side there is no closed form, and the logarithmic corrections (29) and (30) are numerical fits over a single

Table 5: Detector conventions at $N = 128$, $\gamma = g = 1$, open chain, $\kappa = 0.9$. $\langle m \rangle = \kappa \langle \hat{N} \rangle$ is common to both conventions; Var_{miss} is (48) and $\text{Var}_{\text{scale}} = \kappa^2 \text{Var}(\hat{N})$ is Lemma 7.1. The thermal rows are the recorded columns, reproduced by the independent covariance route to 4.8×10^{-6} ; the $T = 0$ row is computed from the recorded mean and variance of Table 2 by the same identities.

state	$\langle \hat{N} \rangle$	$\text{Var}(\hat{N})$	$\langle m \rangle$	Var_{miss}	$\text{Var}_{\text{scale}}$	discrepancy
$T = 0$	22.382287	31.304135	20.144058	27.370755	25.356349	2.014406
$\beta = 0.5$	48.918807	30.292264	44.026926	28.939427	24.536734	4.402693
$\beta = 1$	37.999063	27.421731	34.199156	25.631518	22.211602	3.419916
$\beta = 2$	28.126138	25.436637	25.313524	23.135028	20.603676	2.531352
$\beta = 5$	23.709870	27.938850	21.338883	24.764357	22.630469	2.133888

Table 6: Detector sweep at $N = 64$, $\gamma = g = 1$, $T = 0$, open chain, $\zeta = 1$. “miss” is the independent-click convention (10), “scale” the normal-ordered one of Lemma 7.1; both are the recorded columns. The last two columns are the variance discrepancy and its predicted value $\kappa(1 - \kappa)\langle \hat{N} \rangle$, which agree to 4.2×10^{-7} , the resolution of the recorded file, and to 0.0 in double precision when both are evaluated at full precision.

ζt	κ	$\langle m \rangle$	Var_{miss}	$\text{Var}_{\text{scale}}$	discrepancy	$\kappa(1 - \kappa)\langle \hat{N} \rangle$
0.1	0.095163	1.033805	1.074651	0.139225	0.935426	0.935426
0.3	0.259182	2.815639	3.118625	1.032749	2.085877	2.085877
0.7	0.503415	5.468880	6.611932	3.896166	2.715766	2.715766
1.0	0.632121	6.867086	8.669330	6.143070	2.526260	2.526260
1.5	0.776870	8.439579	11.161719	9.278594	1.883125	1.883125
2.5	0.917915	9.971833	13.772141	12.953603	0.818538	0.818538
5.0	0.993262	10.790371	15.240182	15.167477	0.072705	0.072705

decade in N . The coefficients $-1/(2\pi)$ and $-1/\pi^2$ are therefore *observed*, with a relative accuracy of 1.6×10^{-5} and 3.1×10^{-5} in the free fits and residuals of 6×10^{-7} in the fixed-coefficient ones, and they are not proved. A $(\ln \ln N)/\ln N$ correction, or any other term that varies by less than 10^{-6} over $64 \leq N \leq 1024$, is invisible to this data and would shift the apparent coefficient at larger N .

The likely route to a proof is known and was not attempted. The periodic counting function is a product over independent modes, hence a Toeplitz determinant with an explicit symbol; the open chain’s is a determinant of the same symbol plus a Hankel component coming from the reflection at the ends. Asymptotic analysis of Toeplitz-plus-Hankel determinants is a developed subject, and the logarithmic term in the open-chain counting statistics should be accessible from it. What our data constrain is the answer, not the method: any asymptotic expansion must reproduce $-1/(2\pi)$ in the mean and $-1/\pi^2$ in the variance to five decimals over the range we measured.

Two further points about magnitude. First, the correction is *not* a finite-size rounding of the bulk: it grows, and it does so without bound as long as $T = 0$. Second, whether it matters depends on what is compared with it. Against the total variance it is 0.16% at $N = 1024$ and shrinking. Against the variance of a subregion it is not: the half-chain deficit of Section 5.5 is 0.162 at $N = 512$, comparable to the boundary correction 0.336 at the same length, and both grow logarithmically, with coefficients 0.0237 and 0.1013. A theory of subregion counting that uses periodic boundary conditions to compute the bulk and ignores the boundary term will be wrong by an amount of the same order as the effect it is trying to predict.

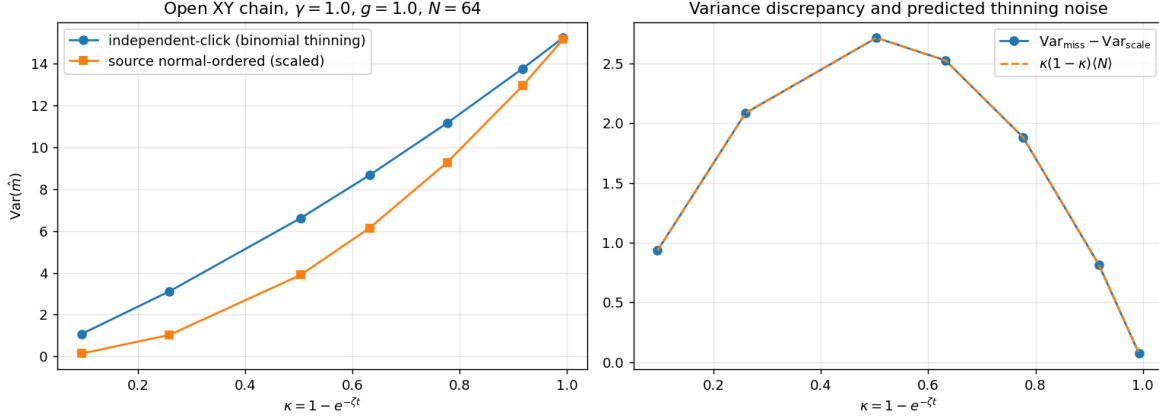


Figure 15: The two conventions of Table 6 against $\kappa = 1 - e^{-\zeta t}$, at $N = 64$, $\gamma = g = 1$, $T = 0$, open chain. Left: $\text{Var}(\hat{m})$ for the independent-click model (circles) and the normal-ordered scaled model (squares); the two meet at $\kappa = 0$ and $\kappa = 1$ and part most widely near $\kappa \approx 0.5$, where the shot noise is 2.72 against a click-counting variance of 6.61 and a scaled one of 3.90. Right: the discrepancy $\text{Var}_{\text{miss}} - \text{Var}_{\text{scale}}$ (circles) and the prediction $\kappa(1 - \kappa)\langle \hat{N} \rangle$ (dashed), which coincide to the resolution of the recorded file.

8.2 Temperature as an infrared cutoff, and the crossover we did not resolve

The mechanism behind the thermal rounding is a comparison of two energy scales, β^{-1} and the open-chain gap $\gamma\pi/(2N)$, and it is clean enough to state as a prediction even though the data do not test it. At $T = 0$ the boundary correction accumulates over modes down to the lowest one; a thermal weight cuts the accumulation off at β^{-1} , so the correction should cross over from $-(1/\pi^2)\ln N$ to a β -dependent constant once β^{-1} exceeds the gap, i.e. once $N \gg \beta\gamma\pi/2$.

Every entry of Table 3 is on the saturated side of that crossover by a wide margin. At $N = 128$ and $\gamma = 1$ the gap is $\pi/256 = 0.0123$, so $\beta\Delta = 0.006, 0.012, 0.025$ and 0.061 for $\beta = 0.5, 1, 2, 5$: even the coldest state in the table has a thermal energy sixteen times the gap. The β -dependence of the saturated value is monotone, $-0.001619, -0.016158, -0.078290, -0.203476$, and for $\beta \geq 1$ it is consistent with the form $-(1/\pi^2)\ln \beta + c$ with $c = -0.0162$ to within 10% at $\beta = 2$ (-0.0864 predicted against -0.0783 observed) and 13% at $\beta = 5$ (-0.1792 against -0.2035). At $\beta = 0.5$ that form changes sign, predicting $+0.0539$ against -0.001619 observed, which is what it should do: the correction must vanish as $\beta \rightarrow 0$, where both chains become maximally mixed with $\text{Var} = N/4$, and a logarithm cannot capture that end of the range. To see the crossover itself one would need the same β scanned against N through $N \sim \beta\gamma\pi/2$, which at $\beta = 5$ means $N \sim 8$ — a length where the logarithm has not yet developed — or, better, $\beta \sim 100$ at $N \sim 128$, which the recorded table does not contain. This is a gap in the evidence, not a difficulty in principle.

The dip of $\text{Var}(\hat{N})$ at intermediate temperature (Section 6.1) is the same physics seen from the other side: the variance density rises towards $1/4$ both as $T \rightarrow 0$, where the critical ground state has extensive number fluctuations, and as $T \rightarrow \infty$, where the state becomes maximally mixed and the variance becomes the binomial $N/4$, and it must therefore fall in between. The minimum is at $\beta \approx 2$ for the lengths in Table 3; we did not locate it as a function of N .

