

Flux-Enhanced Entanglement with an Unchanged Zero-Magnetization Spectrum in a Motzkin Ring

Qihang Wang¹ and Zhiyuan Yao^{2,*}

¹*School of Mathematical Sciences, Peking University, Beijing 100871, China*

²*Lanzhou Center for Theoretical Physics, Key Laboratory of Theoretical Physics of Gansu Province,
Key Laboratory of Quantum Theory and Applications of MoE,*

Gansu Provincial Research Center for Basic Disciplines of Quantum Physics, Lanzhou University, Lanzhou 730000, China

(Dated: October 19, 2026)

Threading a flux through a ring often shifts its energy levels. In the N -site spin-one Motzkin ring, a boundary flux leaves the entire zero-magnetization spectrum unchanged, yet increases the ground-state entanglement by giving each spin path a phase set by its lowest height. For equal halves and fixed nontrivial flux, the excess von Neumann entropy tends to 1.581 nats, while the excess logarithmic negativity grows as $2 \log \log N + O(1)$. The limiting Schmidt weights come in pairs, so local operations that conserve each half's magnetization extract a Bell pair with fidelity approaching one.

Introduction.— Threading a flux Φ through a ring is the standard probe of how a many-body ground state responds to a phase accumulated around it. The energy response defines the stiffness and Drude weight [1, 2], and twisted boundary conditions constrain which ground states can be unique and gapped [3]. Flux insertion also leaves fingerprints in the entanglement spectrum, such as the quantized charge pumped across a cut in topological phases [4]. Energy and entanglement need not respond together, however. A unitary acting across a cut preserves every energy level but can change entanglement [5], and sign or phase structure can change entanglement at fixed configuration probabilities [6, 7]. Can a physical boundary flux, which modifies a single bond, entangle a ground state while leaving every energy of its symmetry sector exactly unchanged?

The spin-one Motzkin chain [8] makes this question exactly solvable. Its local moves conserve magnetization, so the ground state is a superposition of spin paths whose entanglement and correlations can be counted [9–11]. On the unit-weight ring, the $2N + 1$ equal-weight sector ground states were described in Ref. [12], and Appendix D of Ref. [13] proved that they exhaust the ground space, settling a conjecture of Ref. [14]. Area deformations change the path probabilities [15, 16]; a boundary phase instead leaves every probability unchanged. Appendix A compares it with other Motzkin and XXZ deformations.

In this letter, we show that a boundary flux strictly increases the entanglement of the zero-magnetization, or neutral, Motzkin ground state while leaving the entire neutral spectrum unchanged. The flux attaches to each spin path a phase set by its lowest height [Fig. 1]. Because this minimum depends on the whole configuration, the phase couples the two sides of a cut, although a diagonal unitary that preserves every neutral energy removes it exactly. For equal halves at fixed nontrivial flux, the extra von Neumann entropy tends to the flux-independent constant $C_1 \simeq 1.581$, whereas the extra logarithmic negativity grows without bound. The limiting Schmidt weights

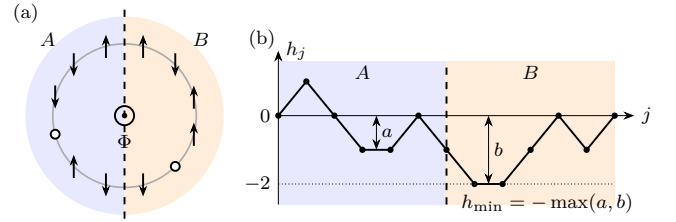


FIG. 1. **How the flux couples the two halves.** (a) A ring of $N = 12$ spin-one sites threaded by a flux Φ ; up arrows, open circles, and down arrows denote u , 0 , and d . The dashed line marks a contiguous cut into regions A and B . (b) Height profile h_j of the neutral configuration in (a). The path dips to depth a in A and b in B , and its global minimum $h_{\min} = -\max(a, b)$ fixes the amplitude phase $e^{-i\Phi h_{\min}}$ of Eq. (5). Because the minimum can lie on either side of the cut, the phase entangles A and B , whereas every zero-magnetization energy is unchanged. The schematic was drawn with generative-AI assistance.

occur in degenerate pairs, so local operations that conserve each region's magnetization extract a logical Bell pair with fidelity approaching one.

Controlled comparisons show what sustains and what removes this effect. The entropy offset and the Bell extraction survive short-range occupation correlations, whereas a uniform neutral gap suppresses the von Neumann response. Adding a reverse pair conversion leaves every zero-flux ground state unchanged but restores a neutral energy response: the missing energy response is a property of the allowed local processes, not of the ground state.

Complex-weight Motzkin ring.— On a ring of $N \geq 2$ spin-one sites, let $\{|u\rangle, |0\rangle, |d\rangle\}$ carry charges $+1, 0, -1$. On ordered bond $(i, i + 1)$, with indices modulo N , choose $z_i \in \mathbb{C} \setminus \{0\}$ and define

$$\begin{aligned} |U_i\rangle &= \frac{|u0\rangle - z_i^*|0u\rangle}{\sqrt{1 + |z_i|^2}}, & |D_i\rangle &= \frac{|0d\rangle - z_i^*|d0\rangle}{\sqrt{1 + |z_i|^2}}, \\ |F_i\rangle &= \frac{|ud\rangle - z_i^*|00\rangle}{\sqrt{1 + |z_i|^2}}. \end{aligned} \quad (1)$$

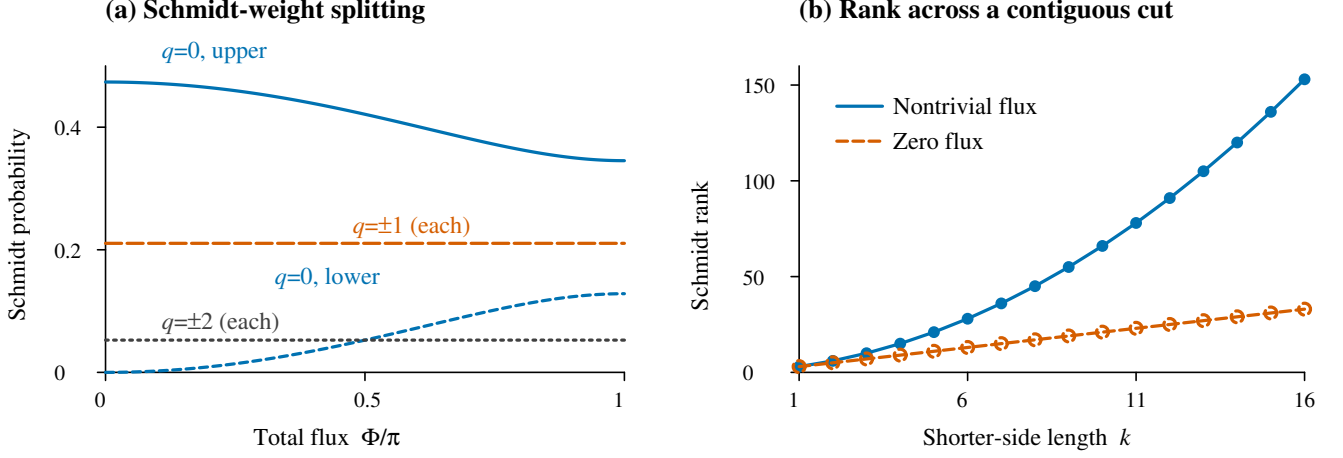


FIG. 2. **The neutral entanglement responds to flux while the neutral spectrum stays fixed.** (a) Exact Schmidt probabilities of the neutral ground state of a four-site ring cut into halves of two sites. The two $q = 0$ probabilities are $(9 \pm \sqrt{49 + 32 \cos \Phi})/38$; each $q = \pm 1$ block has probability $4/19$ and each $q = \pm 2$ block $1/19$. The lower $q = 0$ probability, absent at zero flux, turns on continuously. (b) Exact Schmidt rank for contiguous cuts with shorter-side length k , valid for every ring length: $(k + 1)(k + 2)/2$ at any nontrivial flux (filled circles) and $2k + 1$ at zero flux (open circles). All curves and markers are exact analytic evaluations.

The periodic projector Hamiltonian and its bond holonomy are

$$H_N(\mathbf{z}) = \sum_{i=1}^N \sum_{X=U,D,F} |X_i\rangle\langle X_i|, \quad \mathcal{Z}(\mathbf{z}) = \prod_{i=1}^N z_i. \quad (2)$$

Every projector has unit coefficient, which sets the energy unit, and every local term conserves the total magnetization M . For unit-modulus bonds we write $z_i = e^{i\phi_i}$ and call $\Phi = \sum_i \phi_i \pmod{2\pi}$ the total boundary flux; onsite spin rotations redistribute the individual phases without changing any entanglement. For every flux the neutral sector $M = 0$ contains exactly one zero-energy state. A nontrivial flux removes all nonextreme charged zero modes, leaving this neutral state and the two polarized product states, and the charged spectra shift with flux; the complete complex-bond classification and its winding argument are given in Appendix A.

Neutral spectrum without phase response.— Write $H_N^{(0)}(\mathbf{z})$ for the restriction to $M = 0$ and $z_i = r_i e^{i\phi_i}$ with $r_i > 0$. For a neutral spin configuration x , with $x_i \in \{1, 0, -1\}$, define the heights $h_j = \sum_{i=1}^j x_i$ ($h_0 = h_N = 0$), their minimum $h_{\min} = \min_{0 \leq j \leq N} h_j$, and the cyclic area $A_{\text{cyc}}(x) = \sum_{j=1}^N h_j - N h_{\min}$. Each forward move $u0 \rightarrow 0u$, $0d \rightarrow d0$, or $ud \rightarrow 00$ lowers A_{cyc} by exactly one, also on the wrapping bond. Choose real phase lifts ϕ_i , set $\theta = \sum_i \phi_i / N$, and choose periodic β_i with $\beta_i - \beta_{i+1} = \phi_i - \theta$. The diagonal unitary

$$\mathcal{U}|x\rangle = e^{i\theta A_{\text{cyc}}(x) + i \sum_i \beta_i x_i} |x\rangle \quad (3)$$

obeys $\mathcal{U}_{xx}/\mathcal{U}_{yy} = e^{i\phi_i}$ for every allowed move $x \rightarrow y$ on bond i . The diagonal entries of each projector depend

only on r_i , and its source–target entry is $-z_i/(1 + r_i^2)$. Hence, term by term,

$$H_N^{(0)}(\mathbf{z}) = \mathcal{U} H_N^{(0)}(\mathbf{r}) \mathcal{U}^\dagger. \quad (4)$$

The whole neutral spectrum, not only its ground state, is independent of the bond phases. Because $h_{\min}$ depends on the entire configuration, $\mathcal{U}$ need not factorize across a spatial cut, so equal energies need not imply equal entanglement. The identity survives arbitrary nonnegative coefficients on the individual projectors and any potential diagonal in the S^z basis. Neutral thermal partition functions and multitime correlations of S^z -diagonal observables are therefore flux independent as well. Charged-sector energies, in contrast, can respond to the flux.

Entanglement without spectral change.— Now take $|z_i| = 1$ and unit coefficients. An onsite gauge moves the entire flux onto bond $(N, 1)$ without changing Schmidt probabilities, and the unique neutral zero mode becomes

$$|\Psi_\Phi\rangle = \frac{1}{\sqrt{T(N, 0)}} \sum_{M(x)=0} e^{-i\Phi h_{\min}(x)} |x\rangle, \quad (5)$$

where $T(n, q) = [s^q](s^{-1} + 1 + s)^n$ counts unrestricted walks. Consider a contiguous cut into $1 \leq L < N$ and $R = N - L$ sites, and let $k = \min(L, R)$. Besides the charge q crossing the cut, the decomposition must track how far each segment dips below its starting height. If the first segment ends at height q with depth $a = -\min h_A$, and $b = -q - \min h_B$ for the second, then $h_{\min}(x) = -\max(a, b)$ [Fig. 1(b)]. The reflection principle counts the words of fixed charge and depth, $c_n(q, a) = T(n, q + 2a) - T(n, q + 2a + 2)$ for

$\max(0, -q) \leq a \leq \lfloor (n - q)/2 \rfloor$. Normalized equal-weight sums of these words form orthonormal bases, in which the coefficient matrices are

$$K_q(a, b) = \sqrt{\frac{c_L(q, a)c_R(q, b)}{T(N, 0)}} e^{i\Phi \max(a, b)}. \quad (6)$$

Their squared singular values are the complete Schmidt probabilities. At zero flux every charge block has rank one. At nontrivial flux, words with different depths acquire relative phases, and a nonvanishing phase minor shows that every block attains its maximal rank, for every cut and every $\Phi \notin 2\pi\mathbb{Z}$, including roots of unity. Summing over q gives

$$\chi_\Phi = \begin{cases} 2k + 1, & \Phi \in 2\pi\mathbb{Z}, \\ (k + 1)(k + 2)/2, & \Phi \notin 2\pi\mathbb{Z}. \end{cases} \quad (7)$$

The rank alone does not fix how the entropy grows.

The block weights $p_q = T(L, q)T(R, q)/T(N, 0)$ do not depend on flux; the flux only splits p_q into $p_q r_{q,j}$, with r_q a probability vector. The zero-flux Schmidt vector therefore majorizes the flux-dependent one, and every Rényi entropy $S_\alpha = (1 - \alpha)^{-1} \log \text{Tr} \rho_A^\alpha$, with natural logarithms and its von Neumann limit, obeys

$$S_\alpha(\Phi) > S_\alpha(0), \quad 0 < \alpha < \infty, \quad k \geq 2, \quad \Phi \notin 2\pi\mathbb{Z}. \quad (8)$$

Strictness comes from the $q = 0$ block, which is mixed for $k \geq 2$; for $k = 1$ every block keeps rank one and the Schmidt spectrum is unchanged. Operationally, a measurement of the subsystem magnetization has flux-independent outcome probabilities, but at nontrivial flux the postmeasurement state with $q = 0$ is entangled whenever $k \geq 2$. Appendix D gives the proofs, including negative charges and unequal cuts.

The smallest example is a four-site ring with $L = R = 2$, whose purity is

$$\text{Tr} \rho_A^2 = \frac{99 + 16 \cos \Phi}{361}. \quad (9)$$

It varies continuously with flux, following the Schmidt-weight splitting in Fig. 2(a), although every neutral energy stays fixed.

Thermodynamic excess entanglement.— Does the quadratic Schmidt rank change the leading growth of the von Neumann entropy? It does not. With normalized blocks $\hat{K}_q = K_q/\sqrt{p_q}$ and $\sigma_q = \hat{K}_q \hat{K}_q^\dagger$, the excess is $\Delta S_1 \equiv S_1(\Phi) - S_1(0) = \sum_q p_q S(\sigma_q)$. In the decomposition of entanglement into number and configurational, or symmetry-resolved, parts [17–19], the flux changes only the configurational part; the number entanglement $-\sum_q p_q \log p_q$ is flux independent. The ordered conditional probabilities obey the size-independent envelope $\lambda_j \leq 128/j^2$, which stays summable after multiplication

by the logarithm entering the von Neumann entropy. The extra entropy is therefore bounded uniformly in size, flux, and cut; a low-rank approximation of the coefficient kernel proves this bound (Appendix G).

For equal halves, $N = 2n$ and $L = R = n$, the excess approaches the same constant at every fixed nontrivial flux. The limiting conditional spectrum consists of two copies of $b_j = 4/[\pi^2(2j - 1)^2]$, so that

$$\begin{aligned} \Delta S_1 &\longrightarrow C_1 = -2 \sum_{j \geq 1} b_j \log b_j \simeq 1.581, \\ \Delta S_2 &\longrightarrow \log 3, \quad \frac{\text{Tr} \rho_A(\Phi)^2}{\text{Tr} \rho_A(0)^2} \longrightarrow \frac{1}{3}. \end{aligned} \quad (10)$$

The full entropy is $S_1(\Phi) = \frac{1}{2} \log N + \frac{1}{2} \log(\pi e/3) + C_1 + o(1)$: the flux changes the constant, not the coefficient of the logarithm.

The doubling has a simple origin: the lowest point of the path lies either in A or in B . Expressed through its cumulative probability, the depth turns these two cases into upper- and lower-triangular integration kernels. At fixed nontrivial flux their relative phase oscillates ever faster as the typical depth grows as $\sqrt{N}$, so the two branches become orthogonal; each has spectrum b_j and carries half the weight. This limit requires control of the small Schmidt weights, not only of the largest ones. The same construction gives a bounded continuous crossover $\Delta S_1 \rightarrow F(\kappa)$ when $\Phi\sqrt{n} \rightarrow \kappa$, with $F(0) = 0$ and $F(\kappa) \rightarrow C_1$ as $|\kappa| \rightarrow \infty$, so the zero-flux and thermodynamic limits do not commute. Appendix G proves these statements, expresses F through an explicit continuum operator, and derives the offset for every Rényi order $\alpha > 1/2$. The offsets refer to balanced cuts; the entropy bound holds for every contiguous cut.

Below $\alpha = 1/2$ the small weights dominate. At fixed nontrivial flux and equal halves, we find the sharp growth laws

$$\Delta S_\alpha = \begin{cases} \frac{1 - 2\alpha}{2(1 - \alpha)} \log N + O(1), & 0 < \alpha < \frac{1}{2}, \\ 2 \log \log N + O(1), & \alpha = \frac{1}{2}, \end{cases} \quad (11)$$

with remainders bounded at fixed α and Φ . These laws are governed by $O(\sqrt{N})$ Schmidt weights. A tridiagonal inverse of the depth-phase matrix [20] supplies the lower bound, and Gaussian depth tails the matching upper bound (Appendix G). For a pure bipartition the logarithmic negativity is $E_{\mathcal{N}} = \log \|(|\Psi_\Phi\rangle\langle\Psi_\Phi|)^{T_B}\|_1 = S_{1/2}$ [21], so its flux excess grows without bound while the von Neumann excess stays finite. This statement concerns the full pure-state bipartition, not two separated regions.

Extracting the additional entanglement.— Can this phase response supply an entangled register pair without moving magnetization across the cut? Operations that conserve each region's magnetization access only the entanglement within charge sectors [22, 23], not the full Schmidt spectrum [24]. Give each side a charge-neutral

logical qubit, order the conditional weights $r_{q,j}$, and map consecutive Schmidt states locally onto logical 0 and 1 with a common residual state. Retaining every charge outcome yields the Bell fidelity

$$F_{\text{Bell}} = \frac{1}{2} + \sum_q p_q \sum_{j \geq 1} \sqrt{r_{q,2j-1} r_{q,2j}} \longrightarrow 1. \quad (12)$$

The limit holds for equal halves at fixed nontrivial flux, where the limiting conditional weights occur in equal pairs; neither communication nor postselection is needed. At zero flux the charge blocks are products, and any such charge-preserving local protocol gives $F_{\text{Bell}} \leq 1/2$. Appendix H proves both statements and constructs a logical-register Bell witness. The operations may act on entire halves; we assume neither an efficient circuit nor a fundamental superselection rule for spins. Unlike the protected doublets of the Haldane chain [25], these pairs emerge asymptotically from path depths.

Robustness and its limits.— The phase removal relies on the allowed moves, not on the magnitudes of the matrix elements. Assign positive onsite weights $g_i(s)$ and choose, on each allowed pair $s \rightarrow t$, the source-to-target ratio $e^{i\phi_i} g_i(s_1) g_{i+1}(s_2) / [g_i(t_1) g_{i+1}(t_2)]$. This compatible change of hopping and pair conversion gives neutral amplitudes $\prod_i g_i(x_i)$ times the same phase and preserves the neutral phase conjugacy. Positive depth weights only rescale the rows and columns of Eq. (6), so the ranks, the strict entropy enhancement, and the uniform excess bound survive at fixed weights. For uniform $g_i(u) = g_i(d) = 1$, $g_i(0) = r > 0$, the constants C_1 and $\log 3$ also survive; only the entropy baseline and the crossover scale change (Appendix I). No universal offset is established for arbitrary spatial profiles.

Short-range correlations leave the offset unchanged as well. For a word x , put $\tau_i = |x_i|$ and assign probability weight $w_W(x) = \prod_i W_{\tau_i \tau_{i+1}}$, with a fixed real symmetric 2×2 matrix W of strictly positive entries; the neutral amplitudes are proportional to $\sqrt{w_W(x)} e^{-i\Phi h_{\min}(x)}$. For $N \geq 4$ a four-site frustration-free parent follows from the Rokhsar–Kivelson (RK) construction [26, 27], with local ratios that now depend on the two neighboring spins. For equal halves $N = 2n$ and fixed $\Phi \notin 2\pi\mathbb{Z}$,

$$\Delta S_{W,\alpha} \longrightarrow C_\alpha, \quad \frac{1}{2} < \alpha < \infty, \quad (13)$$

where $\Delta S_{W,\alpha} = S_\alpha(\Psi_{W,\Phi}) - S_\alpha(\Psi_{W,0})$ and C_α are the constants of Appendix G. The weights change the walk variance and add a finite boundary spectrum, which factors from the depth response and cancels in the entropy difference (Appendix J). The limiting weights remain paired, so Eq. (12) again tends to one, although the zero-flux no-go does not extend to this family, in which a charge block can already be entangled.

The offset is nevertheless not a consequence of neutral isospectrality alone. A uniform neutral gap forces the von Neumann flux response to vanish faster than any power

of N , and sufficiently strong positive single-ion anisotropy produces such a gap while preserving the isospectrality; the response at weak fixed anisotropy remains open (Appendix K). A finite offset thus requires gapless neutral excitations, consistent with the $O(N^{-5/2})$ gap bound of the unperturbed ring [28].

Role of the local processes.— Is the absence of a neutral energy response a property of the zero-flux ground state or of the dynamics? In a uniform-flux gauge, add $\epsilon \sum_i |G_i\rangle \langle G_i|$ with $|G_i\rangle = (|du\rangle - e^{i\Phi/N} |00\rangle) / \sqrt{2}$ and fixed $\epsilon > 0$. Every zero-flux ground vector, and hence its entanglement, is unchanged. Yet the neutral ground-energy curvature $\mathcal{K}_{N,\epsilon} = \partial_\Phi^2 E_{N,\epsilon}^{(0)}(0)$ and the full zero-flux gap obey

$$\mathcal{K}_{N,\epsilon} = \Theta_\epsilon(N^{-1}), \quad \gamma_{N,\epsilon}(0) = \Theta_\epsilon(N^{-2}). \quad (14)$$

The new $du \leftrightarrow 00$ step allows an $ud \leftrightarrow du$ exchange through 00 and lifts the kinetic constraint that kept the neutral spectrum flux independent. Series-conductance bounds give the curvature, and a comparison with unrestricted exchanges bounds the gap (Appendix L). The curvature refers to the neutral sector; the polarized states keep the full ground energy at zero. Identical zero-flux ground states therefore need not share their flux response.

A separate four-process gradient family [29] provides the converse comparison. After a measurement of the regional magnetization, the remaining average entanglement is $\mathcal{E}_{\text{ch}} = S(\rho_A) - H(p_q)$. At a tuned flat-state weight, $\mathcal{E}_{\text{ch}}$ of the actual lowest neutral state vanishes at large size, while the neutral energy curvature stays of order N^{-1} . This suppression survives bounded common bond strengths; stability under arbitrary perturbations is not established. Appendix A states the precise limits and rates, including the distinct finite-size weak-flux coefficient.

