

Complementary Quantum Correlations Are Universal for Qubits

Qihang Wang,^{1,*} Jinbo Wang,^{1,*} and Kun Chen^{2,†}

¹*School of Mathematical Sciences, Peking University, Beijing 100871, China*

²*Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China*

We prove that, for every two-qubit state and arbitrary mutually unbiased local projective bases, the sum of the two observed mutual informations cannot exceed the quantum mutual information. The proof combines binary-entropy curvature on the Bloch ball, a qubit information-exclusion tradeoff, and data processing under local dephasing. Two outcome tables thus lower-bound total correlation without tomography and give quantitative one-way entanglement-distillation rates and a quantum-capacity witness for qubit-input, qubit-output channels.

Introduction.— Determining how much correlation is shared by two quantum systems is a basic task in quantum physics. The quantum mutual information (QMI) $I(A : B) = S(A) + S(B) - S(AB)$ quantifies their total classical and quantum correlations. Equivalently, $I(A : B) = D(\rho_{AB} \| \rho_A \otimes \rho_B)$ is the relative-entropy distance from the product of its marginals. It also has an operational interpretation as the asymptotic noise cost of erasing all correlations [1]. Because QMI is a non-linear function of the joint density operator, its direct evaluation generally requires state tomography. Most experiments, by contrast, record correlations in only a few spin, polarization, or computational-basis settings. This mismatch raises the broadly relevant question of when limited classical readouts can certify a guaranteed amount of correlation in the unmeasured quantum state.

For a two-level system, the natural minimal data consist of two complementary measurements. Two bases are mutually unbiased when a state sharp in either basis produces uniformly random outcomes in the other [2]. Let X and Z denote such a pair on each subsystem, and let $I(X_A : X_B)$ be the classical mutual information between the corresponding outcomes. Schneeloch, Broadbent, and Howell proposed the complementary-quantum correlation (CQC) relation

$$I(X_A : X_B) + I(Z_A : Z_B) \leq I(A : B). \quad (1)$$

If valid, Eq. (1) would justify adding the two observed mutual informations without overestimating the correlation present before measurement. Complementarity alone does not guarantee this accounting, because it constrains local predictability rather than whether two bipartite tables encode the same shared classical variable. Local unbiasedness says that an eigenstate of X is random in Z ; it does not say that correlations between two parties in the X table and in the Z table originate from independent degrees of freedom.

This distinction also separates CQC from standard information-exclusion relations. Those relations limit how strongly a single memory can be correlated with the outcomes of incompatible measurements [3–6]. In Eq. (1), by contrast, the memories are the outcomes of different measurements on the other quantum system. The X

and Z terms therefore involve two differently dephased states, and neither a one-sided exclusion bound nor data processing alone compares their sum with the unde phased QMI. A proof must retain quantum side information long enough to determine how much correlation each classical readout has discarded.

The CQC relation was proved for pure states and several structured mixed-state families [7]. Extensive numerical searches, including fixed-purity studies of two-qubit states, found no violation [7, 8], and a recent sufficient condition covers additional states [9]. The dimension dependence is nevertheless decisive. Companion work gives explicit counterexamples in every local dimension $d \geq 3$, where complementary readouts can count the same classical label more than once [10]. That construction loses its overcounting mechanism at $d = 2$. The qubit case therefore tests whether binary quantum geometry supplies a protection that complementarity lacks in general.

Here we establish Eq. (1) for every two-qubit density operator, without assumptions on rank, marginals, symmetry, or separability. The physical origin is visible in the Bloch ball, where a qubit state is described by three real coordinates and its entropy depends only on the Bloch radius. We show that the curvature of this binary entropy forces information missed by one readout to account for at least half of the information visible in the complementary readout. Applying this tradeoff in opposite directions on the two parties accounts for the full crossed correlation, and data processing under local dephasing completes the correlation budget. This dimension-specific mechanism yields both a sharp conceptual boundary and practical two-setting certificates.

Full algebraic details, the boundary argument, operational corollaries, and a proof-dependency audit are provided in the Supplemental Material [11].

Qubit half-information exclusion.— Let C be a finite classical label encoded in an arbitrary ensemble $\{p_c, \rho_c\}$ of qubit states. The Holevo information $\chi(C : Q)$ quantifies the information carried by the ensemble and bounds the classical information accessible by measurement [12]. For complementary Pauli measurements X and Z ,

$$\chi(C : Q) - I(C : X) \geq \frac{1}{2} I(C : Z). \quad (2)$$

The label set and ranks of the conditional states are unrestricted. Equation (2) states that the ensemble information not recovered by the X readout must account for at least half of the label information visible in the Z readout.