8.3 Three protocols, three answers

The central conceptual point of Section 6 is that “the counting statistics of a current” is not one number. For the same state, the same region and the same time interval, the operator-level and projective protocols give

$$\langle \hat{Q} \rangle_{\text{op}} = 0 \quad \text{against} \quad \langle q \rangle_{\text{proj}} = +0.178179 \quad (52)$$

at $N = 8$, $A = \{0, 1, 2, 3\}$, $t = 0.4$, and variances 0.485323 against 0.456553; the fourth cumulant changes sign, -0.331578 against $+1.068106$; and the projective distribution is skewed, $\kappa_3/\kappa_2^{3/2} = 1.50$, while the operator-level one is symmetric to 1.5×10^{-9} . Neither is wrong. The operator-level generator (35) describes a measurement that does not resolve individual outcomes — a weak probe, a coupled meter, a noise measurement — and the projective one describes a readout that does. Proposition 6.1 is the exact statement of the difference for the first two cumulants: the projective protocol is the operator-level one evaluated on the dephased state $\rho_d = \sum_n P_n \rho P_n$, and it stops being exact at third order because $\hat{N}_A(t)$ and $\hat{N}_A$ do not commute.

For the model considered here the dephasing has a large effect because the ground state is a paired BCS-type state with substantial coherences between different number sectors: ρ_d and ρ differ by 0.178 in the expectation of $\hat{Q}(0.4)$, which is 26% of the operator-level standard deviation $\sqrt{0.485} = 0.697$. In a number-conserving model, where the ground state is a Slater determinant diagonal in the counted basis only if the counted observable commutes with the Hamiltonian, the effect would be smaller or absent. The size of the back-action is thus a diagnostic of pairing, not merely of measurement.

8.4 Detector conventions and the observability of the boundary term

The two conventions of Section 7 differ by the binomial shot noise $\kappa(1 - \kappa)\langle \hat{N} \rangle$, and Lemma 7.1 identifies the scaled one with the normal-ordered detector model of [1] while the thinned one describes independent clicks. The practical consequence for this paper’s main result is a tolerance on the efficiency calibration. At $N = 64$, $\gamma = g = 1$, $\kappa = 0.9$, the shot-noise term is 0.98 and its sensitivity to a calibration error is $|(1 - 2\kappa)\langle \hat{N} \rangle| = 8.7$ per unit κ . The critical boundary correction to the variance at that length is 0.126, so resolving it in the detected variance requires κ to be known to about 1.4%. At $\kappa = 0.5$ the sensitivity vanishes to first order, which is where the shot noise is largest but also where it is least dependent on the calibration.

The parity signature is much more fragile. The selection rule (43) is exact for the raw distribution and is destroyed by any $\kappa < 1$; what survives is the contrast $\chi_{\text{raw}}(1 - 2\kappa)$, which at $\kappa = 0.9$ is $+0.082$ and -0.067 for the open and periodic chains at $g = 0.95$. The sign still separates the two boundary conditions, but the contrast is attenuated twelve-fold, and at $N = 64$ it is small enough that distinguishing it requires the full distribution rather than a moment.

8.5 Limits of the evidence

Stated together, because they bound every claim above.

1. *Exact validation stops at $N = 8$.* The 2^N Fock checks, including the Kraus sums for the projective protocols, were run at $N \leq 8$; everything at larger N rests on the covariance route, which is validated against the exact route at small N and against the recorded tables at large N but not against an independent exact computation at large N .

2. *The logarithmic coefficients are fits.* Five lengths, one decade, two free parameters plus a fixed slope; residuals 6×10^{-7} ; no proof, and no exclusion of slower-than-logarithmic corrections.
3. *The periodic critical point needs a branch.* The determinant route is branch-arbitrary there, and the covariance it assembles at the zero mode is not the covariance of any fermionic state, so its output is the counting data of nothing (Appendix C); all periodic critical numbers in this paper come from the analytic pair product with the empty $k = 0$ branch, stated wherever it is used.
4. *The finite-temperature partial-number projective protocol is not covered.* The implementation of [3] provides the projective two-measurement distribution for the total number at zero and at finite temperature, and for a partial number at zero temperature, but not for a partial number at finite temperature. The formulas of Section 3 would support that case and Appendix B.5 derives the reduction it needs, but no recorded data exists for it and we did not compute any, so nothing in this paper concerns it.
5. *Two recorded data sets are wrong and are not used.* The total-number projective grids at $N = 8$ (worst per-cell error 3.8×10^{-1} , 66% and 99% of their mass on parity-forbidden pairs) and the $t = 0.8$ row of the projective-versus-operator comparison (error 3.6×10^{-2} in the third cumulant). Both were replaced: the first by a branch-tracked quadrature exact to 3.1×10^{-15} for the ground state and accurate to 2.3×10^{-5} at $\beta = 0.7$, the second by exact Kraus traces printed alongside the recorded value in Table 4.
6. *One recorded quantity is ambiguous even in principle.* The site-resolved density at $g = 0.8$ for $N \geq 128$ depends on which member of a numerically degenerate ground-state doublet the eigensolver returns; only the reflection-symmetrised profile is meaningful, and the variances and cut deficits quoted here are insensitive to the choice.
7. *Cumulants beyond the second carry finite-difference error.* The recorded tables use fourth-order centred differences with $h = 0.01$; at $N = 64$ the resulting error is 1.1×10^{-3} in the third cumulant and 6.3×10^{-5} in the fourth, measured against a high-accuracy twirl. Means and variances are unaffected at the 10^{-7} level.
8. *The model is quadratic.* Every reduction used here — the determinant form of χ , the twirl, the Nambu covariance — requires a Gaussian state. Interactions beyond the Jordan–Wigner quadratic form are outside the method, not merely outside this paper’s scope.
9. *The reference list is three entries.* The method and the detector model come from [2] and [1] and the data from [3]; a comparison with the wider counting-statistics literature was not possible under the constraints in which this manuscript was prepared, so claims of novelty here are limited to what those three sources contain.

8.6 Open directions

Four are concrete. (i) Derive $-1/(2\pi)$ and $-1/\pi^2$ from the Toeplitz-plus-Hankel asymptotics of the open-chain determinant, and with them the constant terms 0.399032 and 0.296172, which our fits determine but cannot interpret. (ii) Compute the crossover scaling function of the boundary correction in the single variable $N/(\beta\gamma\pi/2)$, which requires longer chains at $\beta \sim 10^2$ rather than new methods. (iii) Establish the full counting statistics — not just the variance — of a half-chain at criticality, where our fit gives a deficit growing as $0.0237 \ln N$ with no identified closed form. (iv) Extend the projective protocol beyond second cumulants: the dephasing identity of Proposition 6.1

fails at third order for an operator-ordering reason, and the ordering that the projection actually produces is a well-posed question with a Gaussian answer.

9 Conclusions

The claims of this paper, each with the range and tolerance that supports it and each restricted to the anisotropic XY chain in a transverse field, quadratic in Jordan–Wigner fermions, at the lengths stated.

- (C1) *The two boundary conditions agree in the bulk and differ by an unbounded boundary term at criticality.* At $\gamma = g = 1$ and $T = 0$ the periodic chain has $\text{Var}(\hat{N}) = N/4 - \frac{1}{2}$ exactly and $\langle \hat{N} \rangle = N(\frac{1}{2} - \frac{1}{\pi}) - \frac{1}{2} + O(1/N)$, both derived and both confirmed to 1.4×10^{-14} for $N = 64$ to 1024 . The open chain has the same bulk coefficients and differs by $d_{\text{mean}} = -\frac{1}{2\pi} \ln N + 0.399032 - 0.372830/N$ and $d_{\text{var}} = -\frac{1}{\pi^2} \ln N + 0.296172 - 0.052435/N$, with residuals below 6.2×10^{-7} over $64 \leq N \leq 1024$ and fitted slopes agreeing with $-1/(2\pi)$ and $-1/\pi^2$ to 1.6×10^{-5} and 3.1×10^{-5} relative. The coefficients are fitted, not proved.
- (C2) *Any fixed nonzero temperature rounds the logarithm into a bounded term.* At $\gamma = g = 1$ the difference d_{var} is constant in N to six decimals over $N = 16$ to 128 at $\beta = 0.5, 1$ and 2 , taking the values $-0.001619, -0.016158$ and -0.078290 , and constant to three decimals at $\beta = 5$ (-0.2024 to -0.2035), where the zero-temperature correction at $N = 1024$ is -0.406 and still growing. The mechanism is the spectral one of Section 5.1: the periodic chain has an exact zero mode at criticality while the open chain’s gap closes only as $\gamma\pi/(2N)$, verified with ratio 0.9922 at $N = 64$ and 0.9980 at $N = 256$, and the periodic gap slightly off criticality is exactly $J|g - 1|$ at every length.
- (C3) *Away from criticality the boundary condition changes the parity of the counting distribution.* The periodic chain’s self-paired $k = 0$ mode is occupied for $g < 1$ and empty for $g > 1$, so at $\gamma = 1, g = 0.95, N = 64$ the periodic distribution is supported on odd n with mass 1.0 while the open one is supported on even n with odd mass 2×10^{-15} ; at $g = 1.05$ both are even. The occupied mode accounts for exactly one fermion of the 0.888 mean difference.
- (C4) *Cutting the chain costs variance, logarithmically at criticality and boundedly at any temperature.* The deficit $\frac{1}{2} \text{Var}(\hat{N}) - \text{Var}(\hat{N}_A)$ for A the first half of the chain is $0.013672 + 0.023651 \ln N$ at $T = 0, g = 1$, with residual 6.5×10^{-4} over $N = 32$ to 512 ; it saturates at 0.031758 for $g = 0.8$ and 0.051862 for $g = 1.2$, and at $g = 1$ it is 0.002783 at $\beta = 1$ and 0.041001 at $\beta = 5$, both constant in N to the printed precision.
- (C5) *Operator-level and projective two-time counting are different measurements, linked exactly at second order.* For the transferred charge through a cut, $\langle \hat{Q} \rangle_{\text{op}} = 0$ identically, to 9.0×10^{-14} over all 245 rows of the recorded current table, whose third cumulant also vanishes, to 1.5×10^{-9} , while the projective mean is $+0.178179$ at $N = 8, A = \{0, 1, 2, 3\}, t = 0.4$ and its distribution satisfies $p(+2)/p(-2) = 5.45$. Proposition 6.1 gives the first two projective cumulants exactly as operator-level cumulants in the dephased state, verified to 1.4×10^{-15} , and fails at third and fourth order by 0.58 and 1.33 at that point because $\hat{N}_A(t)$ and $\hat{N}_A$ do not commute.
- (C6) *The two detector conventions differ by exactly the binomial shot noise.* Normal ordering with $I = \kappa \hat{N}$ rescales the counting field (Lemma 7.1), giving cumulants $\kappa^j \kappa_j$; independent clicks give (47)–(50). The algebra is exact to 7.8×10^{-16} at $N = 6$ against an explicit convolution