Conclusion.— A boundary flux can entangle a many-body ground state without shifting a single level of its symmetry sector. In the Motzkin ring the flux attaches to each spin path a phase set by its global minimum; this phase is invisible to every neutral energy and to every S^z -diagonal thermal correlation, but not to entanglement. For balanced cuts the excess von Neumann entropy saturates at $C_1 \simeq 1.581$, the excess logarithmic negativity grows as $2 \log \log N$, and the paired limiting Schmidt weights yield a magnetization-conserving Bell pair. The comparison models show that the allowed local processes, rather than the zero-flux ground state, decide whether the energy responds, and that a nonzero energy response need not entail a conditional-entanglement response.

The specified compatible and correlated families preserve the offset; a uniform neutral gap removes it. Stability under generic weak perturbations remains open. Number-resolved entanglement measurements [19] and proposed Rydberg realizations [30] suggest tests, but implementing the boundary phase remains an open experimental task.

We acknowledge Gewu Intelligence Lab for providing the collaborative research environment in which this project was developed.

Disclosure of automated assistance. OpenAI GPT-5.6 Sol, Anthropic Claude Fable 5, Grok 4.6, and OpenAI GPT-6.1 Sol contributed to the analytical derivations, limiting-regime and finite-size checks, physical interpretation, literature comparison, figure design, and manuscript preparation. The AI-assisted work included the neutral-sector spectral and entanglement analysis, its thermodynamic limit and weak-flux crossover, and revision of the LaTeX text and figures. Anthropic Claude Opus 5.5 rechecked derivations and bibliographic records, revised the manuscript text, and drew the schematic in Fig. 1. The results await further verification by domain experts.

Data availability. All results are analytic and are derived in the Letter and its appendices. Figure 1 is a schematic, Fig. 2 evaluates exact Schmidt probabilities and ranks, Fig. 3 evaluates the displayed spectral formulas, and Fig. 4 shows analytic limiting response coefficients. No experimental data were used. The plotting and finite-checking scripts are available from the authors upon reasonable request.

* Corresponding author: yaozy@lzu.edu.cn

- [1] W. Kohn, Theory of the insulating state, *Phys. Rev.* **133**, A171 (1964).
- [2] B. S. Shastri and B. Sutherland, Twisted boundary conditions and effective mass in Heisenberg–Ising and Hubbard rings, *Phys. Rev. Lett.* **65**, 243 (1990).
- [3] Y. Yao and M. Oshikawa, Twisted boundary condition and Lieb–Schultz–Mattis inapplicability for discrete symmetries, *Phys. Rev. Lett.* **126**, 217201 (2021).
- [4] M. P. Zaletel, R. S. K. Mong, and F. Pollmann, Flux insertion, entanglement, and quantized responses, *J. Stat. Mech.: Theory Exp.* **2014**, P10007.
- [5] A. Hamma and P. Zanardi, Quantum entangling power of adiabatically connected Hamiltonians, *Phys. Rev. A* **69**, 062319 (2004).
- [6] Y. Zhang, T. Grover, and A. Vishwanath, Entanglement entropy of critical spin liquids, *Phys. Rev. Lett.* **107**, 067202 (2011).
- [7] T. Grover and M. P. A. Fisher, Entanglement and the sign structure of quantum states, *Phys. Rev. A* **92**, 042308 (2015).
- [8] S. Bravyi, L. Caha, R. Movassagh, D. Nagaj, and P. Shor, Criticality without frustration for quantum spin-1 chains, *Phys. Rev. Lett.* **109**, 207202 (2012).
- [9] R. Movassagh, Entanglement and correlation functions of the quantum Motzkin spin-chain, *J. Math. Phys.* **58**, 031901 (2017).
- [10] L. Dell’Anna, O. Salberger, L. Barbiero, A. Trombettoni, and V. E. Korepin, Violation of cluster decomposition and absence of light cones in local integer and half-integer spin chains, *Phys. Rev. B* **94**, 155140 (2016).
- [11] V. Menon, A. Gu, and R. Movassagh, Symmetries, correlation functions, and entanglement of general quantum Motzkin spin-chains (2024), arXiv:2408.16070 [quant-ph].
- [12] R. N. Alexander, G. Evenbly, and I. Klich, Exact holographic tensor networks for the Motzkin spin chain, *Quantum* **5**, 546 (2021).
- [13] S. Sengoku and H. Watanabe, Absence of nontrivial local conserved quantities in a class of $U(1)$ -symmetric spin-1 chains (2026), v1, August 18, 2026, arXiv:2608.17548 [cond-mat.str-el].
- [14] A. G. Pronko, Periodic Motzkin chain: Ground states and symmetries, *Nucl. Phys. B* **1017**, 116963 (2025).
- [15] Z. Zhang, A. Ahmadain, and I. Klich, Novel quantum phase transition from bounded to extensive entanglement, *Proc. Natl. Acad. Sci. U.S.A.* **114**, 5142 (2017).
- [16] L. Levine and R. Movassagh, The gap of the area-weighted Motzkin spin chain is exponentially small, *J. Phys. A: Math. Theor.* **50**, 255302 (2017).
- [17] M. Goldstein and E. Sela, Symmetry-resolved entanglement in many-body systems, *Phys. Rev. Lett.* **120**, 200602 (2018), arXiv:1711.09418.
- [18] J. C. Xavier, F. C. Alcaraz, and G. Sierra, Equipartition of the entanglement entropy, *Phys. Rev. B* **98**, 041106 (2018).
- [19] A. Lukin, M. Rispoli, R. Schittko, M. E. Tai, A. M. Kaufman, S. Choi, V. Khemani, J. Léonard, and M. Greiner, Probing entanglement in a many-body-localized system, *Science* **364**, 256 (2019).
- [20] G. Fikioris, Spectral properties of Kac–Murdock–Szegő matrices with a complex parameter, *Linear Algebra Appl.* **553**, 182 (2018).
- [21] G. Vidal and R. F. Werner, Computable measure of entanglement, *Phys. Rev. A* **65**, 032314 (2002).
- [22] H. M. Wiseman and J. A. Vaccaro, Entanglement of indistinguishable particles shared between two parties, *Phys. Rev. Lett.* **91**, 097902 (2003).
- [23] H. Barghathi, C. M. Herdman, and A. Del Maestro, Rényi generalization of the accessible entanglement entropy, *Phys. Rev. Lett.* **121**, 150501 (2018), arXiv:1804.01114 [cond-mat.quant-gas].
- [24] M. A. Nielsen, Conditions for a class of entanglement transformations, *Phys. Rev. Lett.* **83**, 436 (1999).
- [25] F. Pollmann, A. M. Turner, E. Berg, and M. Oshikawa, Entanglement spectrum of a topological phase in one dimension, *Phys. Rev. B* **81**, 064439 (2010).
- [26] C. Castelnovo, C. Chamon, C. Mudry, and P. Pujol, From quantum mechanics to classical statistical physics: Generalized Rokhsar–Kivelson Hamiltonians and the Stochastic Matrix Form decomposition, *Ann. Phys.* **318**, 316 (2005).
- [27] F. Verstraete, M. M. Wolf, D. Perez-Garcia, and J. I. Cirac, Criticality, the area law, and the computational power of projected entangled pair states, *Phys. Rev. Lett.* **96**, 220601 (2006).
- [28] C. McCarthy, H. Singh, S. Gopalakrishnan, and R. Vasseur, Subdiffusive transport in the Fredkin dynamical universality class, *Phys. Rev. B* **111**, 184317 (2025).
- [29] C. Arita, P. L. Krapivsky, and K. Mallick, Variational calculation of transport coefficients in diffusive lattice gases, *Phys. Rev. E* **95**, 032121 (2017), arXiv:1611.07719.
- [30] K. Mukherjee, H. Barghathi, A. Del Maestro, and R. Mukherjee, Quantum simulation of Motzkin spin chains with Rydberg atoms, *Phys. Rev. Lett.* **137**, 150601 (2026).
- [31] N. Reff, Spectral properties of complex unit gain graphs, *Linear Algebra Appl.* **436**, 3165 (2012).

- [32] M. R. Kannan, N. Kumar, and S. Pragada, Bounds for the extremal eigenvalues of gain Laplacian matrices, *Linear Algebra Appl.* **625**, 212 (2021).
- [33] L. Barbiero, L. Dell’Anna, A. Trombettoni, and V. E. Korepin, Haldane topological orders in Motzkin spin chains, *Phys. Rev. B* **96**, 180404(R) (2017).
- [34] R. Andrei, M. Lemm, and R. Movassagh, The spin-one Motzkin chain is gapped for any area weight $t < 1$ (2022), arXiv:2204.04517 [quant-ph].
- [35] F. Sugino and V. Korepin, Rényi entropy of highly entangled spin chains, *Int. J. Mod. Phys. B* **32**, 1850306 (2018).
- [36] J. Wang, Z. Liu, Z. Yan, and C. Wu, Singularity and universality from von Neumann to Rényi entanglement entropy and disorder operator in Motzkin chains, *Phys. Rev. B* **112**, 075128 (2025).
- [37] H. Barghathi and A. Del Maestro, Detection of a Rényi index dependent transition in entanglement entropy scaling, *Quantum* **10**, 2184 (2026).
- [38] L. Dell’Anna, Long-distance entanglement in Motzkin and Fredkin spin chains, *SciPost Phys.* **7**, 053 (2019).
- [39] V. Alba, M. Haque, and A. M. Läuchli, Entanglement spectrum of the Heisenberg XXZ chain near the ferromagnetic point, *J. Stat. Mech.* **2012**, P08011 (2012), arXiv:1206.0131 [cond-mat.str-el].
- [40] S. Dutta, S. Kuhr, and N. R. Cooper, Generating symmetry-protected long-range entanglement, *Phys. Rev. Research* **6**, L012039 (2024), arXiv:2201.10564.
- [41] X.-X. Fang, G. N. M. Tabia, K.-S. Chen, Y.-C. Liang, and H. Lu, Experimental single-copy distillation of quantumness from higher-dimensional entanglement, *Phys. Rev. Lett.* **134**, 150201 (2025), arXiv:2410.06610.
- [42] T. Dörrstel, T. Iadecola, J. H. Wilson, and M. Buchhold, Frustration-free control and absorbing-state transport in entangled state preparation, *Phys. Rev. Lett.* **136**, 120603 (2026).
- [43] F. G. S. L. Brandão, E. Crosson, M. B. Şahinoğlu, and J. Bowen, Quantum error correcting codes in eigenstates of translation-invariant spin chains, *Phys. Rev. Lett.* **123**, 110502 (2019).
- [44] O. Salberger, P. Padmanabhan, and V. Korepin, Non-interacting Motzkin chain—periodic boundary conditions (2018), arXiv:1809.00709 [quant-ph].
- [45] K. Hao, O. Salberger, and V. Korepin, Exact solution of the quantum integrable model associated with the Motzkin spin chain, *J. High Energy Phys.* **2023** (8), 009.
- [46] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed experiment to test local hidden-variable theories, *Phys. Rev. Lett.* **23**, 880 (1969).
- [47] C. Boudreault, C. Berthiere, and W. Witczak-Krempa, Entanglement and separability in continuum Rokhsar-Kivelson states, *Phys. Rev. Research* **4**, 033251 (2022).
- [48] S. Bachmann, S. Michalakakis, B. Nachtergaele, and R. Sims, Automorphic equivalence within gapped phases of quantum lattice systems, *Commun. Math. Phys.* **309**, 835 (2012).
- [49] K. M. R. Audenaert, A sharp continuity estimate for the von Neumann entropy, *J. Phys. A: Math. Theor.* **40**, 8127 (2007).
- [50] D. A. Yarotsky, Ground states in relatively bounded quantum perturbations of classical lattice systems, *Commun. Math. Phys.* **261**, 799 (2006).
- [51] W.-L. You, Y.-W. Li, and S.-J. Gu, Fidelity, dynamic structure factor, and susceptibility in critical phenomena, *Phys. Rev. E* **76**, 022101 (2007).
- [52] X. Chen, E. Fradkin, and W. Witczak-Krempa, Gapless quantum spin chains: Multiple dynamics and conformal wavefunctions, *J. Phys. A: Math. Theor.* **50**, 464002 (2017).
- [53] H. Watanabe, H. Katsura, and J. Y. Lee, Critical spontaneous breaking of U(1) symmetry at zero temperature in one dimension, *Physical Review Letters* **133**, 176001 (2024), arXiv:2310.16881 [cond-mat.stat-mech].
- [54] C. Kipnis, C. Landim, and S. Olla, Hydrodynamical limit for a nongradient system: The generalized symmetric exclusion process, *Commun. Pure Appl. Math.* **47**, 1475 (1994).
- [55] C. Kipnis, C. Landim, and S. Olla, Macroscopic properties of a stationary non-equilibrium distribution for a non-gradient interacting particle system, *Ann. Inst. H. Poincaré Probab. Statist.* **31**, 191 (1995).
- [56] P. Zanardi and L. Campos Venuti, Entanglement susceptibility: area laws and beyond, *J. Stat. Mech.: Theory Exp.* **2013**, P04023, arXiv:1205.2507 [quant-ph].
- [57] V. Alba, M. Haque, and A. M. Läuchli, Boundary-locality and perturbative structure of entanglement spectra in gapped systems, *Phys. Rev. Lett.* **108**, 227201 (2012), arXiv:1107.1726 [cond-mat.str-el].
- [58] V. Balasubramanian, M. B. McDermott, and M. Van Raamsdonk, Momentum-space entanglement and renormalization in quantum field theory, *Phys. Rev. D* **86**, 045014 (2012), arXiv:1108.3568 [hep-th].
- [59] P. Gonçalves and M. Jara, Nonlinear fluctuations of weakly asymmetric interacting particle systems, *Archive for Rational Mechanics and Analysis* **212**, 597 (2014), arXiv:1309.5120.
- [60] P. Gonçalves, M. Jara, and M. Simon, Second order Boltzmann–Gibbs principle for polynomial functions and applications, *Journal of Statistical Physics* **166**, 90 (2017), arXiv:1507.06076.
- [61] H.-T. Yau, Logarithmic Sobolev inequality for generalized simple exclusion processes, *Probab. Theory Relat. Fields* **109**, 507 (1997).
- [62] P. Gonçalves, K. Hayashi, and M. Sasada, Characterization of gradient condition for asymmetric partial exclusion processes and their scaling limits, *ALEA, Lat. Am. J. Probab. Math. Stat.* **22**, 331 (2025), arXiv:2405.15142.
- [63] G. Vidal, Entanglement of pure states for a single copy, *Phys. Rev. Lett.* **83**, 1046 (1999), arXiv:quant-ph/9902033.
- [64] P. Diaconis and L. Saloff-Coste, Logarithmic Sobolev inequalities for finite Markov chains, *Ann. Appl. Probab.* **6**, 695 (1996).
- [65] C. Appert-Rolland, B. Derrida, V. Lecomte, and F. van Wijland, Universal cumulants of the current in diffusive systems on a ring, *Phys. Rev. E* **78**, 021122 (2008), arXiv:0804.2590.
- [66] P. Helms, U. Ray, and G. K.-L. Chan, Dynamical phase behavior of the single- and multi-lane asymmetric simple exclusion process via matrix product states, *Phys. Rev. E* **100**, 022101 (2019), arXiv:1904.07336.
- [67] M. C. Bañuls and J. P. Garrahan, Using matrix product states to study the dynamical large deviations of kinetically constrained models, *Phys. Rev. Lett.* **123**, 200601 (2019).
- [68] S. Yang, Cutoff of the simple exclusion process with inhomogeneous conductances (2024), arXiv:2409.16337 [math.PR].

- [69] M. Sasada, Hydrodynamic limit for two-species exclusion processes, *Stochastic Processes and their Applications* **120**, 494 (2010), arXiv:0910.3994 [math.PR].

Appendix A: Further quantitative results and model comparisons

1. Amplitude transport and complete kernel classification

The following statements complete the complex-bond result used in the Letter. They retain the ordered-bond convention of Eqs. (1) and (2).

The conjugates in Eq. (1) make zero energy equivalent to the amplitude transport

$$\begin{aligned} a(\cdots u0\cdots) &= z_i a(\cdots 0u\cdots), \\ a(\cdots 0d\cdots) &= z_i a(\cdots d0\cdots), \\ a(\cdots ud\cdots) &= z_i a(\cdots 00\cdots). \end{aligned} \tag{S1}$$

Because every z_i is nonzero, the allowed moves are independent of the specific bond values and coincide with the unit-weight legal-move graph. Each fixed-magnetization sector is connected for every $N \geq 2$, as proved in Appendix D of Ref. [13], and therefore supports at most one zero mode. Appendix B recalls the reduction to make the extension to complex bonds self-contained.

The complete kernel is

$$\dim \ker H_N(\mathbf{z}) = \begin{cases} 2N + 1, & \mathcal{Z} = 1, \\ 3, & \mathcal{Z} \neq 1. \end{cases} \tag{S2}$$

For $\mathcal{Z} \neq 1$ the three survivors are the two polarized product states and one neutral parallel section. The key is a winding obstruction. In a closed move sequence, let k_i count signed traversals of the accepted orientation on bond i . Charge-word closure forces all k_i to equal one integer w , so the transported amplitude is $\mathcal{Z}^w$. Every nonextreme charged sector contains an explicit winding-one loop. This excludes even nontrivial roots of unity when $\mathcal{Z} \neq 1$. By contrast, a cyclic-area potential shows that every closed neutral sequence has $w = 0$, so exactly one neutral state always survives. If $\mathcal{Z} = 1$, every loop is consistent and every sector contributes one state. The full construction, including the two oppositely ordered bonds at $N = 2$, is given in Appendix B.

For uniform positive $z_i = t$, the $2N + 1$ states occur only at $t = 1$; a positive wrap defect $s = t^{-(N-1)}$ restores $\mathcal{Z} = 1$. For $|z_i| = 1$, write $z_i = e^{i\phi_i}$ and define the gauge-invariant total flux $\Phi = \sum_i \phi_i \pmod{2\pi}$ [Fig. 3(a)].

2. Explicit size-independent entropy ceiling

The conditional eigenvalue envelope derived in Appendix G gives, for the unmodified unit-modulus model at every size, flux and contiguous cut,

$$0 \leq \Delta S_1 \leq \log \zeta(2) + 2 \log 16 + 2. \tag{S3}$$

3. Charged-sector spectra and one-sided gap bounds

Each one-flat sector $M = \pm(N-1)$ is N -dimensional and invariant, with incidence determinant $|1 - \mathcal{Z}|^2 / \prod_i (1 + |z_i|^2)$. An onsite gauge makes unit-modulus bond phases uniform, giving

$$E_{q,\pm}(\Phi) = 1 - \cos\left(\frac{2\pi q \mp \Phi}{N}\right), \quad q = 0, \dots, N-1. \tag{S4}$$

Their eigenvalue multisets coincide despite opposite momentum shifts [Fig. 3(b)]. These twisted-cycle spectra [31, 32] coexist with the many-body ground-sector selection.

Let $\Delta_N(\Phi)$ be the smallest positive eigenvalue of the full Hamiltonian, and take the principal representative $\bar{\Phi} \in [-\pi, \pi]$. The invariant blocks supply genuine full-system eigenvalues, giving

$$\Delta_N(\Phi) \leq \begin{cases} 1 - \cos(|\bar{\Phi}|/N) \leq \bar{\Phi}^2/(2N^2), & \bar{\Phi} \neq 0, \\ 1 - \cos(2\pi/N), & \bar{\Phi} = 0. \end{cases} \tag{S5}$$

At zero flux the charged ground level $q = 0$ must first be removed. These bounds hold at fixed total flux, not fixed per-bond phase [Fig. 3(c)], and are not the softest known large-ring ceiling. McCarthy *et al.* proved an $O(N^{-5/2})$ upper bound on the original periodic neutral gap [28]. Equation (4) carries that bound to every total flux, so also $\Delta_N(\Phi) \leq CN^{-5/2}$ with a flux-independent constant. This sharper bound uses a neutral excitation, not the one-flat bands.

Let $\lambda_{N,M}(\bar{\Phi})$ be the lowest energy in sector M . An equal-weight untwisted state in the uniform-flux gauge gives, for every nonzero principal flux and $M = -N + 1, \dots, -1, 1, \dots, N - 1$

$$0 < \lambda_{N,M}(\bar{\Phi}) \leq \frac{\bar{\Phi}^2}{4N}. \quad (\text{S6})$$

Thus at least $2N - 2$ positive levels remain in this $O(N^{-1})$ window, counting multiplicity. The one-flat pair obeys the sharper $O(N^{-2})$ bound.

4. Entanglement response in four-process comparison models

The following statements concern the actual lowest neutral states of the specified four-process gradient parents, not the original three-process state or a trial-state entropy.

Entanglement also responds at the separate four-channel gradient point with hopping strengths one and both pair conversions two [29], where the neutral energy does depend on flux. Measuring regional magnetization leaves average entanglement $\mathcal{E}_{\text{ch}} = S(\rho_A) - H(p_q)$. For its actual neutral-sector ground and macroscopic contiguous cuts,

$$\mathcal{E}_{\text{ch}}(\Phi) = -\eta_{N,l}\Phi^4 \log(\Phi^4) + O_N(\Phi^4), \quad \eta_{N,l} = \Theta(1). \quad (\text{S7})$$

For $a_N = l/N$ bounded away from zero and one, $\eta_{N,l} = a_N^3(1 - a_N)^3/576 + O(N^{-1/10})$; the half-ring limit is $1/36864$. One product of regional spin dipoles controls the leading entangling response (Appendix P). Here weak flux is taken before large size. Separately, constants $c, \Phi_*, N_0 > 0$ independent of N give

$$\mathcal{E}_{\text{ch}}(\Phi) \geq c\Phi^4 \log(1/|\Phi|), \quad N \geq N_0, \quad 0 < |\Phi| \leq \Phi_*. \quad (\text{S8})$$

This lower bound also holds for (U, D, F, G) strengths $(1 + t, 1 - t, 2, 2)$ on a small N -independent interval $|t| \leq t_*$. The actual neutral ground retains charge-internal entanglement after charge conjugation is broken. In the same small-flux regime, every nonsingleton fixed- M sector has spacing

$$E_{1,N,M,t}(\Phi) - E_{0,N,M,t}(\Phi) = \Theta(N^{-2}). \quad (\text{S9})$$

Subtracting a common energy shift controls the actual state uniformly; the charge-even dipole correlation changes only at order t^2 and bounds the conditional entanglement. Appendices N and O prove these statements. The gradient relation is retained; generic four-channel rates remain outside this result.

The same gradient family allows a complementary separation. Divide the flat component in both unnormalized pair vectors by $w > 0$, giving zero-flux probabilities proportional to w^{n_0} . At $w = 4$, for any contiguous interval and the actual lowest neutral state of energy $E(\Phi)$,

$$\mathcal{E}_{\text{ch}}(\Phi) \leq C\Phi^4 N^{-1/5} \log(eN/|\Phi|), \quad NE''(0) \longrightarrow \frac{1}{3}. \quad (\text{S10})$$

The small-total-flux window is independent of N . For each fixed $w \neq 4$, macroscopic contiguous cuts instead retain a positive $\Phi^4 \log(1/|\Phi|)$ lower bound. Appendix Q controls the complete entropy tail; the zero-flux baseline itself does not disappear (Fig. 4). The suppression at $w = 4$ also survives arbitrary bounded common bond strengths (Appendix R).