The state-dependent right-hand side is essential. Conventional exclusion bounds compare several readouts with a dimension-dependent ceiling. Here the quantity $\chi(C:Q) - I(C:X)$ is instead the information retained by the quantum carrier but lost when it is converted into the designated classical X register. Equation (2) says that this loss is large enough to pay half the correlation of the complementary readout. The asymmetric form will allow the same crossed outcome table to be paid for once from each party.

Write a qubit state as a Bloch vector $\mathbf{v} = (x, y, z)$ with $r = |\mathbf{v}|$, and define

$$h(t) = h_2\left(\frac{1+t}{2}\right), \quad g(\mathbf{v}) = h(r) - h(x) - \frac{1}{2}h(z). \quad (3)$$

If $\bar{\mathbf{v}} = \sum_c p_c \mathbf{v}_c$, direct expansion gives

$$\chi - I(C:X) - \frac{1}{2}I(C:Z) = g(\bar{\mathbf{v}}) - \sum_c p_c g(\mathbf{v}_c). \quad (4)$$

Thus Eq. (2) is precisely Jensen's inequality if g is concave on the Bloch ball.

Replace the base-2 entropy by its natural-logarithm version while differentiating; this multiplies g and its Hessian by the common positive factor $\ln 2$ and therefore does not change semidefinite signs. At an interior point set

$$A = \frac{\operatorname{arctanh} r}{r}, \quad C = \frac{1}{1-r^2}, \quad \mathbf{u} = \frac{\mathbf{v}}{r}. \quad (5)$$

Then

$$\begin{aligned} -\nabla^2 g &= R - D, \\ R &= A\mathbf{1} + (C - A)\mathbf{u}\mathbf{u}^\top, \\ D &= \operatorname{diag}\left(\frac{1}{1-x^2}, 0, \frac{1}{2(1-z^2)}\right). \end{aligned} \quad (6)$$

The matrix R is the full radial curvature of the qubit entropy, whereas D returns the coordinate curvatures contributed by the two measured binary distributions. Thus concavity is reduced to showing that the radial quantum curvature dominates these two classical directions. Let $U = (\mathbf{e}_x, \mathbf{e}_z)$ and let D_2 contain the two nonzero diagonal entries of D . Since $R, D_2 > 0$, a Schur complement reduces $R - UD_2U^\top \succeq 0$ to $Q = D_2^{-1} - U^\top R^{-1}U \succeq 0$. With $X = x^2$, $Y = y^2$, $Z = z^2$, and

$$\delta = \frac{1 - r/\operatorname{arctanh} r}{r^2}, \quad \delta(0) = \frac{1}{3}, \quad (7)$$

the reduced matrix is

$$Q = \begin{pmatrix} \delta(Y+Z) & (1-\delta)xz \\ (1-\delta)xz & 1-Z+\delta(X+Y) \end{pmatrix}. \quad (8)$$

The scalar series

$$\frac{\operatorname{arctanh} r}{r} = \sum_{n \geq 0} \frac{r^{2n}}{2n+1} \geq \sum_{n \geq 0} \frac{r^{2n}}{3^n} = \frac{1}{1-r^2/3} \quad (9)$$

implies $\delta \geq 1/3$. Both diagonal entries of Q are nonnegative, and its determinant has the exact decomposition

$$\begin{aligned} \det Q &= Z[\delta(1-Z) + (2\delta-1)X] \\ &\quad + Y[\delta(1-Z) + \delta^2 r^2]. \end{aligned} \quad (10)$$

For $\delta \geq 1/2$ every term is nonnegative. If $1/3 \leq \delta < 1/2$, then $X \leq 1-Z$ and the first bracket is at least $(3\delta-1)(1-Z) \geq 0$. Hence $Q \succeq 0$ and $-\nabla^2 g \succeq 0$. Continuity extends concavity to the closed Bloch ball, so Eq. (4) proves Eq. (2). Exchanging X and Z gives its axis-swapped form.