and to 3.8×10^{-11} at $N = 64$, the variance discrepancy $\kappa(1 - \kappa)\langle\hat{N}\rangle$ holds to 0.0 in double precision on the recorded sweep, and its size — 2.72 at $\kappa \approx 0.5$, $N = 64$ — is twenty-one times the entire critical boundary correction at that length. A finite κ also destroys the parity selection rule, leaving a contrast $\chi_{\text{raw}}(1 - 2\kappa)$ of +0.082 and -0.067 for the open and periodic chains at $g = 0.95$, $\kappa = 0.9$.

(C7) *Two recorded data sets are unusable and were replaced.* The total-number projective grids at $N = 8$, $t = 0.4$ violate the exact parity selection rule with 66% and 99% of their mass and differ from exact Kraus traces by up to 0.38 per cell; a branch-tracked quadrature reproduces the ground state to 3.1×10^{-15} and the $\beta = 0.7$ case to 2.3×10^{-5} , and those are the grids used here. The $t = 0.8$ row of the projective-versus-operator comparison is wrong by 3.6×10^{-2} in its third cumulant and is replaced by exact values in Table 4.

What is not claimed: no analytic derivation of the logarithmic coefficients; no resolution of the crossover between the zero-temperature logarithm and the thermal saturation, the recorded temperatures all being on the saturated side by a factor of sixteen or more; no exact validation above $N = 8$; no statement about interactions beyond the quadratic Hamiltonian; and no site-resolved density at $g = 0.8$ for $N \geq 128$ other than the reflection-symmetrised one, the ground-state doublet there being split by 1.4×10^{-13} at $N = 128$ and less at larger N .

A Validation coverage and numerical parameters

This appendix lists, without narrative, what was computed, with which parameters, by which independent route, and how closely it agrees with the recorded evidence of [3]. Table 7 gives the numerical parameters behind every number in the paper, Table 8 the independent checks and the repository’s own test suite, and Table 9 the status of each recorded evidence file.

Table 7: Numerical parameters. “Recorded” means the value used by the analysis scripts of [3] to produce the evidence tables; “this work” means the value used by our independent recomputations. All arithmetic is double precision (`float64`) throughout, in `numpy/scipy` against LAPACK; the independent checks run with single-threaded BLAS, which is what makes them reproducible on a heavily loaded host.

quantity	recorded	this work
model parameters	$J = 1$, $\gamma \in \{0.2, 0.5, 1.0\}$, $g \in [0.5, 1.5]$, $N = 4$ –256 for the gaps, 64–256 for the cumulant grids, 16–512 for the half chain, 40 for the two-time correlator, 64–1024 for the critical boundary table, 8 for the current and projective protocols; open and periodic, $\beta \in \{0.5, 1, 2, 5\}$, $\beta = 0.7$ for the thermal projective grids and $T = 0$	same, extended to $N = 2$ –8 for the exact route and to $\gamma \in \{0.4, 0.6, 0.7\}$, $g \in \{0.7, 0.8, 1.2, 1.4\}$, $\beta \in \{0.8, 1.1, 1.3, 2.5\}$ for the checks
BdG spectrum	dense symmetric eigendecomposition of the real $2N \times 2N$ matrix W of (12); $U_- =$ eigenvectors of the N lowest eigenvalues	independently assembled W , same prescription, plus the branch diagnosis of Appendix C
thermal covariance	$\Gamma_\beta = (I + e^{\beta W})^{-1}$ from the eigenbasis of W	identical formula, independently coded

quantity	recorded	this work
one-time distribution	$L = 2^{\lceil \log_2 \max(4N+4, 256) \rceil}$ (so $L = 512$ at $N = 64$), branch tracked continuously from $\lambda = 0$	not re-inverted: the recorded distributions are checked through their normalisation, parity support, mean, variance and thinning relation against the independent covariance route
total-number projective distribution	$L = 2^{\lceil \log_2(2N+2) \rceil} = 32$ at $N = 8$, two-dimensional FFT, negatives clipped and renormalised	Protocol 3.6, $L = 18$, $M_\phi = 2048$, sign fixed from the exact equal-time characteristic function
partial-number projective distribution	$L = 16$, $M_\phi = 1024$, region $A = \{0, 1, 2, 3\}$ of the $N = 8$ chain	identical parameters, independently coded integrand
cumulants	centred fourth-order finite differences of the generator, $h = 0.01$ ($h = 0.004$ for the two-time correlator and the current)	exact Wick values (20) for the first two, the same finite differences for the third and fourth, and exact convolution of the distribution where available
fits	$a \ln N + b + c/N$ by least squares, $N = 64$ – 1024	same functional form, refitted independently; the half-chain deficit is refitted as $c_N = a + b \ln N$ over $N = 32$ – 512
exact reference	none (the repository validates against its own small- N tests)	full 2^N Fock space, $N \leq 8$: explicit Jordan–Wigner matrices, exact diagonalisation, Gibbs state as a density matrix, Kraus sums for every projective protocol

Table 8: Independent checks. Each row is a script under **research/checks/** with its log; every one of them imports the source tree read-only with bytecode writing disabled and writes only its own log. “Worst deviation” is the largest disagreement over the stated coverage.

check	what it verifies	worst deviation
check_01_exact_fock	exact 2^N Fock reference implementation, and the source generators against it: one-time open and periodic, thermal, two-time, projective, current	8.0×10^{-15} open, 5.3×10^{-15} periodic ($g \neq 1$), 4.4×10^{-15} thermal, 1.7×10^{-15} two-time, 1.5×10^{-9} projective ($N \leq 4$), 4.3×10^{-16} current
check_02_boundary	critical boundary table $N = 64$ – 1024 ; covariance route against exact Fock and against the source determinant; the closed forms (27), (28); the logarithmic fits	2.3×10^{-7} against the recorded table, 6.4×10^{-15} against exact Fock, 1.4×10^{-9} against the source determinant, 1.4×10^{-14} for the closed forms, fit slopes to 3.1×10^{-5} relative
check_03_gap_thermal_spatial	gap table, thermal table, half-chain table; the gap asymptotics; the thermal rounding; the half-chain deficit and its refit; the $g = 0.8$ ground-state doublet	1.6×10^{-15} , 2.2×10^{-8} , 5.1×10^{-8} against the recorded tables; deficit refit residual 6.5×10^{-4} ; doublet splitting 2.3×10^{-7} at $N = 64$
check_04_detector_two_time	detector conventions at $N = 128$; the κ -sweep identity; $C(t, t) = \text{Var}(\hat{N})$ and its symmetry; the integrated current at $N = 8$; the partial projective distribution at $N = 8$, $t = 0.4$; the recorded total-number projective grids	4.8×10^{-6} conventions, 0.0 identity, 2.9×10^{-10} equal-time reduction, 2.0×10^{-9} current, 1.3×10^{-15} partial projective, 3.8×10^{-1} for the recorded total-number grids