5. Anisotropy, related deformations and remaining questions

Beyond compatible frustration-free deformations, finite diagonal potentials retain strict integer-order Rényi enhancement. A uniform neutral gap instead forces the von Neumann flux response to vanish at large N ; sufficiently strong positive single-ion anisotropy realizes this regime. At zero anisotropy, the magnitude of the neutral energy curvature grows at least as N^2 , and the fidelity susceptibility at least as N^3 , throughout the fixed- r family with bounded positive couplings. These lower bounds leave the entropy at weak fixed anisotropy undetermined (Appendix K).

Existing Motzkin deformations mostly change the magnitudes of path amplitudes. Real area weights alter entanglement and can change the gap [15, 16, 33, 34]. Rényi-index-dependent scaling and noncommuting entropy limits are also known, both in colored open Motzkin chains [35, 36] and in number-conserving state constructions [37]. Negativity between disjoint regions has also been calculated in untwisted Motzkin and Fredkin chains [38]. Unlike anisotropy detuning near the XXZ ferromagnetic point [39], the boundary phase in our uncolored ring preserves all configuration probabilities and the entire neutral energy spectrum. It couples both sides of the cut, whereas the subsystem disorder operator of Ref. [36] acts within one region.

Timed pulses can transfer population between symmetry sectors [40], and single-copy filters can distill mixed-state correlations [41]. Our charge-preserving channel instead retains all outcomes of a pure state.

These states offer benchmarks for preparation [42] and Rydberg proposals [30], neither of which establishes this flux deformation. Nontrivial holonomy removes the growing zero-energy manifold used for approximate codes [43]; integrable variants [44, 45] use different Hamiltonians. Entropy at generic reverse-conversion rates or weak single-ion anisotropy, unequal-cut offsets, nonunit-modulus entropy and coding with soft charged states remain open.

Appendix B: Complex-bond kernel and charged-sector spectra

This appendix proves fixed-sector legal-move connectivity, the complete arbitrary-complex-bond kernel theorem, the one-flat determinant and flux spectra, the full-system gap upper bound, and a charge-resolved low-energy counting bound. It also records positive-bond defect bounds. Ordered bonds are retained separately at $N = 2$.

1. Fixed-sector legal-move connectivity

The unit-weight ground states and their $2N + 1$ degeneracy are known [12]; the connectivity proof is given in Appendix D of Ref. [13]. We recall the cyclic-word reduction here as a self-contained starting point for the complex-bond analysis, not as a new proof of a previously open degeneracy conjecture.

Write a computational-basis word as $x \in \{u, 0, d\}^N$ and let $M(x) = \#u(x) - \#d(x)$. Ignoring the nonzero bond values leaves the reversible moves

$$u0 \leftrightarrow 0u, \quad 0d \leftrightarrow d0, \quad ud \leftrightarrow 00 \quad (\text{S11})$$

on every ordered cyclic bond.

Delete the zeros while retaining the cyclic order of the nonzero letters. If both u and d occur, the resulting cyclic binary word has a transition $u \rightarrow d$. Flat-step swaps expose that pair on an ordered physical bond, possibly across $(N, 1)$, and $ud \rightarrow 00$ removes it. Repeating decreases $\min(\#u, \#d)$ until only one nonzero sign remains. Nonwrapping adjacent swaps then sort the result to

$$c_m = \begin{cases} u^m 0^{N-m}, & m \geq 0, \\ d^{-m} 0^{N+m}, & m < 0. \end{cases} \quad (\text{S12})$$

Thus every word of magnetization m reaches c_m , while every move preserves M . Each fixed- M sector is therefore one connected component. At $N = 2$, the bulk and wrap terms are the two oppositely ordered bonds; neither is deduplicated.

For nonzero complex bonds the graph is unchanged. A zero-energy amplitude is determined throughout a connected sector by its value at one word if and only if its parallel transport is consistent around every closed move sequence. Hence each sector contributes either zero or one kernel vector.

2. Complete complex-bond holonomy classification

Use the accepted orientations

$$u0 \rightarrow 0u, \quad 0d \rightarrow d0, \quad ud \rightarrow 00. \quad (\text{S13})$$

For a closed sequence let k_i be the signed number of traversals on ordered bond $(i, i+1)$. Each forward move changes the charge word by $e_{i+1} - e_i$. Closure gives

$$\sum_{i=1}^N k_i(e_{i+1} - e_i) = 0, \quad (\text{S14})$$

so $k_1 = \dots = k_N = w \in \mathbb{Z}$. At $N = 2$, this reads $k_1(e_2 - e_1) + k_2(e_1 - e_2) = 0$ and gives the same conclusion. The complex amplitude accumulated around the loop is

$$\prod_i z_i^{k_i} = \left(\prod_i z_i \right)^w = \mathcal{Z}^w. \quad (\text{S15})$$

The complex case requires control of w , because $\mathcal{Z}^w$ can equal one at a nontrivial root of unity. Let $0 < m < N$ and start from the binary word $u^m 0^{N-m}$. Apply the directed swaps successively on bonds

$$m, m-1, \dots, 1; \quad m+1, m+2, \dots, N-1; \quad N, \quad (\text{S16})$$

omitting empty ranges. Every step is legal, the word returns to itself, and each ordered bond is traversed exactly once. Thus $w = 1$. Ring reflection together with $u \leftrightarrow d$ maps this loop to the accepted $0d \rightarrow d0$ orientation and supplies a winding-one loop in every nonextreme negative sector. At $N = 2$ the construction is the two-step loop using the two oppositely ordered bonds. Consequently $\mathcal{Z} \neq 1$ kills every nonextreme charged sector, with no root-of-unity exception. The polarized words u^N, d^N have no legal pair and always remain zero modes.

It remains to treat the neutral sector. For a neutral word, let $h_0 = 0$, $h_j = \sum_{k=1}^j x_k$, and define the cyclic area

$$A_{\text{cyc}}(x) = \sum_{j=1}^N (h_j - h_{\min}), \quad h_{\min} = \min_{0 \leq k \leq N} h_k. \quad (\text{S17})$$

On a nonwrapping bond, every forward move in Eq. (S13) lowers exactly one intermediate prefix height by one while leaving the minimum unchanged. Cyclic rotation only permutes the vertex heights and adds a common constant, which subtraction of $h_{\min}$ removes; the same unit drop therefore holds on the wrap bond. Summing around a closed neutral sequence yields

$$0 = \sum_i k_i = Nw, \quad (\text{S18})$$

so $w = 0$. Every neutral loop has trivial complex transport, and connectivity gives exactly one neutral parallel section for arbitrary nonzero $\mathbf{z}$.

If $\mathcal{Z} = 1$, choose nonzero site factors with $q_i/q_{i+1} = z_i$ cyclically. Then the sector- m state

$$|\Omega_m(\mathbf{q})\rangle = \sum_{M(x)=m} \left(\prod_{i=1}^N q_i^{x_i} \right) |x\rangle, \quad x_i \in \{1, 0, -1\}, \quad (\text{S19})$$

obeys all three amplitude relations: the ratios for $u0/0u$, $0d/d0$, and $ud/00$ are all $q_i/q_{i+1} = z_i$. A common rescaling of all q_i only rescales a fixed- M state. Connectivity proves uniqueness. This establishes the $2N+1$ versus 3 classification.

For positive uniform $z_i = t$, Eq. (S17) also gives the explicit neutral state

$$|\Omega_0(t)\rangle = \sum_{M(x)=0} t^{A_{\text{cyc}}(x)} |x\rangle. \quad (\text{S20})$$

With bulk weight t and wrap weight $s = t^{-(N-1)}$, a cyclic choice of the q_i restores all sectors. This is the positive radial compensation quoted in Appendix A.

3. Complex one-flat determinant

Let $|j\rangle$ denote the positive one-flat word whose unique flat letter is at site j . The local ket has coordinates $-z_i^* e_i + e_{i+1}$, so the corresponding constraint row is

$$b_i^+ = \frac{-z_i e_i + e_{i+1}}{\sqrt{1 + |z_i|^2}}. \quad (\text{S21})$$

In the negative block the row is $b_i^- = (e_i - z_i e_{i+1})/\sqrt{1 + |z_i|^2}$. Hence $H_\pm = B_\pm^\dagger B_\pm$.

After removing row normalizations, a nonzero determinant permutation chooses either all diagonal entries or all cyclic successors; any mixture repeats or omits a column. Up to an irrelevant sign,

$$|\det B_\pm| = \frac{|1 - \mathcal{Z}|}{\sqrt{\prod_i (1 + |z_i|^2)}}. \quad (\text{S22})$$

Taking the Gram determinant proves

$$\det H_+ = \det H_- = \frac{|1 - \mathcal{Z}|^2}{\prod_i (1 + |z_i|^2)}. \quad (\text{S23})$$

At $\mathcal{Z} \neq 1$ each square incidence matrix is invertible. At $\mathcal{Z} = 1$ its nonzero-bond recursion fixes every component from one initial value, so the nullity is exactly one. For $N = 2$, $B_+ \propto \begin{pmatrix} -z_1 & 1 \\ 1 & -z_2 \end{pmatrix}$; the two ordered bonds give the same formula directly.

4. Uniform complex band and unit-modulus gauge reduction

For $z = \rho e^{i\theta}$, Fourier waves $a_j = e^{ik_q j}$ with $k_q = 2\pi q/N$ diagonalize both circulants. Direct multiplication gives

$$\begin{aligned} E_{q,+} &= 1 - \frac{2\rho}{1 + \rho^2} \cos(k_q - \theta), \\ E_{q,-} &= 1 - \frac{2\rho}{1 + \rho^2} \cos(k_q + \theta). \end{aligned} \quad (\text{S24})$$

Equivalently, the positive branch is

$$E_{q,+} = \frac{(\rho - 1)^2}{1 + \rho^2} + \frac{2\rho}{1 + \rho^2} [1 - \cos(k_q - \theta)], \quad (\text{S25})$$

with the opposite phase in the negative branch. At uniform positive $\rho = t$, this reduces to the accepted radial mass $(t - 1)^2/(1 + t^2)$ at $q = 0$.

Now take $z_i = e^{i\phi_i}$ and choose a representative $\Phi = \sum_i \phi_i$. Periodic site phases α_i satisfying

$$\alpha_{i+1} - \alpha_i = \phi_i - \Phi/N \quad (\text{S26})$$

exist because the right-hand side sums to zero. The associated diagonal unitary makes every positive one-flat bond phase $e^{i\Phi/N}$; the inverse diagonal choice does the same for the negative block. Equation (S24) yields

$$E_{q,\pm}(\Phi) = 1 - \cos\left(\frac{2\pi q \mp \Phi}{N}\right). \quad (\text{S27})$$

Changing Φ by 2π merely relabels q . Reflection $q \mapsto N - q$ shows that the two eigenvalue multisets coincide. Their common determinant is consistently

$$\prod_q E_{q,\pm} = \frac{4 \sin^2(\Phi/2)}{2^N}. \quad (\text{S28})$$

5. Full-system gap upper bound

Magnetization conservation makes each complete one-flat block an invariant block of the full Hamiltonian. Let $\bar{\Phi} \in [-\pi, \pi]$ be the principal total flux. If $\bar{\Phi} \neq 0$, the closest allowed twisted momentum is $q = 0$, and Eq. (S27) contains the positive full-system eigenvalue

$$\varepsilon_{1f}(N, \Phi) = 1 - \cos(|\bar{\Phi}|/N). \quad (\text{S29})$$

The kernel theorem removes every charged zero mode at nontrivial flux, so this level is excited. The smallest positive full-system eigenvalue therefore satisfies

$$\Delta_N(\Phi) \leq \varepsilon_{1f}(N, \Phi) \leq \frac{\bar{\Phi}^2}{2N^2}. \quad (\text{S30})$$

At $\bar{\Phi} = 0$, the $q = 0$ level is a ground state; the first positive one-flat level instead gives

$$\Delta_N(0) \leq 1 - \cos(2\pi/N). \quad (\text{S31})$$

For fixed nonzero total flux, Eq. (S30) forces the full gap to close as $N \rightarrow \infty$. It gives neither a lower bound nor the exact global asymptotic. A family with fixed per-bond phase has $\Phi_N = N\theta \pmod{2\pi}$ and must select the zero- or nonzero-flux branch at each N separately.

6. Charge-resolved low-energy family

We now prove that every nonextreme charged sector lifted by nonzero flux still contains a vanishing-energy excitation. Take unit-modulus bonds and choose real phase lifts $z_i = e^{i\phi_i}$ such that $\sum_i \phi_i = \bar{\Phi}$, where $\bar{\Phi} = \text{Arg}(\prod_i z_i) \in [-\pi, \pi] \setminus \{0\}$. An onsite diagonal unitary

$$U_{\alpha}|x_1 \cdots x_N\rangle = \exp\left(i \sum_{j=1}^N \alpha_j x_j\right) |x_1 \cdots x_N\rangle, \quad x_j \in \{1, 0, -1\}, \quad (\text{S32})$$

preserves every magnetization sector. Choosing periodic phases such that

$$\alpha_{i+1} - \alpha_i = \phi_i - \bar{\Phi}/N \quad (\text{S33})$$

sends all three projectors on every bond to the uniform representative $z_i = e^{i\varphi}$, where $\varphi = \bar{\Phi}/N$. The right-hand side of Eq. (S33) sums to zero, so this gauge exists on the full ring, including the two ordered bonds at $N = 2$.

For $n \geq 0$ and integer m , define the trinomial count

$$T(n, m) = [x^m](x^{-1} + 1 + x)^n, \quad D(N, M) = T(N, M), \quad (\text{S34})$$

with $T(n, m) = 0$ outside $|m| \leq n$, and set

$$A(N, M) = T(N - 2, M - 1) + T(N - 2, M) + T(N - 2, M + 1). \quad (\text{S35})$$

In the uniform gauge, use the normalized untwisted sector state

$$|\Omega_{N,M}\rangle = \frac{1}{\sqrt{D(N, M)}} \sum_{M(x)=M} |x\rangle \quad (\text{S36})$$

as a trial vector. On one ordered bond, each of the U , D , and F move pairs has equal trial amplitudes. For a fixed outside word, its contribution to the corresponding normalized twisted projector is

$$\frac{|1 - e^{i\varphi}|^2}{2D(N, M)} = \frac{1 - \cos \varphi}{D(N, M)}. \quad (\text{S37})$$

The outside magnetization is respectively $M - 1$, $M + 1$, or M . Summing the three context counts and then the N ordered bonds gives the exact Rayleigh quotient

$$\langle \Omega_{N,M} | H_N(e^{i\varphi}) | \Omega_{N,M} \rangle = N[1 - \cos(\bar{\Phi}/N)] \frac{A(N, M)}{D(N, M)}. \quad (\text{S38})$$

Splitting the final two letters of a length- N word yields

$$\begin{aligned} D(N, M) &= T(N - 2, M - 2) + 2T(N - 2, M - 1) \\ &\quad + 3T(N - 2, M) + 2T(N - 2, M + 1) + T(N - 2, M + 2). \end{aligned} \quad (\text{S39})$$

Consequently,

$$D(N, M) - 2A(N, M) = T(N - 2, M - 2) + T(N - 2, M) + T(N - 2, M + 2) \geq 0. \quad (\text{S40})$$

Rayleigh–Ritz, Eq. (S40), and $1 - \cos y \leq y^2/2$ now give, for every nonextreme charged sector,

$$0 < \lambda_{N,M}(\bar{\Phi}) \leq N[1 - \cos(\bar{\Phi}/N)] \frac{A(N, M)}{D(N, M)} \leq \frac{N}{2}[1 - \cos(\bar{\Phi}/N)] \leq \frac{\bar{\Phi}^2}{4N}. \quad (\text{S41})$$

Strict positivity follows from the kernel theorem: at nonzero flux no nonextreme charged sector contains a zero mode. The $2N - 2$ sectors in Eq. (S41) are mutually orthogonal invariant subspaces. Selecting one sector eigenvector from each proves that the full Hamiltonian has at least $2N - 2$ positive eigenvalues in $(0, \bar{\Phi}^2/(4N)]$, counting multiplicity.

At $M = \pm(N - 1)$, $D(N, M) = N$ and $A(N, M) = 1$. The trial quotient is then $1 - \cos(|\bar{\Phi}|/N)$, and the one-flat Fourier calculation proves that this trial vector is the exact $q = 0$ eigenvector. Thus the edge pair retains the sharper $O(N^{-2})$ scale. Outside the edge sectors, Eq. (S38) is only a variational quotient: we do not claim exact energies, distinct levels, a matching lower bound, or an exhaustive description of the low-energy spectrum.

7. Positive one-defect bounds

For completeness, set $z_1 = \dots = z_{N-1} = t > 0$ and $z_N = s > 0$. Trial vectors

$$a^+ = (1, t, t^2, \dots, t^{N-1}), \quad a^- = (t^{N-1}, \dots, t, 1) \quad (\text{S42})$$

are annihilated by all bulk incidence rows. Their wrap-row Rayleigh quotient gives, for the two one-flat minima $\lambda_{\pm}(t, s)$,

$$\lambda_{\pm}(t, s) \leq \frac{(1 - st^{N-1})^2}{(1 + s^2) \sum_{j=0}^{N-1} t^{2j}}. \quad (\text{S43})$$

Each $H_{\pm}$ is a sum of N normalized projectors, so $0 \leq H_{\pm} \leq NI$. Using $\det H_{\pm} = \lambda_1 \dots \lambda_N$ gives the finite-size lower bound

$$\lambda_{\pm}(t, s) \geq \frac{(1 - st^{N-1})^2}{N^{N-1}(1 + t^2)^{N-1}(1 + s^2)}. \quad (\text{S44})$$

The two bounds locate the unique fixed- N, t closing at $s = t^{-(N-1)}$ and its quadratic dependence on radial mismatch. The lower bound is nonuniform in N and implies no thermodynamic gap.

8. Scope of analytic and computational evidence

All all- N conclusions above follow from the analytic arguments. Local ancillary checkers independently enumerate winding-one cycles through $N = 30$, compute exact complex-rational ranks and determinants through $N = 5$, compare ten numerical phase spectra, and test the charge-resolved counting and Rayleigh formulas in bounded sectors. These finite checks corroborate the proofs but are not used in them. No result here establishes a global gap lower bound, an exact global-gap asymptotic, a bulk phase transition, or a quantized response.

Appendix C: Complete neutral-sector phase removal

Keep the ordered-bond projectors and conjugation convention of Eq. (1). Write $H_N^{(0)}$ for their restriction to $M = 0$. For $z_i = r_i e^{i\phi_i}$, $r_i > 0$, choose arbitrary real lifts ϕ_i and put $\theta = N^{-1} \sum_i \phi_i$. Periodic site phases β_i satisfying $\beta_i - \beta_{i+1} = \phi_i - \theta$ exist because the right sides sum to zero. On neutral words define

$$\mathcal{U}|x\rangle = \exp \left[i\theta A_{\text{cyc}}(x) + i \sum_i \beta_i x_i \right] |x\rangle. \quad (\text{S45})$$

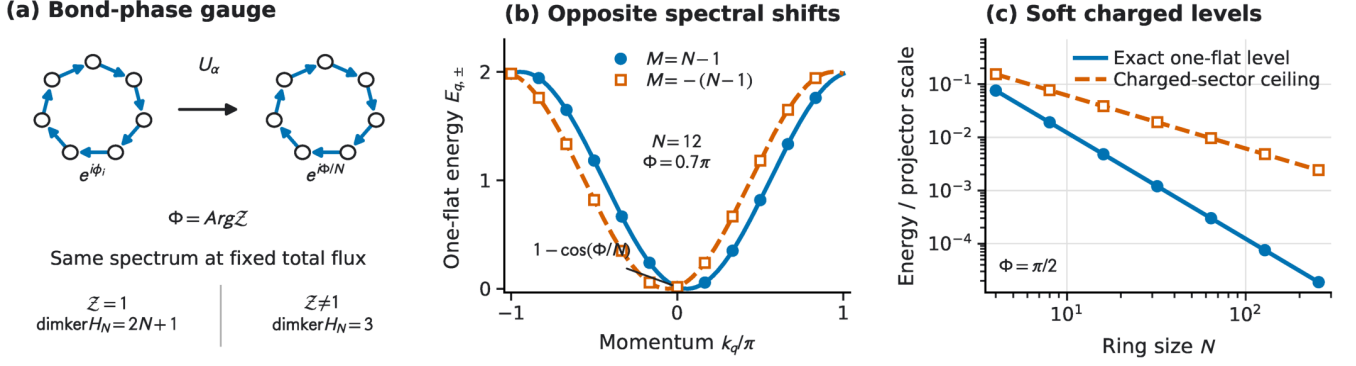


FIG. 3. **Exact zero-mode selection and its gapless remnant.** (a) Onsite gauge rotations redistribute unit-modulus bond phases uniformly, leaving the total flux $\Phi = \text{Arg } \mathcal{Z}$; the displayed kernel classification holds for arbitrary nonzero complex z_i . (b) At $N = 12$ and $\Phi = 0.7\pi$, the two complete one-flat sectors flow in opposite momentum directions (markers are the finite-ring momenta), although their eigenvalue multisets coincide. (c) For fixed $\Phi = \pi/2$, the exact one-flat level $1 - \cos(\Phi/N)$ gives the N^{-2} full-gap upper bound, while at least $2N - 2$ positive eigenvalues lie at or below the N^{-1} charged-sector ceiling. Neither curve is claimed to be the exact global gap. All curves are direct evaluations of Eqs. (S4)–(S6).

This is a unitary on the entire neutral Hilbert space, not an onsite unitary.

For clarity, the unit-drop identity of Eq. (S17) is local and exact. On an interior bond i , the three accepted source pairs have intermediate height respectively $h_{i-1} + 1$, h_{i-1} , and $h_{i-1} + 1$. After the move they equal respectively h_{i-1} , $h_{i+1} = h_{i-1} - 1$, and h_{i-1} . Thus the new intermediate height equals an already present neighboring height; the global minimum does not change. Exactly one height is lowered by one. A neutral cyclic rotation translates all vertex heights by a common constant and permutes them, preserving A_{cyc} . Rotate to make a wrapping move interior. This also applies at $N = 2$, with the two ordered bonds kept separately.

For any accepted move on bond i , source minus target has charge difference $e_i - e_{i+1}$ and area difference one. Equation (S45) therefore gives

$$\frac{U_{xx}}{U_{yy}} = e^{i(\theta + \beta_i - \beta_{i+1})} = e^{i\phi_i}. \quad (\text{S46})$$

The local ket is $(|x\rangle - z_i^*|y\rangle)/\sqrt{1 + r_i^2}$, so its source–target matrix entry is $-z_i/(1 + r_i^2)$. The diagonal entries are $1/(1 + r_i^2)$ and $r_i^2/(1 + r_i^2)$. Conjugating the positive-bond projector by $\mathcal{U}$ gives those same three entries. Consequently, term by term,

$$H_N^{(0)}(\mathbf{z}) = \mathcal{U} H_N^{(0)}(\mathbf{r}) \mathcal{U}^\dagger. \quad (\text{S47})$$

All neutral eigenvalues and multiplicities agree. Phase lifts only affect the representative $\mathcal{U}$, not this identity.