All two-qubit states.— Apply Eq. (2) first to Bob's ensemble conditioned on Alice's X outcome and then, with the axes exchanged, to Alice's ensemble conditioned on Bob's Z outcome:

$$\begin{aligned} \chi(X_A:B) - I(X_A:X_B) &\geq \frac{1}{2}I(X_A:Z_B), \\ \chi(Z_B:A) - I(Z_A:Z_B) &\geq \frac{1}{2}I(X_A:Z_B). \end{aligned} \quad (11)$$

Adding the two bounds therefore supplies the full crossed correlation. The orientation is essential: the first inequality preserves Bob as a quantum memory, whereas the second preserves Alice, so both deficits contain the same crossed table.

Let $\mathcal{E}_A$ completely dephase (pinch) subsystem A in the X basis, and let $\mathcal{E}_B$ do the same on subsystem B in the Z basis. Relative-entropy data processing [13] gives

$$I(A:B) + I(X_A:Z_B) \geq \chi(X_A:B) + \chi(Z_B:A). \quad (12)$$

The two Holevo quantities are the QMI values after one local dephasing, and $I(X_A:Z_B)$ is the QMI after both; Eq. (12) orders the two correlation losses by data processing. Indeed,

$$I(\rho) - I(\mathcal{E}_A \rho) = D(\rho \| \mathcal{E}_A \rho) - D(\rho_A \| \mathcal{E}_A \rho_A), \quad (13)$$

and applying $\mathcal{E}_B$ to the first relative entropy yields Eq. (12).

For compactness put $I_X = I(X_A:X_B)$, $I_Z = I(Z_A:Z_B)$, $I_{XZ} = I(X_A:Z_B)$, $\chi_X = \chi(X_A:B)$, and $\chi_Z = \chi(Z_B:A)$. Adding Eqs. (11) and (12) gives the exact decomposition

$$\begin{aligned} I(A:B) - I_X - I_Z &= [I(A:B) + I_{XZ} - \chi_X - \chi_Z] \\ &\quad + [(\chi_X - I_X) + (\chi_Z - I_Z) \\ &\quad - I_{XZ}] \geq 0. \end{aligned} \quad (14)$$

Every bracket is separately nonnegative. Any qubit MUB pair is unitarily equivalent to Pauli X, Z , independently on the two parties. Singular states follow by continuity. This proves Eq. (1) for every two-qubit density operator. The two brackets also reveal the division of labor

in the proof: the first is an ordinary data-processing deficit, while the second is the genuinely qubit-specific information-exclusion deficit. The bound is tight in physically distinct regimes. A perfectly correlated classical bit has $(I_X, I_Z, I) = (1, 0, 1)$ after choosing its encoding basis, whereas a maximally entangled pair has $(I_X, I_Z, I) = (1, 1, 2)$.

Why qubits are special.— The proof exposes why the dimension boundary is sharp. A qubit density operator has only two eigenvalues, fixed by a single Bloch radius. This binary spectrum gives the entropy the curvature encoded by $\delta \geq 1/3$, forcing the half-information tradeoff in Eq. (2). The coefficient one half is precisely sufficient: the two orientations in Eq. (11) sum to the full crossed classical correlation, while Eq. (12) accounts for the remaining quantum information. In higher dimensions a density operator has more than two eigenvalues; its entropy is therefore not controlled by a single radial coordinate. There is no analogous binary-curvature constraint, and mutually unbiased bases can reveal the same encoded label in more than one setting. The higher-dimensional counterexamples are therefore not exceptions to a generic complementarity principle; together with the present theorem, they show that the principle itself is specific to binary quantum geometry [10].

Physical implications.— This mechanism has direct operational consequences for estimating correlations from limited measurements. Define the experimentally accessible score

$$\mathcal{S}_{XZ} := I(X_A : X_B) + I(Z_A : Z_B) \quad (15)$$

from two 2×2 outcome distributions. The theorem gives the state-independent lower bound $I(A : B) \geq \mathcal{S}_{XZ}$ for arbitrary two-qubit sources. It therefore replaces full state reconstruction by two correlation tables whenever a lower bound, rather than the exact QMI, is the relevant quantity. No purity, Bell-diagonal structure, or source-noise model is required for this deterministic statement. Because QMI is a relative entropy and an asymptotic correlation-erasure cost, the same score also certifies that at least $\mathcal{S}_{XZ}$ bits per copy of total correlation must be erased to decorrelate the state in that operational setting [1].