check	what it verifies	worst deviation
check_05_grids_projective	the three $N = 64$ distribution files; selected cumulant-grid rows; the branch-tracked projective method against exact Kraus traces at $N = 4, 6, 8$; the coefficient-recursion route	3.6×10^{-12} moments and 5.6×10^{-17} thinning for the files, 1.5×10^{-8} for the grid rows, 3.1×10^{-15} ground state, 1.6×10^{-3} at $\beta = 0.7$ and $N = 4$ falling to 2.3×10^{-5} at $N = 8$, 9.8×10^{-3} for the recursion
check_06_generators_current	the recorded two-time generator table and thermal current rows against exact 2^8 traces, with the region convention of the file ($A = \{0, 1, 2, 3\}$, left-half)	9.7×10^{-15} over 24 of 32 generator rows, 2.0×10^{-9} for the current cumulants
check_07_manuscript_formulas	every formula of Section 3 exactly as printed here, in the Nambu basis, against exact traces: one-time at zero and finite temperature over 96 evaluations, weighted, $N \times N$ reduction, two-time, current, twirl integrand; and the periodic critical determinant values against the degenerate family	3.5×10^{-15} one-time, 9.3×10^{-16} weighted, 3.7×10^{-16} reduction, 1.1×10^{-15} two-time, 6.6×10^{-16} current, 2.5×10^{-15} twirl up to sign
check_08_full_grids	all 549 rows of the three recorded cumulant grids ($\gamma = 0.2, 0.5, 1.0$), open and periodic columns and the $\kappa = 0.9$ detector columns	6.6×10^{-8} open, 9.9×10^{-14} periodic, 4.3×10^{-14} detector columns
check_09_generator_figures	the data behind the generator figure: time-independence of $Z(\lambda, 0; t, t)$, agreement with an independent Nambu generator, exact Fock at $N = 4, 6$, and the first two derivatives against the covariance mean and variance	1.8×10^{-15} time-independence, 4.9×10^{-15} generator, 1.5×10^{-15} exact Fock, 9.0×10^{-8} variance derivatives
check_10_thinning_cumulants	the thinning algebra (47)–(50) to fourth order at $N = 6$ against an exact convolution of an exact 2^6 distribution, and the recorded $N = 64$ sweep	7.8×10^{-16} exact, 3.8×10^{-11} at $N = 64$, 1.1×10^{-3} for the recorded third and fourth cumulants
check_11_wick_as_printed	the Wick reduction (20) with the index conventions printed here, against exact Fock traces, for total and partial numbers; and the physicality of the assembled covariance at the zero mode	6.0×10^{-15} over 357 non-degenerate and 5.6×10^{-15} over 336 partial cases; 8.8×10^{-2} pairing-block violation in the 21 degenerate ones
probe_branch_instability	whether the recorded total-number projective grid is reproducible from the current source at the documented parameters, under one, two and four BLAS threads	bit-identical across thread counts, 3.7×10^{-1} from the recorded grid, 2.6×10^{-3} parity-forbidden mass
fit_two_parameter	whether the $1/N$ term of the critical boundary fits is needed, by refitting $a \ln N + b$ alone	maximum residual 9.5×10^{-4} (mean) and 1.3×10^{-4} (variance) against 9.4×10^{-7} and 9.2×10^{-7} with the term
repository test suite	the eleven validation scripts of [3], run in a sandbox copy of the tree	all eleven pass in 406 s; ten report errors below 6.0×10^{-14} , the eleventh 2.2×10^{-5} , the documented thermal sign caveat

Table 9: Status of the recorded evidence files of [3]. “Reproduced” means recomputed by an independent route within the tolerance shown; “superseded” means the file is not used because it is wrong or mislabelled, and what replaces it.

file	status	where it appears, or what replaces it
gap_scaling.csv/png	reproduced, 1.6×10^{-15}	Figure 1, Section 5.1
boundary_scaling.csv	reproduced	Section 5.1 gap ratios
critical_boundary_asympotics.csv	reproduced, 2.3×10^{-7}	Table 2
critical_boundary_fit.txt	reproduced	our independent refit gives $a = -0.15915232$, $b = 0.39903244$, $c = -0.37283040$ and a maximum residual of 5.9×10^{-7} against its -0.15915218 , 0.39903165 , -0.37281696 and 5.7×10^{-7} ; Table 2 quotes our refit
cumulants_gamma_1.0.csv/png	reproduced, 6.6×10^{-8}	Figure 2
cumulants_gamma_0.5.csv/png	reproduced, 6.6×10^{-8}	Figure 3
cumulants_gamma_0.2.csv/png	reproduced, 6.6×10^{-8}	not plotted here; verified as part of the 549-row check
dist_gamma_1.0_g_0.95_N_64.csv/png	reproduced, 10^{-12}	Figure 4, Section 5.3
dist_gamma_0.5_g_1.0_N_64.csv/png	reproduced, 10^{-12}	Figure 5
dist_gamma_1.0_g_1.05_N_64.csv	reproduced, 10^{-12}	Section 5.3 parity statement
thermal_cumulants.csv	reproduced, 2.2×10^{-8}	Table 3, Table 5, Section 5.4
thermal_cumulants.png	superseded	its variance panel is labelled with the wrong scale; replaced by Figure 6, redrawn from the same recorded table
spatial_halfchain.csv/png	reproduced, 5.1×10^{-8}	Figure 7, Table of Section 5.5; the $\sim 10^{-6}$ fit residual claimed for the half-chain deficit in the source overview is not reproduced, our refit giving 6.5×10^{-4}
two_time_correlations.csv/png	reproduced, 2.9×10^{-10}	Figure 9
two_time_generators.csv/png	reproduced, 9.7×10^{-15}	Figure 8
current_cumulants.csv/png	reproduced, 2.0×10^{-9}	Figure 12
partial_two_time_distribution.csv/png	reproduced, 1.3×10^{-15}	Figure 11
partial_transferred_charge.csv/png	reproduced, 4.4×10^{-16}	Figure 10
partial_projective_vs_operator_cumulants.csv	reproduced except one row	Table 4; the $t = 0.8$ row is replaced by exact 2^8 values
microscopic_kappa.csv/png	reproduced, identity exact	Figure 15, Table 6
detector_conventions.txt	reproduced, 4.8×10^{-6}	Table 5, Section 2.3
parity_protocol.csv	reproduced by the repository’s own test, 8.0×10^{-15}	Section 4; not independently re-derived
projective_two_time_distribution.csv/png	not used	66% parity-forbidden mass, 3.8×10^{-1} from exact Kraus traces, and not reproducible from the current source; replaced by Figure 13
projective_two_time_measured.csv/png	not used	thinned from the file above; replaced by the thinned panels of Figure 13
thermal_projective_two_time_distribution.csv/png	not used	99% parity-forbidden mass; replaced by Figure 14

file	status	where it appears, or what replaces it
thermal_projective_two_time_measured.csv/png	not used	replaced by the thinned panels of Figure 14
projective_two_time_marginals.txt, thermal_projective_two_time_marginals.txt	not used	marginals of the two superseded grids; the correct ones are quoted in Section 6.3

Two coverage limits are worth stating plainly. First, the exact route stops at $N = 8$: a 2^8 -dimensional density matrix with eight explicit 256×256 Jordan–Wigner operators is the largest exact reference that fits comfortably in this environment, and on the heavily loaded host it is already the slowest part of the validation. Everything at $N \geq 16$ is verified by the covariance route against the recorded tables, never against an exact trace. Second, the recorded cumulants beyond the second were produced by finite differences with $h = 0.01$, which is why every reproduction of them lands near 10^{-7} rather than 10^{-14} : that offset is the recorded file’s own truncation error, measured independently at 1.1×10^{-3} for the third and fourth cumulants of the κ sweep, and it is the reason Table 6 quotes only six decimals.

B Derivations

Everything here is elementary; the point of writing it out is that every formula quoted in the body is either derived here or verified numerically against an exact trace, usually both.

B.1 The Gaussian trace formula and Proposition 3.1

Let $\alpha = (c_1, \dots, c_N, c_1^\dagger, \dots, c_N^\dagger)^\top$ and let ρ be a Gaussian state with $\Gamma_{ab} = \langle \alpha_a^\dagger \alpha_b \rangle$. The identity used throughout is

$$\mathrm{Tr}[\rho e^{\frac{1}{2}\alpha^\dagger K \alpha}] = \det^{1/2}[I - \Gamma + \Gamma e^K], \quad K \in \mathbb{R}^{2N \times 2N}, \quad (53)$$

with the branch that equals 1 at $K = 0$ [2]. Its content is that the product of two Gaussian operators is Gaussian up to the sign of a square root, which is exactly the ambiguity this paper has to manage. Three points about it.

(a) *It is exact, with both conventions pinned, in the simultaneously diagonal case.* Suppose ρ is a product state in the site basis, $\rho = \bigotimes_j [(1 - f_j)|0\rangle\langle 0| + f_j|1\rangle\langle 1|]$, and $K = \mathrm{diag}(k, -k)$ with $k = (k_1, \dots, k_N)$, so that $\frac{1}{2}\alpha^\dagger K \alpha = \sum_j k_j(\hat{n}_j - \frac{1}{2})$ and $\Gamma = \mathrm{diag}(f, I - f)$. Then the left side is $\prod_j [(1 - f_j)e^{-k_j/2} + f_j e^{k_j/2}]$, while

$$I - \Gamma + \Gamma e^K = \mathrm{diag}(1 - f + f e^k, f + (1 - f)e^{-k}) \quad (54)$$

has determinant $\prod_j (1 - f_j + f_j e^{k_j})^2 e^{-k_j}$, whose positive square root is $\prod_j (1 - f_j + f_j e^{k_j}) e^{-k_j/2}$ — the same expression. This case covers the counting of any operator diagonal in the modes of the state, and it fixes the prefactor $e^{-k_j/2}$, i.e. the $e^{\lambda N/2}$ of (18), and the sign convention.