The calculation is unchanged with arbitrary nonnegative real coefficients $J_{i,X}$ on the corresponding $X = U, D, F$ projectors. A real diagonal potential V commutes with $\mathcal{U}$ and may be added to both restricted Hamiltonians. This robustness does not assert that V or vanishing $J_{i,X}$ preserve the unique frustration-free neutral state used below. No charged-sector phase removal follows: their winding-one loops obstruct it.

In units $\hbar = k_B = 1$, set $\rho_{\mathbf{z}} = e^{-\beta H_N^{(0)}(\mathbf{z})}/Z_0$ and similarly for $\mathbf{r}$. Functional calculus gives equal Z_0 and $\rho_{\mathbf{z}} = \mathcal{U} \rho_{\mathbf{r}} \mathcal{U}^\dagger$. For each computational-basis-diagonal observable O , $\mathcal{U}^\dagger O \mathcal{U} = O$, and $O_{\mathbf{z}}(t) = \mathcal{U} O_{\mathbf{r}}(t) \mathcal{U}^\dagger$. Cyclicity of trace proves equality of any thermal multitime product of such observables. The same statement holds for initial states transported by $\mathcal{U}$. It does not assert equality for an arbitrary fixed off-diagonal initial state or observable.

Appendix D: Neutral state and charge–depth Schmidt blocks

From here on all $|z_i| = 1$, $J_{i,X} = 1$, and $V = 0$. An onsite gauge sends the bonds to $z_1 = \dots = z_{N-1} = 1$, $z_N = e^{i\Phi}$ and preserves the Schmidt spectrum. The equal-weight neutral state is the unique neutral zero mode at zero flux. In

the uniform gauge its coefficients are $e^{i\Phi A_{\text{cyc}}/N}$. Since

$$\sum_{j=1}^N h_j = \sum_{i=1}^N (N - i + 1)x_i, \quad (\text{S48})$$

removing the onsite phase from this expression leaves

$$|\Psi_\Phi\rangle = T(N, 0)^{-1/2} \sum_{M(x)=0} e^{-i\Phi h_{\min}(x)} |x\rangle. \quad (\text{S49})$$

It obeys unit transport on every interior bond and transport $e^{i\Phi}$ on the seam. Normalization is unchanged because every amplitude has unit modulus. Changing Φ by 2π leaves the state unchanged.

Let $N = L + R$, $L, R \geq 1$, with the first L sites comprising A . Any contiguous cut can be brought to this convention by cyclic relabeling and onsite gauge redistribution. For a length- n word ending at q , let $a = -\min h$. Its feasible depths are

$$I_n(q) = \left\{ \max(0, -q), \dots, \left\lfloor \frac{n-q}{2} \right\rfloor \right\}. \quad (\text{S50})$$

The interval is empty when $|q| > n$. A path reaching depth a requires at least a down steps and $a + q$ up steps, hence $2a + q \leq n$. Conversely, those down steps followed by those up steps and zeros realize every listed depth.

First-hit reflection at height $-a - 1$ maps paths ending at q that cross that boundary to unrestricted paths ending at $q + 2a + 2$; reversing step signs on the initial reflected segment preserves the flat steps. Thus the number with minimum at least $-a$ is

$$C_n(q, a) = T(n, q) - T(n, q + 2a + 2). \quad (\text{S51})$$

Subtracting the count at $a - 1$ gives

$$c_n(q, a) = T(n, q + 2a) - T(n, q + 2a + 2) > 0, \quad a \in I_n(q). \quad (\text{S52})$$

At the lower endpoint $a_0 = \max(0, -q)$, the predecessor count is zero. For $q \geq 0$ this follows from $T(n, q) - T(n, q) = 0$; for $q < 0$ it follows from $T(n, q) = T(n, -q)$. There is no omitted boundary term. We use $T(n, q) = 0$ for $|q| > n$, including $n = 0$.

An A word has endpoint q and depth a . A B word, measured from its own initial height zero, has endpoint $-q$ and depth offset $b = -q - \min h_B$. Reversing B and flipping its steps gives endpoint q and minimum $q + \min h_B = -b$, so it has exactly $c_R(q, b)$ possibilities. The full bridge has minimum $-\max(a, b)$.

Define normalized equal sums $|q, a\rangle_A$ and $|-q, b\rangle_B$ over these disjoint classes. They are orthonormal, and the state is exactly

$$|\Psi_\Phi\rangle = \sum_{q=-k}^k \sum_{a \in I_L(q)} \sum_{b \in I_R(q)} K_q(a, b) |q, a\rangle_A |-q, b\rangle_B, \quad k = \min(L, R), \quad (\text{S53})$$

with K_q given by Eq. (6). The complete Schmidt probabilities are the eigenvalues of $K_q K_q^\dagger$ for all q , including their multiplicities. The Frobenius norm of each block is

$$p_q = \|K_q\|_F^2 = \frac{T(L, q)T(R, q)}{T(N, 0)}, \quad \sum_q p_q = 1. \quad (\text{S54})$$

This follows by summing the depth counts and by concatenating the two words.

Appendix E: Exact rank, majorization, and strict entropy enhancement

At $t = e^{i\Phi} = 1$, each positive weighted block has rank one. For $t \neq 1$, its smaller depth interval is contained in its larger one. Restricting both indices to $a_0, \dots, a_0 + d - 1$, subtract preceding rows from succeeding rows, bottom-up. The resulting square phase matrix has determinant

$$\det[t^{\max(a, b)}]_{a, b=a_0}^{a_0+d-1} = t^{a_0 d + d(d-1)/2} (1 - t)^{d-1}. \quad (\text{S55})$$

For example, after the subtraction row $j > 0$ has entries $t^{a_0+j} - t^{a_0+j-1}$ in columns below j and zero entries elsewhere. Expansion along the last column then evaluates the determinant recursively. It is nonzero at every $t \neq 1$, including nontrivial roots of unity. The row and column count factors are strictly positive, so

$$\text{rank } K_q = \left\lfloor \frac{k - |q|}{2} \right\rfloor + 1. \quad (\text{S56})$$

Summing over q counts pairs of nonnegative integers $(a, a+q)$ whose sum is at most k , hence $(k+1)(k+2)/2$. At $t = 1$ the sum is $2k+1$. This proves Eq. (7) for every ring and contiguous cut.

Write the normalized reduced eigenvalues of block q as $r_{q,j}$. At zero flux the block contributes just p_q ; otherwise its probabilities are $p_q r_{q,j}$. Splitting one weight into these components is a doubly stochastic mixing of $(p_q, 0, \dots, 0)$. Indeed, for a block of size $d > 1$ one may use a matrix whose first column is r_q and whose other columns all equal $(1 - r_q)/(d - 1)$. Its row and column sums are one. Direct sums of these mixing matrices prove majorization, with zeros padded as needed; no ordering of different charge blocks is assumed.

For $\alpha > 1$, $\sum_j r_{q,j}^\alpha \leq 1$; for $0 < \alpha < 1$ the inequality is reversed. Multiplying by p_q^α , summing and applying $(1 - \alpha)^{-1} \log$ gives $S_\alpha(\Phi) \geq S_\alpha(0)$. At $\alpha = 1$ the identity is

$$S(\Phi) = - \sum_q p_q \log p_q + \sum_q p_q S(r_q). \quad (\text{S57})$$

If $k \geq 2$ and $t \neq 1$, the $q = 0$ block has rank at least two and positive weight, giving strictness for every finite positive α . If $k = 1$ all blocks have rank one and their eigenvalues remain p_q . Entropy is continuous at zero flux despite the rank jump. No inequality for nonunit moduli, no monotonicity between two nonzero fluxes, and no thermodynamic entropy exponent are needed for this result.

Measuring the charge of A gives q with probability p_q , independent of flux. The conditioned state has coefficient matrix $K_q/\sqrt{p_q}$; it is a product at trivial flux and entangled at $q = 0$ for $k \geq 2$ and $t \neq 1$. This is a conditional-state statement, not an assumption about spin-charge superselection or a proposed distillation protocol.

Appendix F: Four-spin all-flux benchmark and finite corroboration

For $L = R = 2$, $T(4, 0) = 19$, and the $q = 0$ block is

$$K_0 = \frac{1}{\sqrt{19}} \begin{pmatrix} 2 & \sqrt{2}t \\ \sqrt{2}t & t \end{pmatrix}. \quad (\text{S58})$$

Its reduced trace is $9/19$ and determinant $8(1 - \cos \Phi)/361$. The two Schmidt probabilities are therefore

$$\lambda_{\pm} = \frac{9 \pm \sqrt{49 + 32 \cos \Phi}}{38}. \quad (\text{S59})$$

The $q = \pm 1$ probabilities are each $4/19$, and the $q = \pm 2$ probabilities are each $1/19$. Thus the two-spin reduced purity is

$$\text{Tr } \rho_A^2 = \frac{99 + 16 \cos \Phi}{361}. \quad (\text{S60})$$

It changes from $115/361$ to $83/361$ between zero and π flux, while the complete neutral Hamiltonian remains isospectral. These expressions generate Fig. 2(a); no measured-data claim is involved.

Exact finite checks corroborate the proofs: direct neutral-word enumeration through $N = 9$ checks the cyclic-area and rational-complex edge phases; direct path counts through length eight check Eq. (S52); raw coefficient matrices through $N = 8$ check ranks and purity at zero and π flux. Additional exact rank tests through $N = 7$ use $t = i$ and $t = (3 + 4i)/5$. A separate full-state Laurent-polynomial partial trace at $N = 4$ gives purity numerator $99 + 8t + 8/t$. These finite checks are not used to infer the all- N or all-flux statements, which follow from the analytic arguments above.

Appendix G: Bounded excess entropy and its thermodynamic limit

We retain the unmodified, unit-modulus Hamiltonian and neutral state of Appendix D. Entropies use natural logarithms. Write $\hat{K}_q = K_q/\sqrt{p_q}$ and $\sigma_q = \hat{K}_q \hat{K}_q^\dagger$, so $\text{Tr } \sigma_q = 1$. At zero flux every charge block is pure. Therefore

$$\Delta S_1 = \sum_q p_q S(\sigma_q), \quad (\text{S61})$$

$$e^{(1-\alpha)\Delta S_\alpha} = \frac{\sum_q p_q^\alpha \text{Tr } \sigma_q^\alpha}{\sum_q p_q^\alpha}, \quad \alpha \neq 1. \quad (\text{S62})$$

The second formula involves normalized p_q^α , not p_q .

1. A uniform bound that controls the growing Schmidt rank

Consider any finite probability distributions P_a, Q_a on a common ordered index set, with zero entries added if necessary. For $|t| = 1$ define

$$K(a, b) = \sqrt{P_a Q_b} t^{\max(a, b)}, \quad \sigma = K K^\dagger. \quad (\text{S63})$$

Its squared Hilbert–Schmidt norm is one. Split an atom into subatoms with fractions $s_{a,l}$, using the same fractions for P and Q , and retaining the original phase t^a on every copy. The refined matrix is an isometric embedding of K on both sides, so its nonzero singular values are unchanged.

Let $\mu_a = (P_a + Q_a)/2$. By splitting atoms at quantiles, partition the ordered refined indices into r consecutive groups G_g , each of μ -mass $1/r$. Approximate K by using the column phase when its row group is at most its column group, and the row phase otherwise. For each row group this is a sum of two outer products; hence the approximation K_r has rank at most $2r$. Entries in different groups are exact, even when an original atom was split across groups. The remaining error obeys

$$\|K - K_r\|_{\text{HS}}^2 \leq 4 \sum_g P(G_g) Q(G_g) \leq \frac{4}{r}, \quad (\text{S64})$$

because $P(G_g) + Q(G_g) = 2/r$. If λ_j are the eigenvalues of σ in decreasing order, the best-rank approximation inequality gives

$$\sum_{j > 2r} \lambda_j \leq \frac{4}{r}. \quad (\text{S65})$$

For a random index J with probabilities λ_j , choose $r = \lfloor x/2 \rfloor$ when $x \geq 4$. Equation (S65) implies $\Pr(J > x) \leq \min(1, 16/x)$ for all $x \geq 1$, and thus $\mathbb{E} \log J \leq \log 16 + 1$. Nonnegative relative entropy to the reference distribution $j^{-2}/\zeta(2)$ yields the dimension-independent bound

$$S(\sigma) \leq B \equiv \log \zeta(2) + 2 \log 16 + 2. \quad (\text{S66})$$

This deliberately loose bound is not an optimal constant.

There is also a uniform eigenvalue envelope $\lambda_j \leq 128/j^2$. For large j , sum the eigenvalues from $\lfloor j/2 \rfloor + 1$ through j and apply the preceding tail bound; the finitely many smaller indices follow from $\lambda_j \leq 1$. Consequently, for sufficiently large M ,

$$\sum_{j > M} -\lambda_j \log \lambda_j \leq \frac{256(\log M + 1)}{M}, \quad (\text{S67})$$

$$\sum_{j > M} \lambda_j^\alpha \leq \frac{128^\alpha M^{1-2\alpha}}{2\alpha - 1}, \quad \alpha > \frac{1}{2}. \quad (\text{S68})$$

These bounds, rather than continuity in a fixed dimension, will justify passing from spectra to entropies.

For the physical charge blocks take $P_a = c_L(q, a)/T(L, q)$ and $Q_b = c_R(q, b)/T(R, q)$. Equations (S61) and (S66) prove $0 \leq \Delta S_1 \leq B$ for every size, flux, and contiguous cut.

2. Balanced-cut depth and charge distributions

Now set $N = 2n$ and $L = R = n$. Let $P_n(m) = 3^{-n}T(n, m)$ be the endpoint distribution of independent uniform steps in $\{-1, 0, 1\}$. Its characteristic function is $\varphi(x) = (1 + 2\cos x)/3 = 1 - x^2/3 + O(x^4)$. Fourier inversion with $x = v/\sqrt{n}$ gives, uniformly on compact scaled endpoints,

$$\begin{aligned}\sqrt{n}P_n(m) &= g(m/\sqrt{n}) + o(1), \\ n[P_n(m) - P_n(m+2)] &= -2g'(m/\sqrt{n}) + o(1), \\ g(y) &= \frac{\sqrt{3}}{2\sqrt{\pi}}e^{-3y^2/4}.\end{aligned}\tag{S69}$$

For completeness, $|\varphi(x)| < 1$ away from zero on $[-\pi, \pi]$, and $|\varphi(x)| \leq e^{-cx^2}$ near zero. These bounds dominate the rescaled Fourier integrands. In the second line insert $1 - e^{-2ix}$, whose rescaled leading term is $2iv/\sqrt{n}$. The integrable Gaussian bounds also give uniformity in compact $m/\sqrt{n}$.

Shift the lower depth endpoint by $a_0 = \max(0, -q)$ and write $j = a - a_0$. For both signs of q , the conditional depth probabilities are

$$P_{n,q}^{\text{dep}}(j) = \frac{P_n(|q| + 2j) - P_n(|q| + 2j + 2)}{P_n(|q|)}.\tag{S70}$$

For $y = |q|/\sqrt{n}$ and $s = j/\sqrt{n}$, their step densities converge in L^1 to

$$\begin{aligned}f_y(s) &= 3(y + 2s)e^{-3[(y+2s)^2 - y^2]/4}, \\ F_y(s) &= 1 - e^{-3[(y+2s)^2 - y^2]/4}, \quad s \geq 0.\end{aligned}\tag{S71}$$

Indeed Eq. (S69) proves convergence on compact s , while the exact telescoping identity

$$\Pr(j \geq a) = \frac{P_n(|q| + 2a)}{P_n(|q|)}\tag{S72}$$

controls the tail, first taking $n \rightarrow \infty$ and then $a/\sqrt{n} \rightarrow \infty$. All statements are uniform for y in a fixed compact interval. In particular the maximum atom tends to zero, also at $y = 0$ where $f_0(0) = 0$.

We will also need a global, not just central, estimate. Tilt the step law by $\eta = m/n \in [-1, 1]$. Its three probabilities remain bounded below uniformly in η . Fourier inversion bounds its n -fold convolution by $C/\sqrt{n}$. Since $\mathbb{E}e^{\eta X} \leq e^{\eta^2/2}$ for $X \in [-1, 1]$ of mean zero, undoing the tilt gives

$$P_n(m) \leq \frac{C}{\sqrt{n}}e^{-m^2/(2n)}.\tag{S73}$$

Together with Eq. (S69), this shows that $p_q = P_n(q)^2/P_{2n}(0)$ has scaled limiting density

$$w(y) = \sqrt{\frac{3}{2\pi}}e^{-3y^2/2}, \quad y = q/\sqrt{n},\tag{S74}$$

and $p_q \leq Cn^{-1/2}e^{-q^2/n}$. The normalized weights p_q^α also have uniformly vanishing scaled tails for each fixed $\alpha > 0$.

3. Two universal branches of the conditional Schmidt spectrum

In a balanced charge block the two depth laws coincide; denote their probabilities by P_a . Embed the depth basis as normalized indicators of consecutive quantile intervals $I_a \subset [0, 1]$ of lengths P_a . Let U_n be multiplication by t_n^a on I_a , extended as a unitary on all of $L^2[0, 1]$. Introduce the embedded matrices

$$(T_n)_{ab} = \sqrt{P_a P_b} \mathbf{1}_{a \leq b}, \quad D_n = \text{diag}(P_a).\tag{S75}$$

The normalized coefficient matrix satisfies the exact identity

$$\widehat{K}_q U_n^\dagger = T_n + U_n T_n^\dagger U_n^\dagger - D_n.\tag{S76}$$

Let V be upper Volterra integration, $(Vf)(u) = \int_u^1 f(v) dv$. Off diagonal quantile cells, T_n agrees with its kernel; on each diagonal cell they disagree on half the area. Thus

$$\|T_n - V\|_{\text{HS}}^2 = \frac{1}{2} \sum_a P_a^2 \leq \frac{1}{2} \delta_n, \quad \|D_n\|_{\text{HS}}^2 \leq \delta_n, \quad (\text{S77})$$

where $\delta_n = \max_a P_a \rightarrow 0$ on compact scaled charges.

Suppose $\sqrt{n}|1 - t_n| \rightarrow \infty$. Then U_n tends weakly to zero, uniformly on such charge ranges. To see this, approximate a multiplication matrix element by a smooth function h in the quantile coordinate. Equations (S70)–(S71) reduce it, with absolute error $o(1)$, to

$$\frac{1}{\sqrt{n}} \sum_a f_y(a/\sqrt{n}) h(F_y(a/\sqrt{n})) t_n^a. \quad (\text{S78})$$

On a bounded scaled-depth interval its amplitude has bounded variation, uniformly for bounded y , so Abel summation bounds the sum by $C/(\sqrt{n}|1 - t_n|)$. Equation (S72) controls the remaining mass. Density of smooth functions in $L^1[0, 1]$ completes the weak-convergence argument. A common lower-endpoint phase is immaterial.

Set $A = VV^\dagger$ and $B_0 = V^\dagger V$. Equations (S76)–(S77) imply

$$\|\sigma_q - (A + U_n B_0 U_n^\dagger)\|_1 = o(1). \quad (\text{S79})$$

In particular the cross term $VU_n VU_n^\dagger$ tends to zero in trace norm: approximate both Hilbert–Schmidt factors V by finite-rank operators and use weak convergence for their middle matrix elements. The discarded terms are controlled by the product of Hilbert–Schmidt norms.

For fixed finite-rank spectral truncations, the eigenvectors of A and $U_n B_0 U_n^\dagger$ have a Gram matrix tending to the identity. Their eigenvalues therefore tend to the union of the two truncated lists. The remaining operator norm is at most the next eigenvalue of each branch, which tends to zero as the truncation increases. This proves convergence of every fixed ordered eigenvalue to the union of the spectra of A and B_0 , without assuming that the finite-size matrices commute.

The kernel of B_0 is $\min(u, v)$. Its eigenvalue equation differentiates to $\lambda f'' = -f$ with $f(0) = 0$ and $f'(1) = 0$. The eigenvalues are

$$b_j = \frac{4}{\pi^2(2j-1)^2}, \quad j \geq 1. \quad (\text{S80})$$

Reflection interchanges A and B_0 , so each b_j occurs twice in the limiting conditional spectrum. Each branch has trace $1/2$ and the doubled list has trace one. Pointwise convergence and total mass one imply ℓ^1 convergence. Equations (S67) and (S68) then imply conditional entropy convergence for every finite $\alpha > 1/2$, uniformly on compact scaled charges.

4. Entropy constants and charge averaging

The charge tail bound and the uniform conditional entropy bound allow averaging Eq. (S61). For Eq. (S62), use the normalized p_q^α tails; the traces are bounded by one for $\alpha > 1$ and by the uniform power sum for $1/2 < \alpha < 1$. Therefore

$$\Delta S_\alpha \longrightarrow C_\alpha, \quad C_\alpha = \frac{\log[2(4/\pi^2)^\alpha (1 - 2^{-2\alpha}) \zeta(2\alpha)]}{1 - \alpha}, \quad \alpha > \frac{1}{2}, \alpha \neq 1, \quad (\text{S81})$$

$$C_1 = -2 \sum_{j \geq 1} b_j \log b_j = 2 \log \pi - \frac{8}{3} \log 2 - \frac{12}{\pi^2} \zeta'(2). \quad (\text{S82})$$

The value at one is also the continuous limit of C_α . In particular $2 \sum_j b_j^2 = 1/3$, giving $C_2 = \log 3$. These limits hold for fixed nontrivial $t = e^{i\Phi}$, and more generally whenever $\sqrt{n}|1 - e^{i\Phi_n}| \rightarrow \infty$.

At zero flux the entropy is the Shannon entropy of p_q . Equations (S69), (S73), and (S74) give

$$H(p_q) = \frac{1}{2} \log N + \frac{1}{2} \log(\pi e/3) + o(1). \quad (\text{S83})$$

To justify the entropy tail, apply monotonicity of $-x \log x$ for $x < e^{-1}$ to the Gaussian upper bound on $\sqrt{n}p_q$ outside a fixed scaled-charge interval. This yields an integrable Gaussian times a quadratic polynomial. Hence

$$S_1(\Phi) = \frac{1}{2} \log N + \frac{1}{2} \log(\pi e/3) + C_1 + o(1). \quad (\text{S84})$$

The remainder denotes an asymptotic limit, not a fitted or asserted polynomial convergence rate.

5. The low-order Rényi threshold

The limiting list also determines whether a finite offset is possible. For every fixed $0 < \alpha \leq 1/2$, $2 \sum_j b_j^\alpha = \infty$. Given any $A > 0$, choose a finite truncation whose power sum exceeds $2A$. Uniform convergence of those finitely many probabilities implies $\text{Tr } \sigma_q^\alpha > A$ for all sufficiently large n on any fixed compact scaled-charge interval. Its normalized p_q^α weight stays bounded away from zero by the local limit. Taking A arbitrarily large in Eq. (S62) proves

$$\Delta S_\alpha \longrightarrow \infty, \quad 0 < \alpha \leq \frac{1}{2}, \quad (\text{S85})$$

at every fixed nontrivial flux. Thus the finite-offset threshold is $\alpha = 1/2$, not the von Neumann order $\alpha = 1$.