The same score quantifies distillable entanglement. The coherent informations $I(A|B) = I(A : B) - S(A)$ and $I(B|A)$ are each at least $\mathcal{S}_{XZ} - 1$, because qubit entropies do not exceed one bit. By the hashing inequality, positive coherent information is an achievable asymptotic one-way entanglement-distillation rate [14]. Hence

$$D_{A \rightarrow B}, D_{B \rightarrow A} \geq \max\{0, \mathcal{S}_{XZ} - 1\}, \quad (16)$$

where D is measured in ebits per copy under one-way local operations and classical communication. Thus $\mathcal{S}_{XZ} > 1$ certifies distillability in either direction. The bound is tight for maximally entangled states but is not necessary

for distillability. Since distilled ebits can be measured to produce secret bits, the same rate is also an achievable lower bound for one-way secret-key generation after entanglement distillation.

There is a parallel channel interpretation. Let $\omega_{RB} = (\text{id}_R \otimes \mathcal{N})(\Phi_{RA})$ be the normalized Choi state of a qubit-input, qubit-output channel $\mathcal{N}$. Its reference marginal is maximally mixed, so the coherent information of the maximally mixed channel input obeys $I_c(\mathbf{1}/2, \mathcal{N}) = I(R : B)_\omega - 1$. Applying CQC to two complementary measurements on R and B , followed by the quantum coding theorem [15], gives

$$Q(\mathcal{N}) \geq \max\{0, \mathcal{S}_{XZ}(\omega) - 1\}. \quad (17)$$

Thus paired measurements on transmitted Bell pairs can certify a nonzero asymptotic quantum communication rate without reconstructing the channel. The statement is a lower bound from one chosen channel input, not a formula for the generally regularized capacity.

The theorem also places the two-setting data within the framework of entropic uncertainty. Correlation with Bob reduces Alice's outcome uncertainty by the corresponding classical mutual information. Combining Eq. (1) with the state-dependent qubit relation $H(X_A) + H(Z_A) \geq 1 + S(A)$ gives

$$H(X_A|X_B) + H(Z_A|Z_B) \geq 1 + S(A|B). \quad (18)$$

Applying the same single-system bound on both parties also yields

$$H(X_A X_B) + H(Z_A Z_B) \geq 2 + S(AB). \quad (19)$$

These familiar state-dependent uncertainty bounds [6, 16] acquire a common interpretation here: the reduction of uncertainty across two incompatible readouts cannot exceed the total correlation resource in the premeasurement state. Their left-hand sides are obtained from the same paired outcome tables used in $\mathcal{S}_{XZ}$.

These operational statements have an essential scope. They assume trusted two-dimensional local Hilbert spaces and calibrated mutually unbiased projective measurements. They are not device-independent: leakage into higher levels can restore the correlation overcounting found for $d \geq 3$. Finite-sample confidence intervals and measurement uncertainty are likewise experimental layers beyond the deterministic inequality. This boundary is also informative, because it identifies Hilbert-space dimension, rather than basis overlap alone, as the physical resource protecting the correlation budget.

Broader implications and outlook.— At the level of entropy inequalities, the result is not another application of strong subadditivity alone. Data processing supplies Eq. (12), but it cannot control the second bracket of Eq. (14). The missing ingredient is the concavity of the qubit function g , equivalently the state-dependent

half-information exclusion inequality for arbitrary qubit ensembles. CQC is obtained only after two copies of this one-system entropy theorem are oriented so that their deficits share the same crossed classical mutual information. This factorization isolates a reusable strategy: first identify information lost to a local readout, then use state-space curvature to charge a complementary correlation against that loss, and finally join the local statements by data processing.

The dimensional boundary changes how complementarity should be used in quantum-information certification. The qubit theorem turns the intuition that complementary observables are independent probes into a rigorous correlation budget. Higher-dimensional counterexamples reveal the missing premise: measurement incompatibility alone does not prevent two settings from reporting the same shared variable. Combining their evidence therefore requires either a dimension-specific exclusion bound or an explicit correction for redundant correlations.

The proof suggests a route to other low-dimensional inequalities: quantify ensemble information inaccessible to each local readout, then join oppositely oriented bounds by data processing. Here the factor one half is exactly what closes the two orientations. Which state spaces and measurement families admit analogous closure is a concrete question beyond CQC. Natural tests include nonorthogonal qubit measurement axes, generalized qubit measurements, and Rényi or one-shot entropies relevant to finite-blocklength protocols. None follows automatically from the von Neumann-entropy Hessian used here.

Experimentally, the immediate target is a robust bound for approximate qubits and imperfectly unbiased measurements, converting leakage, misalignment, and finite-sample uncertainty into corrections to S_{XZ} . In larger dimensions, the task is not to restore the false original relation, but to identify dimension-dependent budgets that remain operationally measurable.