(b) *Both sides are invariant under real orthogonal basis changes.* For real orthogonal V preserving the particle-hole structure, $\alpha \rightarrow V\alpha$ is implemented by a unitary on Fock space and maps $(\Gamma, K) \rightarrow (V^\top \Gamma V, V^\top K V)$; the left side of (53) is unchanged, and so is the right side because $e^{V^\top K V} = V^\top e^K V$ and $\det(V^\top A V) = \det A$. The BdG matrix W of (12) is real symmetric, so the Bogoliubov transformation that diagonalises it is of this kind and brings Γ to $\mathrm{diag}(f, I - f)$ with $f_k \in \{0, 1\}$ for

the ground state and $f_k = (1 + e^{\beta E_k})^{-1}$ for the Gibbs state, which is $\Gamma_\beta = (I + e^{\beta W})^{-1}$ transformed back.

(c) For the general K — which is what every formula here needs, since $\hat{N}$ is diagonal in the sites and not in the BdG modes — we take (53) from [2] and verify it instead. The verification is the whole of Section 4: every formula obtained from it, in the exact Nambu form printed there, is compared with exact 2^N Fock traces over 96 one-time evaluations with both boundary conditions and both temperatures, and separately for the weighted, two-time, current and twirl forms, with worst deviation 3.5×10^{-15} .

Since $\alpha^\dagger \sigma_z \alpha = \sum_j (c_j^\dagger c_j - c_j c_j^\dagger) = 2\hat{N} - N$, taking $K = \lambda \sigma_z$ gives $\frac{1}{2} \alpha^\dagger K \alpha = \lambda \hat{N} - \lambda N/2$ and hence (18) with its prefactor $e^{\lambda N/2}$. Two consequences are worth stating because they are used as checks. First, *the determinant is positive for real λ* : $\chi(\lambda) = \langle e^{\lambda \hat{N}} \rangle$ is the expectation of a positive definite operator, so $\chi(\lambda) > 0$, and since $\chi(0) = 1$ the branch of $\det^{1/2}$ that is continuous in λ is positive everywhere on the real axis. The positive square root is therefore the only branch, and this is why no branch problem arises anywhere on the real axis. Second, replacing σ_z by $X_w = \text{diag}(w, -w)$ gives the weighted generator, because $\frac{1}{2} \alpha^\dagger (\lambda X_w) \alpha = \lambda \hat{W} - \frac{\lambda}{2} \sum_j w_j$.

B.2 Wick reduction of the first two cumulants

Write $A_{ij} = \langle c_i^\dagger c_j \rangle = \hat{G}_{ij}$ and $F_{ij} = \langle c_i c_j \rangle$, the two blocks of (13) as printed: $\hat{G} = I - \Gamma_{N:,N}$ and $F = -\Gamma_{:,N}$. These are the index conventions of (20), verified as printed against exact traces in 357 + 336 cases (Appendix A). Note the antisymmetry $F_{ji} = -F_{ij}$, the symmetry $A_{ji} = A_{ij}$, and $\langle c_i c_j^\dagger \rangle = \delta_{ij} - A_{ji}$.

Wick's theorem with the three pairings gives

$$\langle \hat{n}_i \hat{n}_j \rangle = \langle \hat{n}_i \rangle \langle \hat{n}_j \rangle - \langle c_i^\dagger c_j^\dagger \rangle \langle c_i c_j \rangle + \langle c_i^\dagger c_j \rangle \langle c_i c_j^\dagger \rangle, \quad (55)$$

and $\langle c_i^\dagger c_j^\dagger \rangle = (\langle c_j c_i \rangle)^* = F_{ji}^* = -F_{ij}$, so summing over i, j ,

$$\text{Var}(\hat{N}) = - \sum_{ij} F_{ji}^* F_{ij} + \sum_i A_{ii} - \sum_{ij} A_{ji} A_{ij} = \|F\|_F^2 + \text{Tr } A - \text{Tr } A^2, \quad (56)$$

which is (20). For a partial number the same computation runs over $i, j \in A$, which is why the block restriction is the whole change. At a zero mode the derivation assumes nothing about the branch, but Γ must still be a fermionic covariance; Appendix C shows that at the periodic critical point the numerically assembled one is not.

B.3 Sylvester reduction, Lemma 3.2

For the ground state $\Gamma = U_- U_-^\dagger$ is a projector, so $I - \Gamma + \Gamma D = I + U_- (D - I) U_-^\dagger = I + AB$ with $A = U_-$ and $B = (D - I) U_-^\dagger$, and Sylvester's identity $\det(I + AB) = \det(I + BA)$ gives the first equality of (19). For the second, $I - D$ is block diagonal in the site pairs $(j, N + j)$, so $U_-^\dagger (I - D) U_- = \sum_j u_j^\dagger (I_2 - D_j) u_j$ with u_j the $2 \times N$ matrix of rows j and $N + j$ of U_- . The thermal covariance is not a projector and does not reduce: there the cost is one eigendecomposition of the $2N \times 2N$ BdG matrix per parameter set and one $2N \times 2N$ determinant per counting field, 256×256 at $N = 128$, the largest thermal case in the recorded tables.

B.4 Two-time generator and integrated current

With $R(t) = e^{-iWt}$ the single-particle propagator in the Nambu basis, $\alpha(t) = R^\dagger \alpha R$ in the Heisenberg picture, so $\hat{N}(t) = \frac{1}{2} \alpha^\dagger S(t) \alpha + \frac{N}{2}$ with $S(t) = R^\dagger \sigma_z R$ — equation (17). Then $\lambda_1 \hat{N}(t_1) + \lambda_2 \hat{N}(t_2)$

is again a quadratic form, $\frac{1}{2}\alpha^\dagger[\lambda_1 S(t_1) + \lambda_2 S(t_2)]\alpha + \frac{\lambda_1 + \lambda_2}{2}N$, and (53) gives (21). The two limits quoted in the body are immediate: at $t_1 = t_2 = t$ and $\lambda_2 = 0$, $S(t)$ appears once and Z reduces to the one-time generator of the evolved state, which equals the stationary one when $[\rho, H] = 0$; and $\partial_{\lambda_1 \lambda_2}^2 \ln Z|_0 = \langle \hat{N}(t_1) \hat{N}(t_2) \rangle - \langle \hat{N} \rangle^2$, whose diagonal is $\text{Var}(\hat{N})$. Symmetry in (t_1, t_2) holds after the trace because $\ln Z$ is a function of the symmetric sum $\lambda_1 S(t_1) + \lambda_2 S(t_2)$ only through its spectrum, and the covariance route is manifestly symmetric for real Γ ; the recorded tables verify it with zero deviation.

For the transferred charge, $\hat{Q}(t) = \hat{N}_A(t) - \hat{N}_A = \frac{1}{2}\alpha^\dagger[S_A(t) - \sigma_z^A]\alpha$ — the constants $|A|/2$ cancel — and (53) gives (22) with no prefactor. Vanishing of the odd cumulants in the recorded current table follows if the distribution of $\hat{Q}$ is symmetric about zero, which is the case here because the reflection $j \rightarrow N + 1 - j$ maps A to its complement and, at half filling and $\gamma = g = 1$, maps $\hat{Q}$ to $-\hat{Q}$ while leaving the state invariant; we did not verify that symmetry as an operator identity, only the vanishing itself, to 9.0×10^{-14} and 1.5×10^{-9} over the 245 recorded rows.

B.5 The twirl identity, Proposition 3.5

Let P_n project onto $\hat{N} = n$ and $\rho_d = \sum_n P_n \rho P_n$. The joint probability of the protocol is $p(n_1, n_2) = \text{Tr}[P_{n_1} U P_{n_2} \rho P_{n_2} U^\dagger]$, so its characteristic function is

$$F(\theta_1, \theta_2) = \sum_{n_1, n_2} e^{i\theta_1 n_1 + i\theta_2 n_2} p(n_1, n_2) = \text{Tr}[e^{i\theta_1 \hat{N}} U e^{i\theta_2 \hat{N}} \rho_d U^\dagger], \quad (57)$$

where the second equality uses $\sum_n e^{i\theta n} P_n \rho P_n = e^{i\theta \hat{N}} \rho_d$ and cyclicity. Next, $\rho_d = \int_0^{2\pi} \frac{d\phi}{2\pi} e^{i\phi \hat{N}} \rho e^{-i\phi \hat{N}}$, because the integral of $e^{i\phi(n-n')}$ over the circle is $\delta_{nn'}$; substituting and cycling once gives

$$F(\theta_1, \theta_2) = \int_0^{2\pi} \frac{d\phi}{2\pi} \text{Tr}[\rho e^{-i\phi \hat{N}} U^\dagger e^{i\theta_1 \hat{N}} U e^{i(\phi + \theta_2) \hat{N}}], \quad (58)$$

the first equality of (23). The integrand is a product of four exponentials of quadratic forms and is therefore, up to a scalar, one such exponential: each $e^{i\theta \hat{N}} = e^{i\theta N/2} e^{\frac{1}{2}\alpha^\dagger(i\theta \sigma_z)\alpha}$ and $U^\dagger(\cdot)U$ conjugates the matrix inside, so the scalars multiply to $e^{i(\theta_1 + \theta_2)N/2}$, the ϕ -scalars cancel, and (53) gives the determinant of the second equality with $U^\dagger e^{i\theta_1 \sigma_z} U \rightarrow R^\dagger e^{i\theta_1 \sigma_z} R$ in the single-particle representation. The partial-number version is identical with $\sigma_z \rightarrow \sigma_z^A$ and $N/2 \rightarrow |A|/2$.