We now obtain matching growth bounds, proving Eq. (11). Throughout this argument the flux is fixed and nontrivial. Constants may depend on that flux and on the fixed entropy order. The effective depth scale is $\sqrt{n}$, smaller than the full depth rank; both an upper bound at every charge and a lower bound over a growing spectral range are needed.

First, the symmetric sequence $T(n, m)$ is log-concave without internal zeros. Indeed the one-step sequence $(1, 1, 1)$ has this property. Its Toeplitz convolution matrix has nonnegative minors of order two, and Cauchy–Binet preserves those minors under multiplication. Convolution therefore preserves log-concavity. Decreasing adjacent ratios imply, for $m, h \geq 0$,

$$\frac{T(n, m+h)}{T(n, m)} \leq \frac{T(n, h)}{T(n, 0)}. \quad (\text{S86})$$

Together with Eqs. (S69), (S72), and (S73), this gives $\Pr(j \geq a) \leq C e^{-2a^2/n}$ for the shifted depth j , uniformly in *every* charge q .

Partition depth into consecutive shells of length $h = \lceil \sqrt{n} \rceil$, with masses $\mu_l \leq C e^{-2l^2}$. Write $\hat{K}_q = D + O$, retaining only same-shell entries in D . A shell is another maximum-phase kernel with total mass μ_l , so the earlier envelope gives $s_j(D_l) \leq \sqrt{128} \mu_l / j$. For $0 < p < 1$, summing within shells yields $\sum_j s_j(D)^p \leq C_p h^{1-p} \sum_l \mu_l^p \leq C_p n^{(1-p)/2}$. At $p = 1$ the harmonic sum instead gives $\|D\|_1 \leq C \log(n+1)$.

The off-shell part has a much smaller effective rank. Restricted to its first R row and column shells, it has rank at most $2R$: each row shell has one outer product for lower column shells and another for higher ones. The discarded squared Hilbert–Schmidt norm is at most $2 \Pr(j \geq Rh) \leq C e^{-2R^2}$. Hence $s_{2R+1}(O) \leq C e^{-R^2}$ and $\sum_j s_j(O)^p \leq C_p$. The inequality $s_{2j-1}(D+O) \leq s_j(D) + s_j(O)$ and $(a+b)^p \leq a^p + b^p$ now give, uniformly over all charges,

$$\begin{aligned} \text{Tr } \sigma_q^\alpha &\leq C_\alpha n^{(1-2\alpha)/2}, & 0 < \alpha < \frac{1}{2}, \\ \text{Tr } \sqrt{\sigma_q} &\leq C \log(n+1). \end{aligned} \quad (\text{S87})$$

Only a two-term singular-value inequality is used; there is no growing sum of quasi-triangle constants.

For the lower bound, consider $M_d(t)_{ab} = t^{\max(a,b)}$ on $a, b = 0, \dots, d-1$. Put $t = e^{2i\theta}$ with $0 < \theta < \pi$, $\rho = e^{i\theta}$, and $U = \text{diag}(\rho^a)$. Then $M_d = U C_d(\rho) U$, where $C_d(\rho)_{ab} = \rho^{|a-b|}$. Both unitary factors preserve singular values. The standard complex Kac–Murdock–Szegő inverse [20] gives

$$C_d^{-1} = \frac{i}{2 \sin \theta} A_d + B_d, \quad \text{rank } B_d \leq 2, \quad (\text{S88})$$

where A_d has diagonal $2 \cos \theta$, adjacent entries -1 , and eigenvalues $2 \cos \theta - 2 \cos(l\pi/(d+1))$, $l = 1, \dots, d$. The correction B_d acts only at the two endpoints. Let a_j denote singular values in *ascending* order. Intersecting the span of the $j+2$ smallest right singular vectors of A_d with $\ker B_d$ leaves dimension at least j . The min–max principle thus yields $a_j(C_d^{-1}) \leq a_{j+2}(A_d)/(2 \sin \theta)$. The equally spaced angles contain $j+2$ points within $C_\theta(j+2)/d$ of θ , for $1 \leq j \leq \eta_\theta d$ and sufficiently large d . Since cosine is Lipschitz, inversion gives the growing-index bound

$$s_j(M_d) \geq \frac{c_\theta d}{j+2}, \quad 1 \leq j \leq \eta_\theta d. \quad (\text{S89})$$

This uses singular values, not eigenvalue moduli of the non-Hermitian matrix.

For $|q| \leq Y\sqrt{n}$, Eq. (S69) implies $P_{n,q}^{\text{dep}}(j) \geq c_Y/\sqrt{n}$ on $\sqrt{n} \leq j \leq 2\sqrt{n}$, uniformly for large n . Compress $\hat{K}_q$ to this principal interval, of dimension $d \asymp \sqrt{n}$. Its phase matrix is M_d up to a common phase. Compression and positive diagonal rescaling therefore imply

$$s_j(\hat{K}_q) \geq \frac{c_{Y,\Phi}}{j+2}, \quad 1 \leq j \leq \eta_\Phi \sqrt{n}. \quad (\text{S90})$$

Summing its 2α powers supplies the matching lower bounds $c_{\alpha,Y,\Phi} n^{(1-2\alpha)/2}$ for $0 < \alpha < 1/2$ and $c_{Y,\Phi} \log n$ at $\alpha = 1/2$. For each fixed $\alpha > 0$, a positive fraction of the normalized p_q^α weights lies in such a compact scaled-charge interval. Combining these lower bounds and the all-charge upper bounds in Eq. (S62), then taking the logarithm, proves Eq. (11); replacing n by $N = 2n$ affects only the bounded remainder.

Finally, for Schmidt probabilities λ_i of a pure bipartition, the partial-transpose eigenvalues are λ_i and $\pm\sqrt{\lambda_i\lambda_j}$ for $i < j$. Thus $\|(|\Psi\rangle\langle\Psi|)^{T_B}\|_1 = (\sum_i \sqrt{\lambda_i})^2$ [21], proving the stated logarithmic-negativity law. This is not a statement about distillable entanglement or separated mixed subregions. The growth estimates keep both the flux and the positive entropy order fixed; they do not specify a constant remainder limit or a vanishing-flux crossover at these low orders.

6. Weak-flux crossover and order of limits

Take principal fluxes with $\Phi_n\sqrt{n} \rightarrow \kappa$. The inverse of the depth cumulative distribution is

$$s_y(u) = \frac{1}{2} \left[\sqrt{y^2 - \frac{4}{3} \log(1-u)} - y \right], \quad 0 \leq u < 1. \quad (\text{S91})$$

Quantile convergence, uniform away from $u = 1$ on compact y , shows that the coefficient kernels converge in Hilbert–Schmidt norm to

$$G_{y,\kappa}(u, v) = e^{i\kappa \max(s_y(u), s_y(v))}. \quad (\text{S92})$$

Their Hilbert–Schmidt norms are one. The uniform rank approximations and entropy tails above pass to this limit, so charge averaging gives

$$\Delta S_1(\Phi_n) \longrightarrow F(\kappa) = \int_{\mathbb{R}} w(y) S(G_{|y|,\kappa} G_{|y|,\kappa}^\dagger) dy. \quad (\text{S93})$$

This is an operator representation, not an elementary closed form. Dominated kernel convergence and the uniform entropy tails make F continuous; complex conjugation makes it even. At $\kappa = 0$ the kernel is rank one, giving $F(0) = 0$. For $\kappa \neq 0$, choose ordered intervals with $u_1 < v_1 < u_2 < v_2$. Their phase minor is proportional to $e^{i\kappa s_y(v_1)} - e^{i\kappa s_y(u_2)}$, nonzero on a positive-measure set. Thus the operator is not rank one and $F(\kappa) > 0$.

For $|\kappa| \rightarrow \infty$, multiplication by $e^{i\kappa s_y(u)}$ tends weakly to zero: change variables $u = F_y(s)$ and use the Riemann–Lebesgue lemma. The same two-Volterra-branch argument and entropy tails give $F(\kappa) \rightarrow C_1$. No monotonicity of F is required or claimed. The crossover width is consequently of order $N^{-1/2}$ for balanced cuts. At each finite size, entropy is continuous at zero flux, whereas any fixed nontrivial flux gives the positive offset C_1 as $N \rightarrow \infty$. The zero-flux and thermodynamic limits do not commute. All neutral energies nevertheless remain unchanged by Eq. (S47).

Appendix H: Magnetization-preserving extraction of a logical Bell pair

The doubled limiting spectrum has an operational consequence beyond the entropy offsets. We specify the operation class first. Each party controls one entire segment of the ring and a local, initially separable register. The register’s two logical states both have magnetization zero; for example, two auxiliary spin-one sites encode $|0\rangle_{\log} = |ud\rangle$ and $|1\rangle_{\log} = |du\rangle$. Local operations preserve the total magnetization of the corresponding segment and its auxiliaries. No shared entangled ancilla or charge-coherent reference is supplied. This is a restriction on the protocol, not a fundamental superselection rule for spins. Charge-resolved operational entanglement and general pure-state conversion are established concepts [22, 24]; here we give an explicit channel for the flux-dependent Motzkin state.

1. An exact deterministic channel at finite size

For any contiguous cut write the neutral input in charge-resolved Schmidt form,

$$|\Psi\rangle = \sum_q \sqrt{p_q} \sum_{j \geq 1} \sqrt{r_{q,j}} |a_{q,j}\rangle |b_{-q,j}\rangle, \quad r_{q,1} \geq r_{q,2} \geq \dots \geq 0, \quad \sum_j r_{q,j} = 1. \quad (\text{S94})$$

Only blocks with $p_q > 0$ are needed. Pad each list by zeros. On the Schmidt support, Alice's isometry sends

$$|a_{q,2j-1}\rangle \mapsto |0\rangle_a |q,j\rangle_{A'}, \quad |a_{q,2j}\rangle \mapsto |1\rangle_a |q,j\rangle_{A'}, \quad (\text{S95})$$

and Bob uses the matching map on $|b_{-q,j}\rangle$ with residual charge $-q$. The residual labels for different q are orthogonal. These maps must be completed on every local charge sector, not only the occupied Schmidt support. Such completion is always possible: within a sector of dimension d_q , at most $\lceil d_q/2 \rceil$ residual basis states are needed for the paired support, and unused input vectors can be mapped to orthogonal vectors in the $2d_q$ -dimensional output. The isometry extends to a unitary on the segment and a register initialized in $|0\rangle$. All extensions are made separately within each charge sector. Thus $V_A Q_A = (I_a \otimes Q_{A'}) V_A$, and similarly for Bob, including unequal cuts and charge sectors absent from the input.

Trace out the two residual systems, retaining every charge sector. The register output is exactly

$$\begin{aligned} \rho_{ab} &= a|00\rangle\langle 00| + (1-a)|11\rangle\langle 11| + c(|00\rangle\langle 11| + |11\rangle\langle 00|), \\ a &= \sum_q p_q \sum_j r_{q,2j-1}, \quad c = \sum_q p_q \sum_j \sqrt{r_{q,2j-1} r_{q,2j}} \geq 0. \end{aligned} \quad (\text{S96})$$

Its fidelity with $|\Phi^+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ is $F_{\text{Bell}} = 1/2 + c$, as in Eq. (12). In particular,

$$1 - F_{\text{Bell}} = \frac{1}{2} \sum_q p_q \sum_j (\sqrt{r_{q,2j-1}} - \sqrt{r_{q,2j}})^2. \quad (\text{S97})$$

The product channel is trace preserving and uses no communication or postselection. Measuring charge and averaging all outcomes would give the same output, but this measurement is unnecessary.

For any normalized reference list τ with $\tau_{2j-1} = \tau_{2j}$, the elementary inequalities $(\sqrt{x} - \sqrt{y})^2 \leq |x - y| \leq |x - \tau_{2j-1}| + |y - \tau_{2j}|$ give

$$1 - F_{\text{Bell}} \leq \frac{1}{2} \sum_q p_q \|r_q - \tau\|_1. \quad (\text{S98})$$

No entropy-continuity estimate or fixed Schmidt-rank assumption is used. The trace distance from the Bell state is at most $\sqrt{1 - F_{\text{Bell}}}$. Directly, its square is $(a - 1/2)^2 + (c - 1/2)^2 \leq 1/2 - c$, using positivity $a(1-a) \geq c^2$ of Eq. (S96).

2. Balanced large rings and correlated amplitudes

For the original state, set $N = 2n$ and keep $\Phi \notin 2\pi\mathbb{Z}$ fixed. Appendix G proves ℓ^1 convergence of r_q to

$$\tau = (b_1, b_1, b_2, b_2, \dots), \quad b_j = \frac{4}{\pi^2(2j-1)^2}, \quad (\text{S99})$$

uniformly for $|q|/\sqrt{n}$ in compact sets. The doubled list has total mass one. Split the average in Eq. (S98) at $|q| = Y\sqrt{n}$. Its central part tends to zero for fixed Y ; the rest is bounded by $\sum_{|q| > Y\sqrt{n}} p_q$, which vanishes as $Y \rightarrow \infty$ by the Gaussian charge bound. Hence $F_{\text{Bell}} \rightarrow 1$, and the register state converges in trace norm to a Bell pair. This is a single-copy extraction statement, not an optimal distillation rate. No size-dependent rate of convergence is asserted.

The same argument applies to the specified correlated states of Appendix J, with fixed real symmetric, entrywise-positive W , its four-site compatible parent, $N \geq 4$, equal halves and fixed nontrivial flux. The conditional limiting list is the sorted multiset $\omega_i \omega_j b_\ell$, each appearing twice. Every nonzero weight therefore has even multiplicity, including accidental degeneracies; zero secondary ω or rank-one W causes no difficulty. The compact-charge convergence and Gaussian tails proved there give the same conclusion through Eq. (S98). This does not establish generic stability under non-frustration-free perturbations. Nor are these gapless, asymptotic doublets symmetry-protected Haldane doublets [25].

3. Why the original zero-flux state cannot supply this pair

At $\Phi \in 2\pi\mathbb{Z}$, every original-model charge block has rank one. The input after a regional-charge twirl is consequently

$$\mathcal{T}_A(|\Psi_0\rangle\langle\Psi_0|) = \sum_q p_q |a_q\rangle\langle a_q| \otimes |b_{-q}\rangle\langle b_{-q}|, \quad (\text{S100})$$

a separable state. Any local charge-preserving operation into neutral registers has the same reduced-register output before and after this twirl: off-diagonal residual-charge blocks disappear on tracing the charged systems. This remains true with initially separable, charge-definite auxiliaries and for charge-preserving local operations with classical communication, including conditioned branches. Such operations preserve separability of the twirled input. A separable two-qubit state has overlap at most $1/2$ with $|\Phi^+\rangle$, proving $F_{\text{Bell}} \leq 1/2$. The explicit channel attains $1/2$. There is no conflict with the original state's ordinary $O(\log N)$ entanglement: it resides in coherences between regional charges. This no-go concerns the original model only. Correlated W can already produce entanglement within a zero-flux charge block.

4. An achieved logical Bell witness and a four-spin check

The extracted registers also provide a direct correlation witness. With logical Pauli operators choose

$$A_0 = Z, \quad A_1 = X, \quad B_0 = \frac{Z + 2cX}{\sqrt{1 + 4c^2}}, \quad B_1 = \frac{Z - 2cX}{\sqrt{1 + 4c^2}}. \quad (\text{S101})$$

The Clauser–Horne–Shimony–Holt (CHSH) combination [46] has local bound two and expectation

$$\mathcal{B} = \langle A_0 \otimes (B_0 + B_1) + A_1 \otimes (B_0 - B_1) \rangle = 2\sqrt{1 + 4c^2} = 2\sqrt{1 + (2F_{\text{Bell}} - 1)^2}. \quad (\text{S102})$$

Indeed $\langle Z \otimes Z \rangle = 1$ and $\langle X \otimes X \rangle = 2c$ in Eq. (S96). For deterministic local outcomes each equal to ± 1 , one of $B_0 + B_1$ and $B_0 - B_1$ vanishes and the other has magnitude two; averaging proves the local bound. At $k \geq 2$ and nontrivial flux, the original $q = 0$ block has rank at least two, so $c > 0$ and $\mathcal{B} > 2$. For balanced large rings it tends to $2\sqrt{2}$. At $k = 1$ or trivial flux, $c = 0$ for this original-model channel.

For $N = 4$ and $L = R = 2$, only the two $q = 0$ weights contribute to c . Their product is $16 \sin^2(\Phi/2)/361$, giving exactly

$$c = \frac{4|\sin(\Phi/2)|}{19}, \quad F_{\text{Bell}} = \frac{1}{2} + \frac{4|\sin(\Phi/2)|}{19}, \quad \mathcal{B} = 2\sqrt{1 + \left(\frac{8\sin(\Phi/2)}{19}\right)^2}. \quad (\text{S103})$$

These are logical-register correlations after local processing, not a claim about bare-spin two-point measurements or a loophole-free experiment. Preprocessing precedes the independent choices of Bell settings. It can depend on the known Schmidt bases and act on an entire half; no shallow circuit, efficient implementation, adiabatic preparation or energy advantage is established. Equation (S102) is an achieved witness, not an optimality theorem.

Appendix I: Compatible local deformations

We now allow spatially nonuniform positive ground-state weights, while retaining the same legal moves. Let $g_i(u), g_i(0), g_i(d) > 0$. For each source–target pair $(s, t) = (u0, 0u), (0d, d0), (ud, 00)$ on ordered bond $(i, i + 1)$, set

$$k_{i,s,t} = \frac{g_i(s_1)g_{i+1}(s_2)}{g_i(t_1)g_{i+1}(t_2)}, \quad |X_{i,s,t}\rangle = \frac{|s\rangle - k_{i,s,t}e^{-i\phi_i}|t\rangle}{\sqrt{1 + k_{i,s,t}^2}}. \quad (\text{S104})$$

Take $H_g = \sum_{i,s,t} J_{i,s,t} |X_{i,s,t}\rangle\langle X_{i,s,t}|$ with every $J_{i,s,t} > 0$. The three ratios on a bond are generally different: this is not obtained merely by choosing another common z_i in Eq. (1). Small changes of the g_i around one give small changes of the local projectors, including their hopping and pair conversion amplitudes. The relations between these changes are essential; independent arbitrary perturbations of the local matrix elements are not being considered.

1. Neutral spectrum and ground state

The unitary in Eq. (3) has the same phase ratio on every allowed edge, independent of its positive magnitude. The diagonal entries of each projector in Eq. (S104) are phase independent. It follows term by term that

$$H_g^{(0)}(\phi) = \mathcal{U} H_g^{(0)}(\mathbf{0}) \mathcal{U}^\dagger. \quad (\text{S105})$$

As before, an onsite gauge puts the total flux on the wrapping bond. In that gauge the unique neutral zero mode is

$$|\Psi_{g,\Phi}\rangle = Z_g^{-1/2} \sum_{M(x)=0} \left[\prod_i g_i(x_i) \right] e^{-i\Phi h_{\min}(x)} |x\rangle, \quad Z_g = \sum_{M(x)=0} \prod_i g_i(x_i)^2. \quad (\text{S106})$$

Indeed its source-to-target amplitude ratio is $k_{i,s,t} e^{i\phi_i}$, so each local bra annihilates it. The state has nonzero amplitude on every neutral word. Positivity of the projector strengths and connectivity of the legal-move graph then prove uniqueness in that sector. This argument includes both ordered bonds at $N = 2$.

An identical computational-basis-diagonal potential added to both sides of Eq. (S105) preserves the spectral identity. It does not in general preserve Eq. (S106); no ensuing ground-state entropy assertion is made for such a potential.

2. Weighted depth blocks and entropy enhancement

For a segment A , define $C_A(q, a)$ to be the sum of $\prod_{i \in A} g_i(x_i)^2$ over words with charge q and depth $a = -\min h_A$. Define $C_B(-q, d)$ similarly for B , whose heights start at zero. At the cut put $b = d - q$; the global minimum is then $-\max(a, b)$. Weighted sums of words at fixed charge and depth give orthonormal segment bases after normalization. The exact coefficient blocks of Eq. (S106) are

$$K_{g,q}(a, b) = \sqrt{\frac{C_A(q, a) C_B(-q, b + q)}{Z_g}} e^{i\Phi \max(a, b)}. \quad (\text{S107})$$

These counts are definitions by weighted sums. The unweighted reflection formula need not hold for spatially varying g_i .

Strict positivity leaves every feasible depth available. The new blocks are therefore obtained from the same phase matrices as before by positive row and column rescalings. The phase minors in Eq. (S55) retain their nonzero determinants, so the Schmidt ranks remain $2k + 1$ at trivial flux and $(k + 1)(k + 2)/2$ otherwise, where $k = \min(L, R)$. Every charge probability

$$p_{g,q} = Z_g^{-1} \left[\sum_a C_A(q, a) \right] \left[\sum_d C_B(-q, d) \right] \quad (\text{S108})$$

is flux independent. At zero flux each block has rank one; at nontrivial flux the $q = 0$ block has rank at least two when $k \geq 2$. Splitting $p_{g,q}$ into its conditional Schmidt probabilities therefore proves strict enhancement of every finite positive-order Rényi entropy for $k \geq 2$, with equality for $k = 1$, exactly as in Appendix D. This compares fluxes at a fixed filter; local filtering alone does not preserve entropy.

After division by $\sqrt{p_{g,q}}$, Eq. (S107) is the general maximum-phase kernel of Eq. (S63), with two possibly different depth probability laws. The distribution-independent argument of Appendix G applies without a bound on their individual weights. Consequently

$$0 \leq S_1(\Psi_{g,\Phi}) - S_1(\Psi_{g,0}) \leq B \quad (\text{S109})$$

uniformly in size, cut, flux and positive filter, with the same B as in Eq. (S66). This is not a uniform positive lower bound as the size grows, nor an extension of the equal-weight thermodynamic constant C_1 . Strong or size-dependent filters may concentrate the state near product configurations. Robustness of that quantitative offset requires control of the new depth and charge distributions, beyond the finite-size statements established here.

3. Uniform flat-state bias preserves the limiting excess

A natural one-parameter subfamily changes the weight of a zero spin: $g_i(u) = g_i(d) = 1$, $g_i(0) = r$, with $0 < r < \infty$ fixed as N grows. The hopping projectors are unchanged in magnitude, whereas the pair conversion ratio is $k_{i,ud,00} = r^{-2}$. The ground-state amplitudes are weighted by r^{n_0} , where n_0 is the number of zero spins. Write this state as $\Psi_{r,\Phi}$ and its reduced density matrix as $\rho_{r,A}(\Phi)$. Arbitrary strictly positive projector strengths remain allowed, since they do not change this state.

For equal halves $N = 2n$, replace the endpoint count by

$$T_r(n, m) = [z^m](z + r^2 + z^{-1})^n, \quad P_{r,n}(m) = \frac{T_r(n, m)}{(2 + r^2)^n}. \quad (\text{S110})$$

Reflection preserves the number of flat steps. Thus the exact depth count is $T_r(n, q + 2a) - T_r(n, q + 2a + 2)$, with the same endpoint shifts as before. In particular $p_{r,q} = P_{r,n}(q)^2 / P_{r,2n}(0)$, and the two conditional depth laws in each balanced charge block coincide.