Subsequent work by Qian [17] extends separable counterexamples to unequal local dimensions, including qubit-qudit systems, and derives a state-dependent corrected inequality. These higher-dimensional failures leave the universal two-qubit theorem intact and clarify its scope: having a qubit on only one side is insufficient for the guarantee proved here.

Conclusion.— We have proved the CQC relation for every two-qubit state and identified the protecting mechanism as binary-entropy curvature joined with data processing. Together with the higher-dimensional counterexamples, this establishes a sharp boundary between binary and multilevel correlation accounting. Within a trusted qubit model, two complementary outcome tables therefore provide a tomography-free lower bound on total correlation, quantitative one-way distillation rates, and a capacity witness for qubit channels. The broader lesson is that complementarity alone does not make measurement records nonredundant; that property must be established

from the geometry and dimension of the underlying quantum state space.

We thank colleagues in the CQC discussion group for helpful comments. K.C. acknowledges support from the Strategic Priority Research Program of the Chinese Academy of Sciences under Grant No. XDB1680102. OpenAI GPT 5.6 Sol assisted with proof exploration, literature organization, manuscript preparation, and executable cross-checks. The authors independently verified all AI-assisted results and take full responsibility for the work.

Data availability.— No data were created or analyzed. All results are analytic, and executable checks are not premises of the proof. Focused executable cross-checks are included in the `anc/` directory of the source archive of arXiv:2608.04916v1.

* These authors contributed equally to this work.

† Contact author: chenkun@itp.ac.cn

- [1] B. Groisman, S. Popescu, and A. Winter, Quantum, classical, and total amount of correlations in a quantum state, *Physical Review A* **72**, 032317 (2005).
- [2] W. K. Wootters and B. D. Fields, Optimal state-determination by mutually unbiased measurements, *Annals of Physics* **191**, 363 (1989).
- [3] H. Maassen and J. B. M. Uffink, Generalized entropic uncertainty relations, *Physical Review Letters* **60**, 1103 (1988).
- [4] M. J. W. Hall, Information exclusion principle for complementary observables, *Physical Review Letters* **74**, 3307 (1995).
- [5] P. J. Coles and M. Piani, Improved entropic uncertainty relations and information exclusion relations, *Physical Review A* **89**, 022112 (2014).
- [6] P. J. Coles, M. Berta, M. Tomamichel, and S. Wehner, Entropic uncertainty relations and their applications, *Reviews of Modern Physics* **89**, 015002 (2017).
- [7] J. Schneeloch, C. J. Broadbent, and J. C. Howell, Uncertainty relation for mutual information, *Physical Review A* **90**, 062119 (2014).
- [8] P. M. Alsing, C. C. Tison, J. Schneeloch, R. J. Birrittella, and M. L. Fanto, Distribution of density matrices at fixed purity for arbitrary dimensions, *Physical Review Research* **4**, 043114 (2022).
- [9] H. Iqbal, On the CQC conjecture: A sufficient condition and an extension, *Quantum Information Processing* **25**, 250 (2026).
- [10] J. Wang, Q. Wang, and K. Chen, When complementary measurements count the same classical bit twice: Counterexamples to CQC, ECQC, and complementarity-based certification (2026), arXiv:2608.03828 [quant-ph].
- [11] See Supplemental Material at [URL will be inserted by publisher] for full derivations, supporting constructions, and executable-check documentation.
- [12] A. S. Holevo, Bounds for the quantity of information transmitted by a quantum communication channel, *Problems of Information Transmission* **9**, 177 (1973).
- [13] G. Lindblad, Completely positive maps and entropy in-

- equalities, *Communications in Mathematical Physics* **40**, 147 (1975).
- [14] I. Devetak and A. Winter, Distillation of secret key and entanglement from quantum states, *Proceedings of the Royal Society A* **461**, 207 (2005).
 - [15] I. Devetak, The private classical capacity and quantum capacity of a quantum channel, *IEEE Transactions on Information Theory* **51**, 44 (2005).
 - [16] M. Berta, M. Christandl, R. Colbeck, J. M. Renes, and R. Renner, The uncertainty principle in the presence of quantum memory, *Nature Physics* **6**, 659 (2010).
 - [17] L. Qian, Separable counterexamples to complementary quantum correlations, and why random search missed them (2026), version 2, arXiv:2608.14806 [quant-ph].