B.6 The dephasing identity, Proposition 6.1

Differentiating F once in each variable and setting $\theta_1 = \theta_2 = 0$,

$$\langle n_1 \rangle_{\text{proj}} = \text{Tr}[U^\dagger \hat{N} U \rho_d] = \text{Tr}[\hat{N}(t) \rho_d], \quad \langle n_1 n_2 \rangle_{\text{proj}} = \text{Tr}[\hat{N}(t) \hat{N} \rho_d]. \quad (59)$$

The second formula needs the ordering: the θ_1 -derivative acts on the leftmost factor and the θ_2 -derivative on the one next to ρ_d . But ρ_d is diagonal in the $\hat{N}$ basis, so $[\rho_d, \hat{N}] = 0$ and $\text{Tr}[\hat{N}(t) \hat{N} \rho_d] = \text{Tr}[\hat{N} \hat{N}(t) \rho_d]$. Hence

$$\langle q^2 \rangle_{\text{proj}} = \text{Tr}[\hat{N}(t)^2 \rho_d] + \text{Tr}[\hat{N}^2 \rho_d] - 2 \text{Tr}[\hat{N}(t) \hat{N} \rho_d] = \text{Tr}[\hat{Q}(t)^2 \rho_d], \quad (60)$$

because the expansion of $\hat{Q}(t)^2 = (\hat{N}(t) - \hat{N})^2$ contains the *anticommutator* of $\hat{N}(t)$ and $\hat{N}$, whose two orderings have the same trace here. Together with $\langle q \rangle_{\text{proj}} = \text{Tr}[\hat{Q}(t) \rho_d]$ this is the proposition. At third order the same step fails: $\hat{Q}^3$ contains $\hat{N}(t)^2 \hat{N}$ and $\hat{N}(t) \hat{N} \hat{N}(t)$ with different coefficients than the projection produces, and the two orderings are no longer related by a single commutation with ρ_d . That is the measured discrepancy of 4.4×10^{-2} and 1.2×10^{-1} at $t = 0.1$ and 0.58 , 1.33 at $t = 0.4$, not a numerical artefact.

B.7 Normal ordering, Lemma 7.1

For a single mode, $: e^{-\alpha \hat{n}} := 1 + (e^{-\alpha} - 1)\hat{n} = P_0 + e^{-\alpha}P_1$, because normal ordering moves every creator to the left and $\hat{n}^m = \hat{n}$. Therefore

$$Q_\kappa(\lambda) = \text{Tr} \left[\rho \prod_k (P_0^{(k)} + e^{-\lambda \kappa} P_1^{(k)}) \right] = \sum_m e^{-\lambda \kappa m} \text{Tr}[\rho P_m] = \chi_{\text{raw}}(e^{-\lambda \kappa}), \quad (61)$$

with P_m the projector onto $\hat{N} = m$; the product over modes is exactly $\sum_m e^{-\lambda \kappa m} P_m$ because the modes are independent projectors. No assumption on ρ beyond the existence of the number projectors was used, so the lemma holds for paired and unpaired states alike. Writing $\ln Q_\kappa(\lambda) = \sum_j (-\lambda \kappa)^j \kappa_j^{\text{raw}} / j!$ and comparing with $\ln Q_\kappa(\lambda) = \sum_j (-\lambda)^j \kappa_j^{\text{scale}} / j!$ gives $\kappa_j^{\text{scale}} = \kappa^j \kappa_j^{\text{raw}}$.

B.8 Thinning cumulants to fourth order

Equation (24) composes the cumulant generating function with $s(\lambda) = \ln(1 - \kappa + \kappa e^\lambda)$, whose own cumulants are those of a Bernoulli variable:

$$s'(0) = \kappa, \quad s''(0) = \kappa(1 - \kappa), \quad s'''(0) = \kappa(1 - \kappa)(1 - 2\kappa), \quad s^{(4)}(0) = \kappa(1 - \kappa)(1 - 6\kappa + 6\kappa^2). \quad (62)$$

Writing $\mathcal{K}_{\text{miss}} = \mathcal{K} \circ s$ and differentiating four times at zero (the Faà di Bruno composition for cumulants),

$$\mathcal{K}'_{\text{miss}} = \kappa_1 s', \quad (63)$$

$$\mathcal{K}''_{\text{miss}} = \kappa_1 s'' + \kappa_2 (s')^2, \quad (64)$$

$$\mathcal{K}'''_{\text{miss}} = \kappa_1 s''' + 3\kappa_2 s'' s' + \kappa_3 (s')^3, \quad (65)$$

$$\mathcal{K}^{(4)}_{\text{miss}} = \kappa_1 s^{(4)} + \kappa_2 [3(s'')^2 + 4s''' s'] + 6\kappa_3 s'' (s')^2 + \kappa_4 (s')^4, \quad (66)$$

all at $\lambda = 0$. Substituting the Bernoulli values and using $3\kappa^2(1 - \kappa)^2 + 4\kappa^2(1 - \kappa)(1 - 2\kappa) = \kappa^2(1 - \kappa)(7 - 11\kappa)$ gives (47)–(50). The same composition applied to the scaled convention yields $\kappa^j \kappa_j$ directly, and subtracting the two at second order gives the discrepancy $\kappa(1 - \kappa)\kappa_1$ quoted in Section 7.

B.9 Periodic closed forms at criticality

On the periodic chain the momenta are $\Phi_k = 2\pi k/N$, $k = 0, \dots, N-1$, and (3) at $\gamma = g = 1$ gives $E_k = \sqrt{(1 - \cos \Phi_k)^2 + \sin^2 \Phi_k} = 2|\sin \frac{\pi k}{N}|$, with Bogoliubov weight

$$v_k^2 = \frac{1}{2} \left(1 - \frac{g - \cos \Phi_k}{E_k} \right) = \frac{1}{2} \left(1 - \sin \frac{\pi k}{N} \right), \quad 1 \leq k \leq N-1, \quad (67)$$

while $k = 0$ and $k = N/2$ are self-paired: $E = 0$ and $E = 2$ respectively, and the $k = 0$ mode is the branch. In the pair product of [1] each pair $(k, N-k)$, $k = 1, \dots, \frac{N}{2} - 1$, contributes either zero or two fermions, with probability $1 - v_k^2$ and v_k^2 , so

$$\chi_{\text{per}}(z) = \zeta \prod_{k=1}^{N/2-1} [(1 - v_k^2) + v_k^2 z^2], \quad \zeta = 1 \text{ or } z, \quad (68)$$

where ζ encodes the $k = 0$ branch and the $k = N/2$ mode has $v^2 = 0$. A pair contributing $\{0, 2\}$ with weight v^2 has mean $2v^2$ and variance $4v^2(1 - v^2)$, hence

$$\text{Var}_{\text{per}} = \sum_{k=1}^{N/2-1} 4v_k^2(1 - v_k^2) = \sum_{k=1}^{N/2-1} \cos^2 \frac{\pi k}{N} = \frac{N/2-1}{2} + \frac{1}{2} \sum_{k=1}^{N/2-1} \cos \frac{2\pi k}{N} = \frac{N}{4} - \frac{1}{2}, \quad (69)$$

because $\sum_{k=1}^{N-1} \cos(2\pi k/N) = -1$ and the terms are symmetric about $k = N/2$, which makes the half-sum vanish. The branch drops out entirely: both self-paired modes are deterministic, so they contribute nothing to the variance. This is (27), and it holds for every even N and every $\gamma > 0$ at $g = 1$, not only at $\gamma = 1$.