The walk now has variance $v_r = 2/(2 + r^2)$ and characteristic function $(r^2 + 2 \cos x)/(r^2 + 2) = 1 - v_r x^2/2 + O_r(x^4)$. For each fixed positive r this function has modulus strictly less than one away from zero on $[-\pi, \pi]$. The Fourier argument for Eq. (S69), including its finite-difference factor, therefore gives

$$\begin{aligned} \sqrt{n}P_{r,n}(m) &= g_r(m/\sqrt{n}) + o(1), \\ n[P_{r,n}(m) - P_{r,n}(m+2)] &= -2g'_r(m/\sqrt{n}) + o(1), \\ g_r(y) &= (2\pi v_r)^{-1/2} e^{-y^2/(2v_r)}. \end{aligned} \quad (\text{S111})$$

Both convergences are uniform on compact scaled endpoints: near zero a Gaussian dominates the rescaled Fourier integrand, and away from zero the contribution decays exponentially. The telescoping depth tail then gives convergence in L^1 of the conditional step densities to

$$f_{r,y}(s) = \frac{2(|y| + 2s)}{v_r} \exp\left[-\frac{(|y| + 2s)^2 - y^2}{2v_r}\right], \quad s \geq 0. \quad (\text{S112})$$

The maximum depth atom tends to zero uniformly on compact $y = q/\sqrt{n}$, including $y = 0$; the exact tail ratio controls the unbounded depth range.

Global charge averaging is also controlled. Use the bounded tilt parameter $\eta = m/n \in [-1, 1]$, rather than a tilt chosen to match the endpoint mean. At fixed $r > 0$ all three tilted probabilities stay uniformly positive. The Fourier convolution bound and the bounded-step exponential-moment inequality used for Eq. (S73) give $P_{r,n}(m) \leq C_r n^{-1/2} e^{-m^2/(2n)}$. Together with Eq. (S111), this yields the scaled charge density $w_r(y) = (\pi v_r)^{-1/2} e^{-y^2/v_r}$ and vanishing tails of normalized $p_{r,q}^\alpha$ for every fixed $\alpha > 0$.

The operator proof of Appendix G now applies with these densities. Identical conditional laws and vanishing maximum atom give the same upper-Volterra limit. Their smooth, bounded-variation densities give the same weak phase cancellation when $\sqrt{n}|1 - e^{i\Phi_n}| \rightarrow \infty$. The distribution-independent spectral tails pass the two-branch spectrum to conditional entropies; the charge bounds just proved pass it to the full entropies. Consequently the limiting probabilities remain two copies of b_j , and

$$\Delta S_\alpha \longrightarrow C_\alpha \quad (\tfrac{1}{2} < \alpha < \infty), \quad \frac{\text{Tr } \rho_{r,A}(\Phi)^2}{\text{Tr } \rho_{r,A}(0)^2} \longrightarrow \frac{1}{3} \quad (\text{S113})$$

at every fixed nontrivial flux, including the von Neumann limit $C_1 \simeq 1.581$. The finite-truncation argument for Eq. (S85) likewise retains the threshold $\alpha = 1/2$. The charge Shannon entropy changes, giving

$$S_1(\Psi_{r,\Phi}) = \tfrac{1}{2} \log N + \tfrac{1}{2} \log \frac{\pi e}{2 + r^2} + C_1 + o(1). \quad (\text{S114})$$

Finally, rescaling y and s by $\sqrt{v_r}$ in Eq. (S112) and in its charge average gives the weak-flux crossover explicitly in terms of the original F :

$$\Phi_n \sqrt{n} \rightarrow \kappa : \quad \Delta S_1 \longrightarrow F\left(\kappa \sqrt{\frac{3}{2 + r^2}}\right). \quad (\text{S115})$$

None of these limits is asserted uniform as $r \rightarrow 0$ or $r \rightarrow \infty$, for size-dependent r , or for general spatially varying filters. Nor does phase independence at fixed r imply an unchanged energy spectrum when r itself changes.

Appendix J: Short-range occupation correlations

The finite entropy offsets persist when neighboring occupations are correlated. The construction of a local parent from Gibbs amplitudes is standard [26, 27]; the issue here is whether its additional boundary entanglement changes the flux increment. We derive the answer directly on the lattice. This is useful because a simple continuum RK description need not reproduce Motzkin block entanglement [47].

1. Correlated state and four-site parent

Fix a real symmetric matrix $W = (W_{ab})_{a,b=0,1}$ with every $W_{ab} > 0$. The entries, rather than the eigenvalues, are required to be positive. For a spin word x , write $\tau_i = |x_i|$ and define

$$\begin{aligned} w_W(x) &= \prod_{i=1}^N W_{\tau_i \tau_{i+1}}, \quad \tau_{N+1} = \tau_1, \\ Z_{N,W}^{\text{ring}}(q) &= \sum_{M(x)=q} w_W(x), \\ |\Psi_{W,\Phi}\rangle &= \frac{\sum_{M(x)=0} \sqrt{w_W(x)} e^{-i\Phi h_{\min}(x)} |x\rangle}{\sqrt{Z_{N,W}^{\text{ring}}(0)}}. \end{aligned} \quad (\text{S116})$$

The partition function here is not the bond holonomy $\mathcal{Z}$. For example, $W_{ab} = \exp[-J_{\text{occ}}ab - \mu(a+b)/2]$ gives $w_W(x) = \exp[-J_{\text{occ}} \sum_i \tau_i \tau_{i+1} - \mu \sum_i \tau_i]$. At $J_{\text{occ}} \neq 0$ this does not factor into onsite weights. The parameters specify the wavefunction probabilities, not an additive single-ion potential in the original Hamiltonian.

For $N \geq 4$, a local parent uses the original oriented pairs $(s, t) = (u0, 0u), (0d, d0), (ud, 00)$. With outside spins ℓ, r held fixed, put

$$\begin{aligned} w_{\ell,r}(s) &= W_{|\ell|,|s_1|} W_{|s_1|,|s_2|} W_{|s_2|,|r|}, \\ k_{\ell,r}(s, t) &= \sqrt{\frac{w_{\ell,r}(s)}{w_{\ell,r}(t)}}, \\ |X_{i;\ell,r;s,t}\rangle &= \frac{|s\rangle - k_{\ell,r}(s, t) e^{-i\phi_i} |t\rangle}{\sqrt{1 + k_{\ell,r}(s, t)^2}}. \end{aligned} \quad (\text{S117})$$

Sum these projectors, controlled by $|\ell\rangle\langle\ell|_{i-1}$ and $|r\rangle\langle r|_{i+2}$, over bonds, outside spins and the three pairs. The four sites are distinct at $N \geq 4$. All coefficients may be chosen to be one; any strictly positive coefficients give the same zero mode. At $W_{ab} = 1$ the controls sum to the identity, recovering the original two-site parent. Small changes of W give small changes of fixed-range local terms, but retain the RK relations between their matrix elements.

Only three activity bonds change in a local move, so $k_{\ell,r}(s, t) = \sqrt{w_W(x_s)/w_W(x_t)}$. In seam gauge the source depth minus the target depth is zero on a nonwrapping move and one on a wrapping move. Thus every local bra annihilates Eq. (S116). Nonzero amplitudes and neutral-sector connectivity prove uniqueness. The same phase ratio also gives, term by term,

$$H_W^{(0)}(\Phi) = \mathcal{D}_\Phi H_W^{(0)}(0) \mathcal{D}_\Phi^\dagger, \quad \mathcal{D}_\Phi |x\rangle = e^{-i\Phi h_{\min}(x)} |x\rangle. \quad (\text{S118})$$

Here the flux is concentrated on the wrapping bond. Onsite gauge rotations recover any other distribution of the same total flux.

2. Exact endpoint-resolved depth matrices

For an open segment, let $Z_n^{ab}(q)$ be the total internal weight $\prod_{i < n} W_{\tau_i \tau_{i+1}}$ at charge q and endpoint activities $(\tau_1, \tau_n) = (a, b)$. The weight of words at depth A is exactly

$$C_n^{ab}(q, A) = Z_n^{ab}(q + 2A) - Z_n^{ab}(q + 2A + 2), \quad A \geq \max(0, -q). \quad (\text{S119})$$

To prove this, reflect step signs up to the first crossing of a height barrier. Every activity, internal bond weight and endpoint activity is unchanged. Barrier subtraction and the difference between two consecutive barriers give Eq. (S119). Global sign flip and reversal give $Z_n^{ab}(-q) = Z_n^{ab}(q) = Z_n^{ba}(q)$.

For $N = 2n$, retain the depth and both endpoint activities on each side. Use $A = -\min h_A$ and $B = -q - \min h_B$, so the full depth is $\max(A, B)$. Reversing and sign-flipping the right segment gives a charge- q word of depth B with its endpoints interchanged. The preceding symmetries give the same depth count for its actual endpoint class (c, d) . Normalized weighted class sums are orthonormal. Their exact coefficient matrix and charge weight are

$$K_{W,q}[(A, a, b), (B, c, d)] = \sqrt{\frac{C_n^{ab}(q, A)C_n^{cd}(q, B)W_{bc}W_{da}}{Z_{2n,W}^{\text{ring}}(0)}} e^{i\Phi \max(A,B)},$$

$$p_{W,q} = \frac{\sum_{a,b,c,d} Z_n^{ab}(q)Z_n^{cd}(q)W_{bc}W_{da}}{Z_{2n,W}^{\text{ring}}(0)}. \quad (\text{S120})$$

Zero-weight classes are omitted. The two crossing factors $W_{bc}W_{da}$ are essential. The charge weights are flux independent, but at zero flux a charge block now has rank at most four, not necessarily one. The finite-size majorization argument for onsite product weights therefore does not apply to this correlated family.

3. Transfer estimates on the common depth lattice

Let $D(\vartheta) = \text{diag}(1, 2 \cos \vartheta)$. Summing the two signs at every active site gives the exact characteristic functions

$$\sum_q Z_n^{ab}(q) e^{i\vartheta q} = [D(\vartheta)(WD(\vartheta))^{n-1}]_{ab},$$

$$\sum_q Z_{N,W}^{\text{ring}}(q) e^{i\vartheta q} = \text{Tr}[(WD(\vartheta))^N]. \quad (\text{S121})$$

The entrywise-positive symmetric matrix $P = \text{diag}(1, \sqrt{2})W \text{diag}(1, \sqrt{2})$ has a simple top eigenvalue Λ and a positive unit eigenvector $\mathbf{v}$. Set $d = (1, 2)$, $\xi_a = \sqrt{d_a}v_a$, and $s_W = v_1^2 \in (0, 1)$. Its Perron projection supplies the endpoint prefactor $\xi_a \xi_b$. The analytic leading eigenvalue of $WD(\vartheta)$ obeys $\Lambda(\vartheta)/\Lambda = 1 - s_W \vartheta^2/2 + O_W(\vartheta^4)$. Before charge conditioning, signs are independently symmetric given the activities. All offsite increment covariances vanish, while the stationary active fraction is v_1^2 , giving this variance. It also follows by differentiating the simple two-dimensional eigenvalue.

Away from zero modulo 2π the spectral radius is strictly smaller than Λ . For $|\cos \vartheta| < 1$ this follows by strict Perron comparison; at $\vartheta = \pi$ the positive 00 and 11 loops have incompatible phases, excluding equality. Compactness gives exponential contraction away from zero. Near zero the leading branch has a Gaussian bound and the other branch is exponentially smaller. Fourier inversion therefore gives, uniformly for $y = q/\sqrt{n}$ in compact sets and all endpoint classes,

$$\sqrt{n} \Lambda^{-(n-1)} Z_n^{ab}(q) \longrightarrow \xi_a \xi_b g_W(y),$$

$$n \Lambda^{-(n-1)} [Z_n^{ab}(q) - Z_n^{ab}(q+2)] \longrightarrow -2\xi_a \xi_b g'_W(y),$$

$$g_W(y) = (2\pi s_W)^{-1/2} e^{-y^2/(2s_W)}. \quad (\text{S122})$$

The second line inserts $1 - e^{-2i\vartheta}$ inside the dominated Fourier integral; it does not differentiate an unspecified remainder. The ring local limit has prefactor Λ^N .

For a global estimate, tilt by $\eta = q/n \in [-1, 1]$ and replace $D(0)$ by $\text{diag}(1, 2 \cosh \eta)$. Its top eigenvalue is Λ_η . Tilted sign probabilities and the Perron active fraction are uniformly positive on this compact interval. Conditional sign variance per active site is $\text{sech}^2 \eta$, giving a positive uniform lower bound on $\partial_\eta^2 \log \Lambda_\eta$. The same loop argument gives uniform contraction away from zero. The analytic projections and Gaussian Fourier bound thus bound each tilted coefficient by $C_W \Lambda_\eta^{n-1}/\sqrt{n}$. Possible nonleading degeneracies are absorbed into a smaller exponential gap in the finite-dimensional power bound. Since $\cosh \eta \leq e^{\eta^2/2}$, entrywise comparison gives $\Lambda_\eta \leq \Lambda e^{\eta^2/2}$. Undoing the tilt yields

$$Z_n^{ab}(q) \leq C_W \Lambda^{n-1} n^{-1/2} e^{-q^2/(2n)}, \quad p_{W,q} \leq C_W n^{-1/2} e^{-q^2/n}. \quad (\text{S123})$$

In particular, $\sqrt{n} p_{W,q} \rightarrow (\pi s_W)^{-1/2} e^{-y^2/s_W}$, and normalized $p_{W,q}^\alpha$ have vanishing scaled tails for every fixed $\alpha > 0$.

Write $A = \max(0, -q) + j$. The conditional depth law in endpoint class (a, b) is

$$P_{n,q}^{ab}(j) = \frac{Z_n^{ab}(|q| + 2j) - Z_n^{ab}(|q| + 2j + 2)}{Z_n^{ab}(q)}. \quad (\text{S124})$$

Its step density on scale $j/\sqrt{n}$ converges in L^1 to the same function for all four classes,

$$f_{W,y}(u) = \frac{2(|y| + 2u)}{s_W} \exp\left[-\frac{(|y| + 2u)^2 - y^2}{2s_W}\right], \quad u \geq 0. \quad (\text{S125})$$

Equation (S122) controls compact depths. The exact telescoping tail $\Pr(j \geq J) = Z_n^{ab}(|q| + 2J)/Z_n^{ab}(q)$ controls the remainder, first taking $n \rightarrow \infty$ at fixed $J/\sqrt{n}$ and then this ratio to infinity. The convergence and vanishing maximum atom are uniform on compact scaled charges, including $y = 0$.

Importantly, the laws agree asymptotically on the same integer lattice, not merely weakly after rescaling. A law supported on even depths would have no phase oscillation at $\Phi = \pi$, regardless of its broad scaled profile. The L^1 statement excludes such a failure here. The squared Hellinger distance is bounded by L^1 distance, so all four depth laws can be replaced by one common law in the normalized coefficient matrix with a Hilbert–Schmidt error $o(1)$, uniformly in the flux.

4. Boundary factor and limiting Schmidt probabilities

Define a two-dimensional boundary matrix and its probability vector by

$$R_{ab} = \sqrt{\xi_a \xi_b W_{ab}}, \quad \omega_i = \frac{s_i(R)^2}{\Lambda}, \quad i = 1, 2, \quad (\text{S126})$$

where s_i denotes singular values. Indeed $\|R\|_{\text{HS}}^2 = \sum_{ab} \xi_a \xi_b W_{ab} = \mathbf{v}^T \mathbf{P} \mathbf{v} = \Lambda$, so $\sum_i \omega_i = 1$. Write $H(\omega) = -\sum_i \omega_i \log \omega_i$, with $0 \log 0 = 0$. Define $\hat{K}_{W,q} = K_{W,q}/\sqrt{p_{W,q}}$. The local limits and charge normalization reduce Eq. (S120) to

$$\begin{aligned} \hat{K}_{W,q} &= \mathbf{B} \otimes L_{n,q}(\Phi) + o_{\text{HS}}(1), \\ \mathbf{B}_{(a,b),(c,d)} &= R_{bc} R_{da} / \Lambda, \\ L_{n,q}(\Phi)_{jk} &= \sqrt{P_j P_k} e^{i\Phi \max(j,k)}. \end{aligned} \quad (\text{S127})$$

A common depth-shift phase has been omitted. Row and column permutations identify $\mathbf{B}$ with $\mathbf{R} \otimes \mathbf{R}/\Lambda$, up to transposes. Its squared singular values are $\omega_i \omega_j$.

The scalar argument in Appendix G uses only a common depth lattice, its smooth L^1 local limit and a vanishing maximum atom. To recall the key step, embed P_j as quantile intervals on $[0, 1]$. The upper-triangular matrix converges in Hilbert–Schmidt norm to upper Volterra integration; its diagonal correction vanishes. The phase multiplier tends weakly to zero at fixed nontrivial Φ : Abel summation bounds each smooth compact-depth Riemann sum by $C/[\sqrt{n}|1 - e^{i\Phi}|]$, and the exact depth tail controls the rest. Finite-rank approximation then removes the cross terms in trace norm. The upper and lower Volterra spectral subspaces become orthogonal, giving two copies of b_ℓ in Eq. (S80). Consequently the conditional limiting probabilities are $\omega_i \omega_j b_\ell$, with every ℓ appearing twice. At zero flux they are just $\omega_i \omega_j$.

5. Entropy tails and cancellation of boundary entanglement

Eigenvalue convergence alone is insufficient for this entropy limit. Apply the rank- $2m$, squared-error- $4/m$ approximation of Appendix G to each of the sixteen endpoint blocks. Their squared block norms sum to one. Thus the normalized full matrix has a rank- $32m$ approximation with squared error at most $4/m$, and its ordered conditional probabilities satisfy

$$\sum_{j > 32m} \lambda_j \leq 4/m, \quad \lambda_j \leq 32768/j^2. \quad (\text{S128})$$

For the second bound, use $m = \lfloor j/64 \rfloor$ at $j \geq 128$ and $\lambda_j \leq 1$ for the remaining indices. This gives uniformly summable VN entropy tails and power tails for $\alpha > 1/2$. Zero-flux rank at most four supplies a uniform bound there too.

Conditional VN entropies therefore tend to $2H(\omega) + C_1$ at nontrivial flux and to $2H(\omega)$ at zero flux. Conditional α -moments tend respectively to $(\sum_i \omega_i^\alpha)^2 2 \sum_\ell b_\ell^\alpha$ and $(\sum_i \omega_i^\alpha)^2$. For the full entropy use the exact identities $S_1 = H(p_W) + \sum_q p_{W,q} S_1(\sigma_{W,q})$ and $\text{Tr } \rho_A^\alpha = \sum_q p_{W,q}^\alpha \text{Tr } \sigma_{W,q}^\alpha$, where $\sigma_{W,q} = \hat{K}_{W,q} \hat{K}_{W,q}^\dagger$. The Gaussian charge tails and uniform conditional bounds justify averaging. The Rényi average uses normalized $p_{W,q}^\alpha$, not $p_{W,q}$. The boundary factor cancels, giving Eq. (13) with the same C_α as Eq. (S81).

Finally the charge local limit and Gaussian bound give $H(p_W) = \frac{1}{2} \log n + \frac{1}{2} \log(\pi e s_W) + o(1)$. For the entropy tail, apply monotonicity of $-x \log x$ below e^{-1} to the envelope for $\sqrt{n} p_{W,q}$; a Gaussian times a quadratic polynomial is integrable. Since $N = 2n$,

$$S_1(\Psi_{W,\Phi}) = \frac{1}{2} \log N + \frac{1}{2} \log(\pi e s_W/2) + 2H(\omega) + C_1 + o(1) \quad (\text{S129})$$

at fixed nontrivial flux; omit C_1 at zero flux. These limits hold for fixed strictly positive W , not uniformly as entries vanish or parameters depend on N . The sharp low-order rates in Eq. (11) concern the unit-weight ring; the correlated result proved here is for $\alpha > 1/2$.

Appendix K: Diagonal perturbations: strict finite-size response and its limits

Frustration freeness is not needed for every entanglement response. Keep positive bond magnitudes and strictly positive projector strengths, and add any finite real computational-basis-diagonal potential V . The neutral phase conjugacy of Appendix C still holds. At zero phase, the Hamiltonian has nonpositive off-diagonal entries and a connected configuration graph. Perron–Frobenius therefore gives a unique neutral ground state with strictly positive normalized coefficients a_x . In seam gauge its flux image has coefficients $a_x e^{-i\Phi h_{\min}(x)}$. These coefficients need not factor into site weights or depth classes.

1. Integer-order Rényi entropies

For subsystem charge q , index the coefficient block by individual words u, v , not by depth classes. Put $a_u = -\min h_u$, $b_v = -q - \min h_v$, and $c_{uv} = a_{uv} > 0$. Then

$$C_q(\Phi)_{uv} = c_{uv} e^{i\Phi \max(a_u, b_v)}. \quad (\text{S130})$$

For an integer $m \geq 2$, the corresponding reduced moment is

$$\text{tr}(C_q C_q^\dagger)^m = \sum_{\mathbf{u}, \mathbf{v}} W(\mathbf{u}, \mathbf{v}) e^{i\Phi \Omega(\mathbf{u}, \mathbf{v})}, \quad (\text{S131})$$

$$W = \prod_{j=1}^m c_{u_j v_j} c_{u_{j+1} v_j} > 0, \quad u_{m+1} = u_1, \quad (\text{S132})$$

$$\Omega = \sum_{j=1}^m [\max(a_{u_j}, b_{v_j}) - \max(a_{u_{j+1}}, b_{v_j})]. \quad (\text{S133})$$

The moment is real. Taking real parts term by term bounds it above by its zero-flux value. This is the usual sign/modulus decomposition of replica moments [6, 7], applied here to the exact height phase.

Strictness follows from four explicitly available configurations. If $L, R \geq 2$, choose in the $q = 0$ block $u_0 = 0^L$, $u_1 = du 0^{L-2}$, $v_0 = 0^R$, and $v_1 = du 0^{R-2}$. Their matrix of maximum depths is $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Its rectangle has $\Omega = -1$, with strictly positive weight. For $m > 2$, repeating u_0, v_0 closes a cycle with the same phase. Thus, for every integer $m \geq 2$ and $\Phi \notin 2\pi\mathbb{Z}$,

$$S_m(\Phi; V) > S_m(0; V), \quad L, R \geq 2. \quad (\text{S134})$$

In particular, writing $P(\Phi) = \text{tr } \rho_A(\Phi)^2$ gives

$$P(0) - P(\Phi) \geq 4c_{u_0 v_0} c_{u_1 v_0} c_{u_1 v_1} c_{u_0 v_1} (1 - \cos \Phi) > 0. \quad (\text{S135})$$

This is a finite-size lower bound; its coefficient need not stay positive in the thermodynamic limit. The block operator-norm inequality $\|C_q(\Phi)\| \leq \|C_q(0)\|$ also gives nondecreasing min-entropy. Neither argument establishes all real Rényi orders or von Neumann monotonicity for this general V .

2. Why a uniform neutral gap suppresses the response

There is a complementary thermodynamic statement. Consider a sequence of these rings with uniformly bounded finite-range interactions, including a diagonal V of this type. Suppose the gap above the unique neutral ground state at zero flux is bounded below by $\gamma > 0$, independently of N . Then, for every $p > 0$,

$$\sup_{\substack{1 \leq L \leq N \\ |\Phi| \leq 2\pi}} |S_1(\Phi; V) - S_1(0; V)| = o(N^{-p}). \quad (\text{S136})$$

This follows from standard local spectral flow [48] and entropy continuity [49]; the sector and gauge conditions are important for applying those results.