For the mean,

$$\langle \hat{N} \rangle_{\text{per}} = \sum_{k=1}^{N/2-1} 2v_k^2 + \delta_{\text{branch}} = \sum_{k=1}^{N/2-1} \left(1 - \sin \frac{\pi k}{N}\right) + \delta_{\text{branch}} = \frac{N}{2} - 1 - \frac{1}{2} \cot \frac{\pi}{2N} + \delta_{\text{branch}}, \quad (70)$$

using $\sum_{k=1}^{N-1} \sin(\pi k/N) = \cot(\pi/2N)$ and the symmetry $\sin(\pi k/N) = \sin(\pi(N-k)/N)$, with $\delta_{\text{branch}} = 0$ for the empty $k = 0$ mode and 1 for the occupied one. The cotangent form and the form printed in (28) are the same expression: multiplying the latter by $\sqrt{2} \sin x$ with $x = \pi/2N$ and using $\sqrt{2} \sin(\frac{\pi}{4} - x) = \cos x - \sin x$ turns it into the former. Expanding $\cot x = x^{-1} - x/3 - x^3/45 + \dots$ gives

$$\langle \hat{N} \rangle_{\text{per}} = N \left(\frac{1}{2} - \frac{1}{\pi} \right) - \frac{1}{2} + \frac{\pi}{12N} + O(N^{-3}) \quad (\delta_{\text{branch}} = 0), \quad (71)$$

so the $O(1/N)$ remainder of (28) is not merely bounded but known: $\pi/12N = 1.02297 \times 10^{-3}$ at $N = 256$ against the measured residual $+1.023 \times 10^{-3}$, and 2.5574×10^{-4} at $N = 1024$ against $+2.557 \times 10^{-4}$. The bulk density $\frac{1}{2} - \frac{1}{\pi} = 0.181690$ and the bulk variance density $\frac{1}{4}$ follow from (69) and (70) by division by N .

B.10 The periodic gap away from criticality

At $\Phi = 0$, (3) gives $E_0 = |g - 1|$ for every γ , because $\sin \Phi_0 = 0$; and at $\Phi = \pi$, $E_\pi = g + 1$. For $\gamma = 1$ the function $E^2 = (g - \cos \Phi)^2 + \sin^2 \Phi = 1 + g^2 - 2g \cos \Phi$ is monotone in $\cos \Phi$, so its minimum over the discrete momenta is at $\Phi = 0$ and

$$\Delta_{\text{per}} = J|g - 1| \quad (\gamma = 1, g \geq 0), \quad (72)$$

exactly and for every N , which is what the recorded gap table shows to twelve decimals at $g = 1.05$. For $\gamma < 1$ the stationary point of E^2 as a function of $c = \cos \Phi$ lies at $c = g/(1 - \gamma^2)$, so the same identity holds whenever $g \geq 1 - \gamma^2$, which covers the recorded $\gamma = 0.5$, $g = 1.05$ column, and fails for smaller g , where the minimum moves into the interior of the zone. The open-chain gap has no such closed form; the asymptotics $\Delta_{\text{open}} \rightarrow \gamma\pi/2N$ quoted in Section 5.1 is a numerical observation — the measured ratio is 0.884 at $N = 4$, 0.9922 at $N = 64$ and 0.9980 at $N = 256$, approaching 1 from below at $\gamma = 1$ and from above at $\gamma = 0.5$ — and no derivation of it is claimed here.

B.11 Fitting protocol

All logarithmic fits are ordinary least squares in the three parameters of $a \ln N + b + c/N$ over the stated lengths, with no weighting: five points ($N = 64, \dots, 1024$) for the boundary differences of Table 2 and five points ($N = 32, \dots, 512$) for the half-chain deficit of Section 5.5. With five points and three parameters the fit has two degrees of freedom, so the quoted maximum residual is a meaningful measure of misfit but a χ^2 statement would not be. The coefficient a is compared with the predicted value directly, not through the residual; the relative agreement is 1.6×10^{-5} for $-1/(2\pi)$ and 3.1×10^{-5} for $-1/\pi^2$. Nothing in this paper proves that the differences have the form $a \ln N + b + c/N$; the evidence is that this three-parameter form fits the measured range of N (one decade, 64 to 1024) to below 6.2×10^{-7} , while dropping the $1/N$ term leaves maximum residuals of 9.5×10^{-4} in the mean and 1.3×10^{-4} in the variance, a factor of 10^3 and 10^2 worse (`research/checks/fit_two_parameter.py`).

C Branches, zero modes and parity

Three separate places in this problem have a choice that the mathematics does not determine and the numerics does not control: the $k = 0$ mode of the periodic chain at $g = 1$, the ground-state doublet of the open chain at $g < 1$, and the square root of the twirl integrand. This appendix states each choice, the convention adopted here, and the measurement of what happens when it is left to the software.

C.1 The periodic zero mode and its three branches

On the periodic chain the momentum $\Phi_0 = 0$ has $E_0 = |g - 1|$ for every γ , so at $g = 1$ the BdG matrix W has an exact two-dimensional null space spanned by the uniform vectors $(1, \dots, 1, 0, \dots, 0)$ and $(0, \dots, 0, 1, \dots, 1)$: the mode $c_0 = N^{-1/2} \sum_j c_j$ has neither kinetic energy nor pairing at criticality. The many-body Hamiltonian then splits as $H = H_{\text{rest}} + 0 \cdot (\hat{n}_0 - \frac{1}{2})$, the ground manifold is two-dimensional, and $\hat{N} = \hat{n}_0 + \hat{N}_{\text{rest}}$ with $\hat{n}_0$ a free fermion uncorrelated with the rest. Consequently the ground states are

$$\psi = a |e\rangle + b |o\rangle, \quad |e\rangle = |\hat{n}_0 = 0\rangle \otimes |R\rangle, \quad |o\rangle = |\hat{n}_0 = 1\rangle \otimes |R\rangle, \quad (73)$$

with opposite parities, and since $\hat{N}$ is diagonal in this basis the counting function is the affine combination

$$\chi_\psi(z) = |a|^2 \chi_e(z) + |b|^2 \chi_o(z), \quad \chi_o(z) = z \chi_e(z), \quad (74)$$

so that $\langle \hat{N} \rangle_\psi = \langle \hat{N} \rangle_e + x$ and $\text{Var}_\psi(\hat{N}) = \text{Var}_e(\hat{N}) + x(1 - x)$ with $x = |b|^2 \in [0, 1]$. The variance is therefore bounded above by $\text{Var}_e(\hat{N}) + \frac{1}{4}$, and the pair $(\langle \hat{N} \rangle, \text{Var})$ determines x ; no ground state lies off that curve. Both parity branches have the same variance, which is why (27) is branch-independent while (28) is not.

The convention of this paper is the *empty* branch, $x = 0$: it is the one the analytic pair product of [1] gives when the self-paired $k = 0$ mode is left unoccupied, it is the one the recorded critical boundary table uses — our reproduction of that table with the empty branch agrees to 2.3×10^{-7} while the occupied branch would be off by exactly 1 in the mean — and it is the one the repository’s own `test_periodic_k0` pins against the $g \neq 1$ determinant, to 4.7×10^{-14} at $N = 400$.

The line $\gamma = 0$ is a different degeneracy and is excluded from every critical statement here: there the pairing vanishes, $\hat{N}$ commutes with H , and for $|g| \leq 1$ the ground manifold is degenerate in a way that depends on the filling rather than on a single self-paired mode. The recorded parity-protocol table covers that case, flags its two degenerate rows, and reports cumulant errors up to 1.2×10^{-14} on the other 52; running the repository’s own test reproduces the table entrywise to 8.0×10^{-15} .

C.2 What the numerical projector returns at the zero mode

U_- is assembled from the N lowest eigenvectors of W , and when two of those eigenvalues are exactly zero the eigensolver returns an arbitrary orthonormal basis of the null space, so the column that enters U_- is an arbitrary real unit vector $\cos \theta (1, 0) + \sin \theta (0, 1)$. Its contribution to the pairing block $B = \langle c^\dagger c^\dagger \rangle$ of $\Gamma = U_- U_-^\dagger$ is $\cos \theta \sin \theta J/N$, which is symmetric; a fermionic covariance needs B antisymmetric. Table 10 measures the result on the critical line $\gamma = g = 1$.

At $N = 3$ and 4 the arbitrary vector happens to land on a branch and the result is exact; from $N = 5$ on it does not. Over the 21 degenerate cases examined ($N = 3$ to 8, $\gamma \in \{0.4, 0.7, 1\}$) the antisymmetry violation reaches 8.8×10^{-2} and 16 give a $(\langle \hat{N} \rangle, \text{Var})$ pair outside the range (74), with excess variance up to +0.0744. Such a Γ is a projector of the right rank and trace but is not

Table 10: Periodic chain at $g = 1$, $T = 0$, covariance assembled by an independent eigendecomposition. $\|B + B^T\|$ is the antisymmetry violation of the pairing block, which is 10^{-15} in every non-degenerate control case; the two branch rows are the exact 2^N Fock values of the empty and occupied $k = 0$ states, whose variances are identical by (69); “allowed” is $\text{Var}_e + x(1 - x)$ with $x = \langle \hat{N} \rangle_{\text{Wick}} - \langle \hat{N} \rangle_e$, the most any state of the ground manifold can have at that mean, from (74).

N	$\ B + B^T\ $	$\langle \hat{N} \rangle_{\text{Wick}}$	Var_{Wick}	$\langle \hat{N} \rangle_e$	$\langle \hat{N} \rangle_o$	allowed Var	excess
3	1.9×10^{-16}	1.133975	0.250000	0.133975	1.133975	0.250000	0
4	1.1×10^{-16}	0.292893	0.500000	0.292893	1.292893	0.500000	0
5	8.8×10^{-2}	0.512161	0.846803	0.461158	1.461158	0.798401	+0.0484
6	4.3×10^{-3}	0.634140	1.000330	0.633975	1.633975	1.000165	+0.0002
7	7.8×10^{-2}	1.728404	1.398798	0.809357	1.809357	1.324399	+0.0744
8	2.9×10^{-2}	1.973054	1.526200	0.986330	1.986330	1.513100	+0.0131

the covariance of any fermionic state, so the determinant formulas evaluated on it produce numbers that are not the counting statistics of anything.