First move the flux, by onsite charge rotations, onto a bond in the middle of the larger subsystem. Such rotations factor across the cut and preserve entanglement. The flux-dependent terms now have a fixed-size support X at distance proportional to N from both cuts, and their derivative has norm bounded independently of N . Neutral phase conjugacy makes the assumed neutral gap the same throughout the flux path.

Choose an exact spectral-flow filter with cutoff below γ , using the dynamics of the full local Hamiltonian. The flux derivative and its time evolution conserve total magnetization. Matrix elements connecting the neutral ground state to other sectors therefore vanish; the filter identity is needed only for neutral excitation energies above γ . It transports the neutral ground projection without requiring a gap to charged levels. No commutation between the Hamiltonian and its flux derivative is assumed.

The locality proof of Theorem 3.4 in Ref. [48] now applies to this full-dynamics generator. Truncate it by conditional expectation onto an R -neighborhood of X , with R proportional to N and the neighborhood wholly inside the larger subsystem. For every k , the generator tail, and hence the difference between the integrated unitaries, is at most $C_k N^{-k}$. The truncated unitary acts on only one side of the cut and cannot change entanglement. The trace distance between the resulting reduced states is at most $\epsilon_N = C_k N^{-k}$. Entropy continuity gives

$$|\Delta S_1| \leq \epsilon_N N \log 3 + h_2(\epsilon_N), \quad (\text{S137})$$

where $h_2(t) = -t \log t - (1-t) \log(1-t)$. Taking $k > p + 2$ proves Eq. (S136). A nonzero thermodynamic response, such as the offset derived above, consequently excludes a uniform neutral gap under these locality assumptions. This implication does not determine a gap exponent or a phase boundary.

3. A natural perturbation with a rigorous large-anisotropy limit

For the original unit-weight projectors, consider the single-ion term

$$H_{N,D}(\Phi) = H_N(\Phi) + D \sum_i Q_i, \quad Q_i = (S_i^z)^2. \quad (\text{S138})$$

At every finite D , Eq. (S134) applies. On the other hand, there exists a finite D_* such that for each fixed $D > D_*$ the von Neumann response satisfies Eq. (S136).

To see this, write $h_i(\Phi/N)$ for the sum of the three orthogonal local projectors in uniform gauge, so that $\|h_i\| = 1$. Rescaling gives

$$\frac{2}{D} H_{N,D}(\Phi) = \sum_i (Q_i + Q_{i+1}) + \frac{2}{D} \sum_i h_i(\Phi/N). \quad (\text{S139})$$

Each classical pair term has unique ground state 00 and local gap one. The perturbation norm per pair is at most $2/D$. Theorem 1 of Ref. [50], for translation-invariant periodic finite volumes, therefore supplies a unique ground state and a volume-independent gap for sufficiently large D , uniformly in Φ . Along the interpolation that turns on the second sum, the gap remains open and magnetization is conserved. The ground-state charge cannot change from that of 0^N without a level crossing, so the ground state is neutral. The preceding locality argument applies.

No numerical value of D_* is supplied by this application. In particular, small-system spectra cannot identify its threshold. The fixed nonzero entropy offset of the compatible frustration-free family is thus not a universal consequence of neutral isospectrality: a natural diagonal perturbation can preserve the latter while suppressing the thermodynamic von Neumann response. The fate of the offset at arbitrarily weak fixed $D > 0$ remains undetermined.

4. Singular response to weak single-ion anisotropy

The response at the frustration-free point can nevertheless be bounded without assuming an effective dynamical theory. Use the fixed uniform bias $g(u) = g(d) = 1$, $g(0) = r > 0$ of Appendix I, and allow arbitrary strictly positive channel strengths $J_{i,s,t} \leq J_*$, with J_* independent of N . Let H_r denote this zero-phase parent and set $Q = \sum_i (S_i^z)^2$. Write $E_{N,r}^{(0)}(D)$ for the lowest energy of $H_r + DQ$ in the neutral sector. At every finite N this branch is simple. Define its fidelity susceptibility by

$$|\langle \psi_{r,0}(D) | \Psi_{r,0} \rangle|^2 = 1 - \chi_{F,N} D^2 + O_N(D^3), \quad (\text{S140})$$

where $\psi_{r,0}(D)$ is normalized and $\Psi_{r,0}$ is the state in Eq. (S106). This varies the added anisotropy D , not the compatible-parent parameter r . For every fixed $r \in (0, \infty)$ there are positive constants c, c' such that

$$-\partial_D^2 E_{N,r}^{(0)}(D) \Big|_{D=0} \geq cN^2, \quad \chi_{F,N} \geq c'N^3 \quad (\text{S141})$$

for all sufficiently large N . The constants depend on r, J_* .

The mechanism is an exact overlap of the local driving term with a long-wavelength quadratic density mode. Under $\mu_r(x) = |\Psi_{r,0}(x)|^2$, put

$$k = \frac{2\pi}{N}, \quad f_k(x) = \left| \sum_{j=1}^N e^{ikj} x_j \right|^2, \quad \mathcal{V}_N = \text{Var}_{\mu_r}(Q). \quad (\text{S142})$$

The ground-state measure is exchangeable even when the strengths are not. Because $\sum_j x_j = 0$,

$$\mathbb{E} f_k = \frac{N}{N-1} \mathbb{E} Q, \quad \text{Cov}(Q, f_k) = \frac{N}{N-1} \mathcal{V}_N =: \mathcal{C}_N. \quad (\text{S143})$$

Indeed, for $i \neq j$, $\mathbb{E}[Qx_i^2] = \mathbb{E}Q^2/N$ and $\mathbb{E}[Qx_i x_j] = -\mathbb{E}Q^2/[N(N-1)]$; summing the Fourier coefficients uses $\sum_{i \neq j} e^{ik(i-j)} = -N$.

Every legal move changes the Fourier amplitude by $s_i = e^{iki} - e^{ik(i+1)}$, including the wrapping bond. Set $c_N = |s_i|^2 = 4 \sin^2(\pi/N)$. Then

$$(f_k(x) - f_k(y))^2 \leq 8c_N f_k(x) + 2c_N^2 \quad (\text{S144})$$

on an accepted source–target edge $x \rightarrow y$. Its exact Dirichlet weight is $J_{i,s,t} \mu_r(x)/(1 + k_{i,s,t}^2)$, with the normalized ratios in Eq. (S104). The three source patterns on a fixed bond are disjoint. Writing $\gamma_r = \max\{1/2, r^4/(1 + r^4)\}$ therefore gives

$$\begin{aligned} \mathcal{A}_N &:= \langle (f_k - \mathbb{E} f_k) \Psi_{r,0} | H_r | (f_k - \mathbb{E} f_k) \Psi_{r,0} \rangle \\ &\leq \gamma_r J_* N (8c_N \mathbb{E} f_k + 2c_N^2) =: \mathcal{B}_N. \end{aligned} \quad (\text{S145})$$

Both ordered bonds are retained when $N = 2$.

Let $b = (Q - \mathbb{E}Q) \Psi_{r,0}$, and let H_r^+ be the inverse on the neutral subspace orthogonal to its ground state. The usual spectral perturbation identities [51] give

$$-\partial_D^2 E_{N,r}^{(0)}(0) = 2\mathcal{R}_N, \quad \mathcal{R}_N = \langle b | H_r^+ | b \rangle, \quad \chi_{F,N} = \langle b | (H_r^+)^2 | b \rangle. \quad (\text{S146})$$

Two Cauchy–Schwarz inequalities, first in the Dirichlet inner product and then in the spectral measure of b , yield the finite-size bounds

$$\mathcal{R}_N \geq \frac{\mathcal{C}_N^2}{\mathcal{B}_N}, \quad \chi_{F,N} \geq \frac{\mathcal{R}_N^2}{\mathcal{V}_N} \geq \frac{\mathcal{C}_N^4}{\mathcal{B}_N^2 \mathcal{V}_N}. \quad (\text{S147})$$

It remains to control the variance, rather than extrapolate it from sizes. For $v = 2/(2+r^2) \in (0, 1)$, the fixed-endpoint counts in Eq. (S110) satisfy, for fixed s, q ,

$$\frac{T_r(N-s, q)}{T_r(N, 0)} = (2+r^2)^{-s} \left[1 + \frac{s/2 - q^2/(2v)}{N} + O_{r,s,q}(N^{-2}) \right]. \quad (\text{S148})$$

Fourier inversion proves this directly: near zero, $\log(1 - v + v \cos \theta) = -v\theta^2/2 + (v - 3v^2)\theta^4/24 + O_r(\theta^6)$. After $\theta = t/\sqrt{N}$, the Gaussian expansion has an integrated relative remainder $O(N^{-2})$; outside a fixed neighborhood of zero the characteristic function has modulus strictly less than one. The relative N^{-1} coefficient in $T_r(N, q)$ is $(1 - 3v)/(8v) - q^2/(2v)$, which gives the displayed ratio. No derivative of a local-limit error is required. The exact identities

$$\mathbb{E}Q = \frac{2NT_r(N-1, 1)}{T_r(N, 0)}, \quad (\text{S149})$$

$$\mathbb{E}Q(Q-1) = \frac{2N(N-1)[T_r(N-2, 0) + T_r(N-2, 2)]}{T_r(N, 0)} \quad (\text{S150})$$

then imply $\mathbb{E}Q = vN + (v-1)/2 + O_r(N^{-1})$, $\mathbb{E}Q^2 = v^2N^2 + O_r(1)$, and $\mathcal{V}_N = v(1-v)N + O_r(1)$. Consequently $\mathcal{C}_N = v(1-v)N + O_r(1)$ while $\mathcal{B}_N = 32\pi^2\gamma_r J_* v + O_{r,J_*}(N^{-1})$. Equation (S147) proves Eq. (S141).

Neutral phase conjugacy commutes with Q and is independent of D . Thus both the entire neutral energy curve and this anisotropy fidelity susceptibility are identical at every fixed flux. The growing derivatives are nevertheless not a fixed- D entropy theorem: they can be confined to a shrinking perturbative window. Nor are 2 and 3 established critical exponents. Distinguishing an exact ground-state walk law from lattice dynamics is essential in these chains [52]. Determining the von Neumann flux response at arbitrarily weak fixed $D > 0$ still requires control of the perturbed ground state beyond these response bounds.

Appendix L: Adding the reverse pair conversion

The neutral conjugacy uses the allowed moves, not only the zero-flux ground state. To isolate this distinction, retain the original unit weights, put $z_i = e^{i\Phi/N}$, and define

$$H_{N,\epsilon}(\Phi) = H_N(e^{i\Phi/N}, \dots, e^{i\Phi/N}) + \epsilon \sum_i |G_i\rangle\langle G_i|, \quad |G_i\rangle = \frac{|du\rangle - e^{i\Phi/N}|00\rangle}{\sqrt{2}}, \quad \epsilon > 0. \quad (\text{S151})$$

The opposite sign of the phase relative to $|F_i\rangle$ is fixed by the opposite orientation of the charge transfer. Both ordered bonds are retained at $N = 2$. There is no single-ion term in this appendix.

At $\Phi = 0$ every uniform magnetization-sector vector is annihilated by the added projector. Positivity and the original kernel classification therefore give exactly the same $2N + 1$ ground vectors and the same entanglement. At $\Phi \notin 2\pi\mathbb{Z}$, only the two polarized vectors remain at zero energy. Indeed, the original kernel contains just those vectors and its unique neutral vector. In the seam gauge, select a du valley which is the unique global minimum, with every other letter flat. Removing the valley raises the minimum by one. Equation (5) then violates the new reverse relation by the factor $e^{i\Phi}$. The seam can be placed away from the chosen bond by a charge gauge; the mismatch is invariant. This argument includes $N = 2$ and shows that even one positive reverse term removes the old neutral zero mode. The bounds below instead use the same positive ϵ on every bond.

1. Neutral energy curvature

Let $E_{N,\epsilon}^{(0)}(\Phi)$ be the lowest neutral energy and define

$$\mathcal{K}_{N,\epsilon} = \left. \frac{\partial^2 E_{N,\epsilon}^{(0)}}{\partial \Phi^2} \right|_{\Phi=0}, \quad v_N = \frac{2T(N-1, 1)}{T(N, 0)}. \quad (\text{S152})$$

For every finite $N \geq 2$,

$$\frac{\epsilon}{1+\epsilon} \frac{v_N}{N-1} \leq \mathcal{K}_{N,\epsilon} \leq \frac{2T(N-2, 1) + (1+\epsilon)T(N-2, 0)}{NT(N, 0)}. \quad (\text{S153})$$

Thus $\mathcal{K}_{N,\epsilon} = \Theta_\epsilon(N^{-1})$ for each fixed $\epsilon > 0$. More explicitly,

$$\liminf_{N \rightarrow \infty} N\mathcal{K}_{N,\epsilon} \geq \frac{2\epsilon}{3(1+\epsilon)}, \quad \limsup_{N \rightarrow \infty} N\mathcal{K}_{N,\epsilon} \leq \frac{3+\epsilon}{9}. \quad (\text{S154})$$

These are sector-resolved curvatures. The full ground energy remains zero because of the polarized states.

To prove the bounds, let μ be the uniform measure on the $d = T(N, 0)$ neutral words. Treat each bond, outside word, and channel as a separate undirected edge, including parallel edges at $N = 2$. The original three channels have conductance $c_e = 1/2$, and the reverse channel has $c_e = \epsilon/2$. Orient the original moves as in Eq. (S1), with phase slope $a_e = 1/N$; orient $du \rightarrow 00$ with $a_e = -1/N$. Connectedness makes the neutral ground state simple at zero flux. Analytic perturbation and complex conjugation make its energy analytic and even near zero. Its first state derivative can be chosen $if(x)/\sqrt{d}$, with real f : the Hamiltonian derivative is purely imaginary, and a real tangent component only adds a nonnegative quadratic term. Expanding the edge energy and minimizing this tangent gives

$$\mathcal{K}_{N,\epsilon} = 2 \min_{f: \mathcal{W}_{N,0} \rightarrow \mathbb{R}} \mathcal{F}(f), \quad \mathcal{F}(f) = \frac{1}{d} \sum_e c_e [a_e - f(x_e) + f(y_e)]^2. \quad (\text{S155})$$

An additive constant in f has no effect. This is the usual magnetic Laplacian curvature variational principle.

For each fixed bond and outside word, eliminate the intermediate 00 configuration by weighted Cauchy-Schwarz:

$$\begin{aligned} \frac{1}{2} [N^{-1} - f_{ud} + f_{00}]^2 + \frac{\epsilon}{2} [-N^{-1} - f_{du} + f_{00}]^2 \\ \geq c_\epsilon [2N^{-1} - f_{ud} + f_{du}]^2, \quad c_\epsilon = \frac{\epsilon}{2(1+\epsilon)}. \end{aligned} \quad (\text{S156})$$

Each labelled pair edge is used once, even when different contexts share a 00 word. Since the hopping conductance is at least c_ϵ , the resulting form dominates equal-conductance exchanges of all three colors. For an exchange on $(i, i+1)$ its phase slope is $(x_i - x_{i+1})/N$. The cross term between this slope and f vanishes for every f : its divergence at each word is proportional to $\sum_i (x_i - x_{i+1}) = 0$. Consequently

$$\mathcal{F}(f) \geq \frac{c_\epsilon}{2N^2} \mathbb{E}_\mu \sum_i (x_i - x_{i+1})^2 = \frac{c_\epsilon v_N}{N-1}. \quad (\text{S157})$$

The last equality uses $\mathbb{E}x_i^2 = v_N$ and $\mathbb{E}x_i x_j = -v_N/(N-1)$ for distinct sites. This proves the lower bound in Eq. (S153). Setting $f = 0$ in Eq. (S155) proves the upper bound: each hopping channel has $T(N-2, 1)$ source words per bond, and each pair channel has $T(N-2, 0)$. Finally $v_N \rightarrow 2/3$ and $T(N-2, q)/T(N, 0) \rightarrow 1/9$ for fixed $q = 0, 1$ give Eq. (S154). The finite- N expansion $E_{N,\epsilon}^{(0)}(\Phi) = \mathcal{K}_{N,\epsilon} \Phi^2/2 + O_N(\Phi^4)$ does not supply a size-uniform finite-flux estimate.

2. Zero-flux spectral gap

Let $\gamma_{N,\epsilon,M}(0)$ be the first positive eigenvalue in a nonsingleton magnetization sector, and $\gamma_{N,\epsilon}(0)$ the full positive gap. For $|M| < N$,

$$\gamma_{N,\epsilon,M}(0) \geq \frac{\epsilon}{64(1+\epsilon)N^2}, \quad \frac{\epsilon}{64(1+\epsilon)N^2} \leq \gamma_{N,\epsilon}(0) \leq 1 - \cos(2\pi/N). \quad (\text{S158})$$

The constants are deliberately coarse. A self-contained comparison proof also identifies which local process restores the N^{-2} scale.

On words of fixed magnetization M , use the uniform measure μ_M . An involution T acting at rate r has Dirichlet form $\frac{r}{2} \mathbb{E}_{\mu_M} |f - f \circ T|^2$. Let $\mathcal{Q}_{\text{loc}}$ contain, on every bond of the open path $1, \dots, N$, rate-one color exchange S , $ud \leftrightarrow 00$ conversion F , and $du \leftrightarrow 00$ conversion G , each acting as the identity on other pairs. Discard the ring seam from $H_{N,\epsilon}(0)$ and split the remaining form into two equal halves. Retain U, D, F, G from one half, with conductances $1/4, 1/4, 1/4, \epsilon/4$. In the other half the series inequality produces the missing $ud \leftrightarrow du$ exchange. Hence

$$\mathcal{E}_\epsilon(f) \equiv \langle f, H_{N,\epsilon}(0)f \rangle_{\mu_M} \geq c_0 \mathcal{Q}_{\text{loc}}(f), \quad c_0 = \frac{\epsilon}{4(1+\epsilon)}. \quad (\text{S159})$$

Next let $\mathcal{Q}_{\text{comp}}$ apply each of S_{ij}, F_{ij}, G_{ij} at rate $1/N$ for every unordered site pair. We prove

$$\text{Var}_{\mu_M} f \leq 2 \mathcal{Q}_{\text{comp}}(f). \quad (\text{S160})$$

Charge conjugation preserves this equal-rate comparison process, so take $M \geq 0$ and condition on $m = n_d$, with $n_u = m + M$ and $z = n_0 = N - 2m - M$. In a fixed-population slice, uniform permutations satisfy

$$\text{Var}(f \mid m) \leq \frac{1}{N} \sum_{i < j} \mathbb{E}[|f - f \circ S_{ij}|^2 \mid m]. \quad (\text{S161})$$

For completeness, prove this non-sharp bound on labelled permutations by induction. Condition on the label at a distinguished site k . The conditional variance uses the $(N - 1)$ -site bound. Couple two values of the label at k by transposing k with the position of the second label; Jensen bounds the variance of the conditional mean by $(2N)^{-1} \sum_{j \neq k} \mathbb{E}|f - f \circ S_{kj}|^2$. Averaging over k gives coefficient $(N - 2)/[N(N - 1)] + 1/N^2 \leq 1/N$ for every pair square. The $N = 2$ base is immediate. Pulling a multiset-valued function back to labelled permutations gives Eq. (S161), so the averaged conditional variance is at most twice the exchange part of $\mathcal{Q}_{\text{comp}}$.

The remaining count variable has stationary weights $\pi_m \propto N!/[m!(m + M)!z!]$ and birth/death rates

$$\lambda_m = \frac{z(z - 1)}{N}, \quad \mu_m = \frac{m(m + M)}{N}, \quad 0 \leq m \leq \lfloor (N - M)/2 \rfloor. \quad (\text{S162})$$

There are two possible pair creations from each 00 pair; the total birth rate includes both. Detailed balance follows from the factorial ratio. This finite chain has gap at least one. Indeed, for its generator $\mathcal{L}$ and $g_m = h_{m+1} - h_m$, $\nabla \mathcal{L}h = \mathcal{L}_{\text{edge}}g - V_m g$, where the edge generator has birth rate λ_{m+1} , death rate μ_m , and

$$V_m = \lambda_m - \lambda_{m+1} + \mu_{m+1} - \mu_m = \frac{4N - 6m - 3M - 5}{N} \geq 1 \quad (\text{S163})$$

on every count edge. For a nonconstant real eigenfunction of $-\mathcal{L}$, choose an edge maximizing $|g|$ and fix its sign. At that edge $-\mathcal{L}_{\text{edge}}g \geq 0$, so its eigenvalue is at least one. The count-chain variance is therefore bounded by its Dirichlet form. For $h(m) = \mathbb{E}[f \mid m]$, Jensen bounds that form by the reaction part of $\mathcal{Q}_{\text{comp}}$: reaction-edge endpoint marginals are uniform on each population slice, since every word in a slice has the same number of 00 and opposite-charge pairs. If only one slice exists, the count variance vanishes. Variance decomposition proves Eq. (S160).

Finally implement a remote S_{ij} , F_{ij} or G_{ij} by adjacent exchanges moving site j to $i + 1$, applying the local involution at $(i, i + 1)$, and undoing the exchanges. Each path has length at most $2N$. Every prefix is a bijection of the fixed- M word set, including after a reaction, so uniform averaging is preserved. Cauchy-Schwarz bounds a remote square by the path length times the local squares. Across the three remote types, a local exchange occurs at most six times per site pair; with rate $1/N$ and at most $N(N - 1)/2$ pairs, its coefficient is at most $6N^2$. A local reaction occurs only for paths anchored at its left endpoint, giving coefficient at most $2N \leq N^2$. The safe common bound is

$$\mathcal{Q}_{\text{comp}} \leq 8N^2 \mathcal{Q}_{\text{loc}}, \quad \text{Var}_{\mu_M} f \leq \frac{16N^2}{c_0} \mathcal{E}_\epsilon(f). \quad (\text{S164})$$

This proves the sector lower bound. The polarized singleton sectors have no positive internal level. The one-flat sector is unchanged by G and supplies the full-gap upper bound in Eq. (S158).

The neutral gap also has the N^{-2} scale. For $f(x) = \sum_j e^{2\pi i j/N} x_j$ in the uniform neutral ensemble, $\mathbb{E}f = 0$ and $\mathbb{E}|f|^2 = N^2 v_N/(N - 1)$. Direct edge counting gives

$$\mathcal{E}_\epsilon(f) = 4 \sin^2(\pi/N) \frac{N}{T(N, 0)} \left[T(N - 2, 1) + \frac{1 + \epsilon}{2} T(N - 2, 0) \right]. \quad (\text{S165})$$

The Rayleigh quotient is $O_\epsilon(N^{-2})$. Together with the lower bound this gives $\gamma_{N, \epsilon, 0}(0) = \Theta_\epsilon(N^{-2})$.