The implementation supplied with the data has the same problem and a different symptom. Its determinant at the periodic critical point always lands on the affine segment (74) pointwise in λ — at $N = 3$ on the occupied branch, at $N = 4$ on the empty one, at $N = 5$ and 6 in between, within 2.6×10^{-2} relative of the nearer branch — but the implied weight is not the same at different counting fields: 0.061 at $\lambda = +0.35$ against 0.040 at $\lambda = -0.60$ for $N = 5$. For a genuine state the weight would be λ -independent, because $\hat{N}$ is diagonal in the branch basis and (74) is then a classical mixture. The returned generating function is therefore not that of any single ground state either.

This is why every periodic critical number in this paper comes from the analytic pair product of Appendix B.9 with the branch stated, and why the validation of the one-time generators excludes the critical zero mode from the periodic cases.

C.3 The open-chain doublet at $g < 1$

For $g < 1$ the open chain is in the phase with exponentially localised Majorana end modes, and the smallest positive BdG eigenvalue is 2.9×10^{-4} at $N = 32$, 2.3×10^{-7} at $N = 64$, 1.4×10^{-13} at $N = 128$ and 2.6×10^{-15} at $N = 256$ — against 2.4×10^{-2} , 1.2×10^{-2} and 6.1×10^{-3} at $g = 1$ and ≈ 0.2 at $g = 1.2$. Two consequences, of very different severity.

Any quantity that is a trace over the doublet is insensitive to the choice: the total mean, the total variance and the half-chain variance all agree to 5×10^{-8} whichever member is returned. The site-resolved density $\langle \hat{n}_j \rangle$ is not, because the two members are related by an operation that moves weight between the two ends of the chain. The exact finite-chain ground state is reflection symmetric, $\langle \hat{n}_j \rangle = \langle \hat{n}_{N+1-j} \rangle$, and an arbitrary member of a numerically degenerate doublet need not be: the measured asymmetry $\max_j |\langle \hat{n}_j \rangle - \langle \hat{n}_{N+1-j} \rangle|$ is 4.4×10^{-4} at $N = 128$ and 3.7×10^{-2} at $N = 256$, and the raw half-chain mean is correspondingly wrong by 6.1×10^{-4} and 5.1×10^{-2} . Averaging the two members of the doublet restores the symmetry exactly and reproduces the recorded half-chain means to 1.3×10^{-8} ; that symmetrised profile is what Figure 7 and Section 5.5 use. At criticality the splitting is a power law, the doublet is resolved at every length used here, and no symmetrisation is needed.

C.4 The square root of the twirl integrand

Proposition 3.5 reduces the projective two-measurement distribution to a square root of a determinant evaluated on a complex torus. The determinant is the square of an analytic function, but the sign of its square root is a global datum that a pointwise `sqrt` does not supply: unlike the real axis, where $\chi(\lambda) > 0$ pins the branch (Appendix B.1), F is complex and positivity gives nothing. There are three ways to lose it and all three were encountered.

Pointwise square roots. Taking $\sqrt{D(\phi, \theta_1, \theta_2)}$ independently at each grid point produces a grid whose two-dimensional Fourier transform has half its weight in the wrong places. This is what the recorded total-number grids at $N = 8$, $t = 0.4$ show: 0.661 of the ground-state probability and 0.989 of the $\beta = 0.7$ probability sit on pairs with $n_1 - n_2$ odd, which the parity selection rule below forbids exactly, and single entries are wrong by up to 0.378 against exact Kraus traces. Rerunning the same routine at the same documented parameters is deterministic — bit-identical under one, two and four BLAS threads — but returns a third answer, differing from the recorded grid by up to 0.372 and still carrying 2.6×10^{-3} of parity-forbidden mass.

Coefficient recursion. Expanding $D(z) = f(z)^2$ as a Laurent polynomial and recovering the coefficients of f by the recursion implied by squaring is exact in principle and loses precision in practice, because it subtracts large coefficients: at $N = 8$ the resulting distribution is wrong by up to 9.8×10^{-3} in a single entry, and by 2.2×10^{-5} at $N = 4$, $\beta = 0.7$, which is the case the repository’s own thermal test reports and passes.

Protocol 3.6. Continuation along the ϕ circle by unit-circle ratios, with the residual global sign of each (θ_1, θ_2) column fixed from the exactly known $\theta_1 = 0$ row, never subtracts large numbers and is exact for the ground state: 3.1×10^{-15} at $N = 8$ with parity-forbidden mass 8.5×10^{-15} , and 2.1×10^{-15} at $N = 4$ and 6. For the Gibbs state the same protocol leaves 1.6×10^{-3} at $N = 4$, 5.6×10^{-5} at $N = 6$ and 2.3×10^{-5} at $N = 8$, with parity-forbidden mass 4.2×10^{-5} : the sign column is fixed against a row that is itself only as good as the thermal one-time distribution, and the residual is stated in the captions of Figure 14 wherever it matters. Every total-number projective result in this paper comes from this protocol or from exact Kraus traces, never from the recorded grids.

C.5 Parity, and what a detector does to it

H conserves $\Pi = (-1)^{\hat{N}}$, so $[U, \Pi] = 0$ and

$$p(n_1, n_2) = \text{Tr}[P_{n_1} U P_{n_2} \rho P_{n_2} U^\dagger] = \text{Tr}[P_{n_1} U P_{n_2} \rho P_{n_2} U^\dagger P_{n_1}] \quad (75)$$

vanishes unless n_1 and n_2 have the same parity: inserting $\Pi^2 = 1$ and commuting Π through U and through the projectors shows the trace equals $(-1)^{n_1 - n_2}$ times itself. For a one-time distribution the same argument gives support on a single parity sector, which is the statement behind Section 5.3 and behind the parity columns of the recorded distribution files: odd-parity mass 2.1×10^{-15} for the open chain at $\gamma = 1$, $g = 0.95$, $N = 64$, and exactly 1 for the periodic chain, which is the occupied $k = 0$ branch of Appendix C.1 showing up as a shifted sector rather than as a violation.

A detector with efficiency κ under the independent-click convention destroys the selection rule in a computable way. The detected parity contrast is

$$\langle \Pi \rangle_{\text{det}} = \sum_m (-1)^m p_{\text{miss}}(m) = \chi_{\text{miss}}(-1) = \chi_{\text{raw}}(1 - \kappa + \kappa(-1)) = \chi_{\text{raw}}(1 - 2\kappa), \quad (76)$$

one substitution in (10), and it is exact rather than approximate: on the three recorded $N = 64$ distribution files the two sides agree to 1.1×10^{-16} . At $\kappa = 0.9$ it gives +0.0816 for the open chain

and -0.0669 for the periodic one at $\gamma = 1$, $g = 0.95$, the sign flip of Section 7.3 expressed as a single number, with detected even/odd mass $0.541/0.459$ against $0.467/0.533$; at $g = 1.05$ both chains keep a positive contrast, $+0.1742$ and $+0.1642$, and at $\gamma = 0.5$, $g = 1$ they are $+0.2884$ and $+0.2724$. The three limits of (76) are exact and bracket the whole sweep: $\langle \Pi \rangle_{\text{det}} = \chi_{\text{raw}}(1) = 1$ at $\kappa = 0$, where nothing is detected; $\chi_{\text{raw}}(0) = p(n=0)$ at $\kappa = \frac{1}{2}$, the smallest the contrast can be; and $\chi_{\text{raw}}(-1) = \pm 1$ at $\kappa = 1$, the raw selection rule recovered. Evaluated on the recorded $N = 64$, $\gamma = g = 1$, $T = 0$ open-chain distribution, whose $p(n=0) = 1.5309 \times 10^{-3}$, the seven rows of the κ sweep give $+0.1390$, $+0.0087$, $+0.0015$, $+0.0026$, $+0.0146$, $+0.1800$ and $+0.8642$ for $\kappa = 0.0952$, 0.2592 , 0.5034 , 0.6321 , 0.7769 , 0.9179 and 0.9933 : a U shape whose minimum coincides with $p(n=0)$ to four digits, exactly as the identity requires. A detector at half efficiency is parity-blind; recovering the selection rule needs κ close to one.

For a partial number $\hat{N}_A$ none of this holds: $\hat{N}_A$ does not commute with Π , the two-measurement distribution has support on every transferred charge, and the recorded partial grid at $N = 8$, $t = 0.4$ has nonzero weight on all nine values of $q \in \{-4, \dots, 4\}$, from 0.856337 at $q = 0$ down to 4.6×10^{-6} at $q = -4$, and is strongly asymmetric, $p(+2)/p(-2) = 5.45$. That asymmetry between counting the whole chain and counting part of it is physical, and it is the reason the total-number parity rule is such a sharp diagnostic of a broken square root while the partial one has no comparable test.

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