At $\epsilon = 1$ and zero flux, the Hamiltonian is also the spin-one chain of Watanabe, Katsura, and Lee, up to an overall energy scale [53]. The occupations $\eta_i = x_i + 1 \in \{0, 1, 2\}$ identify the zero-flux Hamiltonian with minus one half the bulk generator of the constant-rate generalized symmetric exclusion process [54, 55]. A particle can hop when the departure site is nonempty and the arrival site is not full; each allowed hop has the same rate. Diffusion and spectral-gap estimates for that process are known. These are precisely the site-uniform two-particle-capacity rates studied by Arita, Krapivsky, and Mallick [29], with our generator smaller by a factor of two. Their finite-range variational approximations address bulk diffusivity, not the finite-flux quantum entanglement considered here. The comparisons above give explicit conservative constants here, rather than a new diffusion universality class. The original model instead has the subdiffusive neutral upper gap bound of Ref. [28]. The wavefunction/dynamics distinction is also familiar in Motzkin and Fredkin chains [52]. The present comparison isolates its flux consequence at exactly identical zero-flux states. Neither the zero-flux gap nor the neutral curvature determines the new ground state's entanglement at finite total flux, or a real-time transport coefficient.

Appendix M: Entanglement response after resolving magnetization

The exact phase conjugacy of the three-channel chain need not survive a change of local dynamics. Here we calculate a response that remains after the regional magnetizations have been measured. The special rates below are essential, rather than a general perturbative-stability assumption.

1. Hamiltonian, state, and conditional entanglement

In the uniform gauge $\theta = \Phi/N$, use the normalized vectors U_i, D_i, F_i of Eq. (1) with $z_i = e^{i\theta}$ and $G_i = (|du\rangle - e^{i\theta}|00\rangle)/\sqrt{2}$. Define

$$H_g(\Phi) = \sum_i (|U_i\rangle\langle U_i| + |D_i\rangle\langle D_i| + 2|F_i\rangle\langle F_i| + 2|G_i\rangle\langle G_i|). \quad (\text{S166})$$

A common positive multiplier fixes the energy unit without changing the states. This is not the path obtained by adding an arbitrarily weak G term to the original chain: both pair-conversion strengths are two. Its zero-flux generator is a known gradient member of the capacity-two exclusion family [29], not a new model.

In a fixed total magnetization sector M , the zero-flux ground vector $\Omega_{N,M}$ is uniform on the $T(N, M)$ words of charge M . The comparison in Appendix L, with $\epsilon = 1$, gives an open-path gap at least $1/(128N^2)$ in every nonsingleton sector. Increasing the two pair strengths preserves that lower bound. Thus the actual ground branch $\psi_{N,M}(\Phi)$ is simple and analytic near zero flux at each finite N .

For a contiguous interval A of l sites and its complement B , let $p_q(\Phi)$ be the probability of measuring magnetization q in A , and $\rho_{A,q}(\Phi)$ the normalized conditional reduced state. The average entanglement retained in all outcomes is

$$\mathcal{E}_{\text{ch}}(\Phi) = \sum_q p_q(\Phi) S(\rho_{A,q}(\Phi)) = S(\rho_A(\Phi)) - H(p(\Phi)). \quad (\text{S167})$$

This established operational quantity [22, 23] uses local operations preserving regional magnetization. For spins this is an imposed operation class, not a fundamental superselection rule. Every zero-flux charge block is a product, so $\mathcal{E}_{\text{ch}}(0) = 0$.

For $N \geq 4$, $|M| \leq N - 2$, and $2 \leq l \leq N - 2$, we prove

$$\mathcal{E}_{\text{ch}}(\Phi) = -\eta_{N,M,l} \Phi^4 \log(\Phi^4) + O_{N,M,l}(\Phi^4). \quad (\text{S168})$$

The coefficient obeys the finite lower bound in Eq. (S182). If $M/N \rightarrow \rho \in (-1, 1)$ and $l/N \rightarrow a \in (0, 1)$, then

$$\liminf_{N \rightarrow \infty} \eta_{N,M,l} \geq \frac{v(\rho)^2 v''(\rho)^2 a^3 (1-a)^3}{576}, \quad v(\rho) = \frac{w(w-1)}{3}, \quad w = \sqrt{4 - 3\rho^2}. \quad (\text{S169})$$

Here $v''(\rho) = -2 + 4/w^3$. In the neutral sector the coefficient has an exact macroscopic-cut limit. Uniformly for $a_N = l/N$ in a compact $K \subset (0, 1)$,

$$\eta_{N,0,l} = \frac{a_N^3 (1 - a_N)^3}{576} + O_K(N^{-1/10}). \quad (\text{S170})$$

In particular, $\eta_{N,0,l} = \Theta_K(1)$ and the half-cut limit is $1/36864$. Appendix P proves the exact value and the vanishing error, strengthening the crossed-dipole lower bound. This coefficient is defined by taking the weak-flux limit at finite N . Appendix N additionally proves a size-independent small-flux interval and a neutral fixed-flux entropy lower bound; an exact fixed-flux entropy limit is not asserted.

2. Which part of the response creates new Schmidt weights?

Let μ be the uniform charge- M word measure, $H_0 = H_g(0)$, and $\mathcal{R}$ its diagonal escape rate. Its bond table, in row and column order $-1, 0, 1$, is

$$r(x, y) = \begin{pmatrix} 0 & 1/2 & 1 \\ 1/2 & 2 & 1/2 \\ 1 & 1/2 & 0 \end{pmatrix}_{x,y}, \quad \mathcal{R}(x) = \sum_i r(x_i, x_{i+1}). \quad (\text{S171})$$

Set $\mathbf{r} = \mathcal{R} - \mathbb{E}_\mu \mathcal{R}$ and $\xi = H_0^+ \mathbf{r}$, with $\mathbb{E}_\mu \xi = 0$. The signed bond current $(x_i - x_{i+1})/2$ telescopes around the ring, so $H_g(\Phi)\Omega = (1 - \cos(\Phi/N))\mathcal{R}\Omega$ and

$$\psi_{N,M}(\Phi) = \Omega_{N,M} - \frac{\Phi^2}{2N^2} \xi \Omega_{N,M} + O_{N,M}(|\Phi|^3). \quad (\text{S172})$$

The pseudoinverse acts on functions with the μ inner product; multiplication by Ω identifies them with normalized state vectors.

Given the regional charges, the two words are independent and uniform. Define the orthogonal projection

$$\mathbf{Q}f = f - \mathbb{E}[f \mid x_A] - \mathbb{E}[f \mid x_B] + \mathbb{E}[f \mid q]. \quad (\text{S173})$$

A change depending on only one region changes its local Schmidt vector without creating a new Schmidt weight. Consequently

$$\eta_{N,M,l} = \frac{\|\mathbf{Q}\xi\|_\mu^2}{4N^4}. \quad (\text{S174})$$

The distinction between both-sided entangling perturbations and the full state or fidelity response is standard [56, 57]. Schmidt support also opens under anisotropy detuning near the XXZ ferromagnetic point [39]. Here a boundary flux drives the response, which is resolved within each regional-magnetization block. The finite-size expansion is distinct from the size-uniform fixed-total-flux bound proved in Appendix N.

For the remainder, the block amplitude matrix has the expansion $C_q(\Phi) = C_{q,0} + \Phi^2 C_{q,2} + O_N(|\Phi|^3)$, where $C_{q,0}$ has rank one and a positive singular value. Isolate that singular subspace by smooth row and column unitaries. The small complementary block is $\Phi^2 \mathbf{Q}C_{q,2} + O_N(|\Phi|^3)$; eliminating its large row and column changes it only at order Φ^4 . Weyl's inequality gives small singular values $\Phi^2 s_j + O_N(|\Phi|^3)$. For $s_j > 0$ the probabilities are $s_j^2 \Phi^4 + O_N(|\Phi|^5)$; when $s_j = 0$ they are $O_N(\Phi^6)$. The latter contribute $O_N(\Phi^6 |\log |\Phi||)$ to entropy. For the former the logarithmic error is $O_N(|\Phi|^5 |\log |\Phi||) = o_N(\Phi^4)$. Normalization by the positive block mass changes only the nonlogarithmic Φ^4 term. This proves Eqs. (S168) and (S174). It does not assume analyticity in Φ^2 ; third-order imaginary amplitudes are allowed. Logarithms from opening Schmidt support are known perturbative behavior [58], not by themselves a new criticality criterion.

3. An exact crossed-dipole response

Index sites by $0, \dots, N-1$. For nonzero momenta $k = 2\pi j/N$, put $F_k = \sum_i e^{iki} x_i$, $c_k = 1 - \cos k$, and $G_k = |F_k|^2 - \mathbb{E}_\mu |F_k|^2$. The bond table gives the pointwise identities

$$H_0 F_k = c_k F_k, \quad H_0 G_k = 2c_k (G_k - \mathbf{r}). \quad (\text{S175})$$

The first follows from the bond current; the second follows by applying each move to $|F_k|^2$. Self-adjointness implies

$$\langle \xi, G_k \rangle_\mu = A_{\mathcal{R}} + \frac{\text{Cov}_\mu(\mathcal{R}, G_k)}{2c_k}, \quad A_{\mathcal{R}} = \langle \mathbf{r}, H_0^+ \mathbf{r} \rangle_\mu. \quad (\text{S176})$$

The finite covariance is affine in $\cos k$: $\text{Cov}(\mathcal{R}, G_k) = u_N + \tilde{v}_N \cos k$. An explicit normalized formula is useful. Set $m = N-2$, $Q = M - X - Y$, $\bar{x} = Q/m$, $\nu = v_m(Q)$, $a_X = X - \bar{x}$, $a_Y = Y - \bar{x}$, where

$$e_2(n, q) = \frac{T(n-1, q-1) + T(n-1, q+1)}{T(n, q)}, \quad v_n(q) = e_2(n, q) - (q/n)^2, \quad \mathbb{P}(X, Y) = \frac{T(m, M - X - Y)}{T(N, M)}. \quad (\text{S177})$$

Infeasible Q has probability zero. The conditional Fourier variance is $\mathcal{C}_0 + \mathcal{C}_1 \cos k$, with

$$\begin{aligned} \mathcal{C}_0 &= a_X^2 + a_Y^2 + \nu \frac{m^2 - 2}{m - 1}, & \mathcal{C}_1 &= 2a_X a_Y - \frac{2\nu}{m - 1}, \\ u_N &= N \mathbb{E} \left[r(X, Y) \left(\mathcal{C}_0 - \frac{N^2 v_N(M)}{N - 1} \right) \right], & \tilde{v}_N &= N \mathbb{E} [r(X, Y) \mathcal{C}_1]. \end{aligned} \quad (\text{S178})$$

Write $d_N = u_N + \tilde{v}_N$. This extrapolates the nonzero-momentum formula to $\cos k = 1$; it is not the covariance of the constant $F_0 = M$.

Let $\mathcal{D}_A = \sum_{i \in A} a_i S_i^z$ and $\mathcal{D}_B = \sum_{j \in B} b_j S_j^z$, with separately zero-sum real weights. Set $g = \mathcal{D}_A \mathcal{D}_B$ as a diagonal function, $m_A = \sum i a_i$, and $m_B = \sum j b_j$. Conditional independence gives $\mathbb{Q}g = g$ and

$$\|g\|_\mu^2 = \mathcal{V}_{N,M,l} \left(\sum_i a_i^2 \right) \left(\sum_j b_j^2 \right), \quad \mathcal{V}_{N,M,l} = \sum_q \frac{T(l,q)T(N-l,M-q)}{T(N,M)} \frac{l v_l(q)}{l-1} \frac{(N-l)v_{N-l}(M-q)}{N-l-1}. \quad (\text{S179})$$

Write $A_k = \sum_i a_i e^{ik_i}$ and $B_k = \sum_j b_j e^{ik_j}$. Since ξ is translation invariant, averaging g over translations expresses its overlap with ξ as $N^{-2} \sum_{k \neq 0} A_{-k} B_k \langle \xi, G_k \rangle$. Zero-sum weights remove $k = 0$; disjoint support gives $\sum_k A_{-k} B_k = N \sum_i a_i b_i = 0$. Both the common $A_{\mathcal{R}}$ term and the constant part of $(u_N + \tilde{v}_N \cos k)/c_k$ therefore cancel. The exact cycle Green kernel is

$$\sum_{k \neq 0} \frac{e^{ikd}}{1 - \cos k} = \frac{N^2 - 1}{6} - d(N - d), \quad 0 \leq d < N. \quad (\text{S180})$$

Apply the cycle second difference and fix zero average to verify it. For contiguous disjoint A, B , $0 < j - i < N$. Zero-sum weights kill its constant and linear terms, leaving $-2m_A m_B$. Hence

$$\langle \xi, \mathcal{D}_A \mathcal{D}_B \rangle_\mu = -\frac{d_N m_A m_B}{N^2}, \quad \partial_\Phi^2 \langle \mathcal{D}_A \mathcal{D}_B \rangle_\Phi|_0 = \frac{2d_N m_A m_B}{N^4}. \quad (\text{S181})$$

Cauchy-Schwarz in the $\mathbb{Q}$ subspace bounds η from below by this observable response squared, divided by $16\|g\|_\mu^2$. Centered linear ramps maximize $m_A^2 / \sum a_i^2$ and $m_B^2 / \sum b_j^2$. Taking $a_i = i - (l-1)/2$ and $b_j = j - (l+N-1)/2$ gives

$$\eta_{N,M,l} \geq \frac{d_N^2 L_A L_B}{4N^8 \mathcal{V}_{N,M,l}}, \quad L_A = \frac{l(l^2 - 1)}{12}, \quad L_B = \frac{(N-l)((N-l)^2 - 1)}{12}. \quad (\text{S182})$$

The variance product is positive in the stated domain. The lower bound can vanish if $d_N = 0$. This is a response relation for the specified pure-state branch, not a mixed-state entanglement witness.

4. Uniform canonical expansion and the density limit

We give the counting estimate needed for both thermodynamic bounds. For $\rho = q/n$ in any compact interval inside $(-1, 1)$,

$$e_2(n, q) = u(\rho) + \frac{c(\rho)}{n} + O(n^{-2}), \quad u(\rho) = \frac{4 - \sqrt{4 - 3\rho^2}}{3}. \quad (\text{S183})$$

The remainder is uniform; c is smooth and even. To prove this, tilt the one-spin law to $p_t(x) = e^{tx}/(1 + 2 \cosh t)$, with $\mathbb{E}_t X = \rho$. Conditioning on the sum q leaves the canonical measure unchanged. The tilt is smooth, its variance is bounded below on compact intervals, and all cumulants and their derivatives are uniformly bounded. For $\varphi(h) = \mathbb{E}_t e^{ih(X-\rho)}$ and $m(h) = \mathbb{E}_t [X^2 e^{ih(X-\rho)}]$, Fourier inversion gives

$$e_2(n, q) = \frac{\int_{-\pi}^{\pi} m(h) \varphi(h)^{n-1} dh}{\int_{-\pi}^{\pi} \varphi(h)^n dh}. \quad (\text{S184})$$

Away from a fixed neighborhood of zero, aperiodicity bounds $|\varphi| \leq 1 - \delta$ uniformly in the compact tilt range. Near zero put $h = y/\sqrt{n}$, expand $\log \varphi$ through degree eight and m/φ through degree six, and integrate against $e^{-v(\rho)y^2/2}$. Odd half-integer terms have zero symmetric integral. Uniform Taylor bounds and Gaussian domination control the integrated remainder at relative order n^{-2} . Outside $|y| \leq \sqrt{C \log n}$ the Gaussian tail is smaller than that order for sufficiently large C . The numerator and denominator are

$$n^{-1/2} \sqrt{2\pi/v} [u + n_1/n + O(n^{-2})], \quad n^{-1/2} \sqrt{2\pi/v} [1 + d_1/n + O(n^{-2})].$$

Their coefficients are smooth Gaussian moments with positive v . Division proves Eq. (S183); exact charge conjugation implies $c(-\rho) = c(\rho)$. Finite small n are absorbed in the constant.

The same estimate evaluates Eq. (S178). At formal $\cos k = 1$, subtracting the unconditional Fourier variance from $\mathcal{C}_0 + \mathcal{C}_1$ gives

$$(X + Y - 2\rho)^2 - 2v(\rho) - v'(\rho)(X + Y - 2\rho) + O(N^{-1}).$$

The pair law converges to two independent tilted spins at order N^{-1} . Its density score is $(X + Y - 2\rho)/v$. Differentiating the exact tilted identity $\mathbb{E}_t r(X, Y) = v(\rho)$ twice therefore yields

$$\frac{d_N}{N} = v(\rho)^2 v''(\rho) + O(N^{-1}). \quad (\text{S185})$$

The unknown c cancels in the covariance difference. For macroscopic cuts the regional charge has mean $l\rho$ and variance $v_N(M)l(N-l)/(N-1) = O(N)$. On central charges Eq. (S183) gives $\mathcal{V}_{N,M,l} \rightarrow v(\rho)^2$; outside any fixed central density interval use $0 \leq nv_n(q)/(n-1) \leq 2$ and concentration. Inserting this into Eq. (S182) proves Eq. (S169). At neutrality $v = 2/3$, $v'' = -3/2$, giving the positive half-cut lower bound $1/36864$. A zero of v'' only makes this lower bound inconclusive. Saturation sequences lie outside the compact-density limit.

5. Decomposition and control of the neutral remainder

Here $M = 0$. Set $\mathcal{S}(x) = \sum_i x_i^2$, $s_0 = \mathcal{S} - \mathbb{E}_\mu \mathcal{S}$, $\mathfrak{z} = \mathfrak{r} + 3s_0$, and $\mathcal{F}_N = \sum_{k \neq 0} G_k/c_k$. Summing Eq. (S175) and applying Parseval gives

$$H_0 \mathcal{F}_N = 2Ns_0 - 2(N-1)\mathfrak{r}, \quad \xi = -\frac{3\mathcal{F}_N}{2(4N-3)} + \frac{N}{4N-3} H_0^+ \mathfrak{z}. \quad (\text{S186})$$

We bound the second term by a block argument. Block replacement and equivalence of ensembles are established methods [59, 60]; the estimates below state explicitly what is needed here.

Choose $\ell = \lfloor N^{2/5} \rfloor$ for large N , and partition the whole ring into $k = \lfloor N/\ell \rfloor$ contiguous blocks with lengths in $[\ell, 2\ell]$. Average over all N translations of the partition. Each bond is internal in fraction $1 - k/N$. With $\alpha = N/(N-k)$ and $c' = 3/\alpha$, the reconstruction is exact:

$$\mathcal{R} + 3\mathcal{S} = \alpha \text{Av}_{\text{translations}} \sum_{B'} (\mathcal{R}_{\text{open}, B'} + c' \mathcal{S}_{B'}). \quad (\text{S187})$$

No boundary term has been omitted. Here $\alpha \leq 2$ and $|c' - 3| = O(\ell^{-1})$. For block length L and charge q , the summand's canonical mean is

$$h_L(q) = (1 + c')Le_2(L, q) - q^2/L. \quad (\text{S188})$$

Indeed averaging the bond table gives $\mathbb{E}\mathcal{R}_{\text{open}} = Le_2 - q^2/L$, and $\mathbb{E}\mathcal{S}_{B'} = Le_2$.

For one partition decompose its centered sum as $f + g$: f subtracts every block's conditional mean, and g is the centered sum of $h_L(Q_L)$. Given block charges, the canonical measure factors. Every complete-pair exchange or conversion changes a block summand by at most 14: four affected bonds of range at most two and a change of $\mathcal{S}$ of at most two. Its complete comparison form is at most $147L$, so Eq. (S160) bounds the conditional variance by $294L$. Hence $\|f\|_\mu^2 \leq 294N$. The block open Hamiltonian's entire kernel consists of block-charge functions, to which f is orthogonal. The open-path comparison in Eq. (S164) gives

$$\langle f, H_0^+ f \rangle_\mu \leq CN\ell^2. \quad (\text{S189})$$

The block charges are correlated by the fixed total charge. Rather than bound their variance before conditioning and pay a factor $\sqrt{N}$, we apply a complete-pair Poincaré inequality directly in the canonical ensemble. Appendix P proves $\text{Var}_\mu g \leq CN/\ell^3$. The global gap then gives

$$\text{Var}_\mu g \leq CN/\ell^3, \quad \langle g, H_0^+ g \rangle_\mu \leq CN^3/\ell^3. \quad (\text{S190})$$

Cauchy-Schwarz for the inverse form and convexity under translation averaging in Eq. (S187) imply

$$\langle \mathfrak{z}, H_0^+ \mathfrak{z} \rangle_\mu \leq C(N\ell^2 + N^3/\ell^3) = O(N^{9/5}), \quad \|H_0^+ \mathfrak{z}\|_\mu^2 = O(N^{19/5}). \quad (\text{S191})$$

The second bound uses the spectral gap of this same H_0 ; it does not compare squared inverses of different operators.

Finally $\|G_k\|_\mu \leq CN$ uniformly in nonzero k . For an elementary proof take $X = \text{Re } F_k$ or $\text{Im } F_k$. Exchangeability gives $\mathbb{E}X^2 = O(N)$, and every complete-pair move changes X by at most four. Its form on X^2 is at most $CN(\mathbb{E}X^2 + 1) = O(N^2)$. Equation (S160) gives $\text{Var}(X^2) = O(N^2)$, hence $\mathbb{E}X^4 = O(N^2)$. Since $\sum_{k \neq 0} c_k^{-1} = (N^2 - 1)/6$, $\|\mathcal{F}_N\|_\mu \leq CN^3$. Equation (S186) gives $\|\xi\|_\mu \leq CN^2$, and the projection in Eq. (S174) proves $\eta_{N,0,l} \leq C$. Together with Eq. (S169), this proves the uniform upper and positive lower bounds. The improved remainder in Eq. (S191) also vanishes relative to the leading response scale. Its exact coefficient is evaluated in Appendix P.

The result concerns an actual fixed-magnetization ground branch. It separates within-charge entanglement from changes in charge probabilities and bounds it using a regional observable response. The exact weak-flux coefficient does not transfer the original three-channel fixed-flux entropy law to this Hamiltonian.

Appendix N: A size-independent small-flux entanglement bound

We now keep a small nonzero *total* flux fixed while increasing the ring size. The Hamiltonian is precisely $H_g(\Phi)$ in Eq. (S166), not the original three-channel chain or an arbitrary choice of four rates. Write $H_0 = H_g(0)$, use the canonical uniform measure $\mu_{N,M}$, and retain $\mathcal{R}, \mathfrak{r}, \xi$ from Appendix M. Norms of functions use this measure; multiplication by the normalized uniform vector $\Omega = \Omega_{N,M}$ identifies functions with state vectors.

There are constants $\Phi_0, C > 0$, independent of $N \geq 2$ and M , such that the actual lowest state in each sector is simple for $|\Phi| \leq \Phi_0$ and, with $\langle \Omega, \psi_{N,M} \rangle > 0$,

$$\left\| \psi_{N,M}(\Phi) - \Omega + \frac{\Phi^2}{2N^2} \xi \Omega \right\| \leq C|\Phi|^3. \quad (\text{S192})$$

Singleton sectors are understood separately. For each $a_0 \in (0, 1/2)$ there are $N_0, c, \Phi_1 > 0$, independent of N and l , such that

$$\mathcal{E}_{\text{ch}}(\Phi) \geq c\Phi^4 \log(1/|\Phi|), \quad N \geq N_0, \quad a_0 \leq l/N \leq 1 - a_0, \quad 0 < |\Phi| \leq \Phi_1, \quad M = 0. \quad (\text{S193})$$

The constants are not optimized. In particular, this gives a positive large-size lower bound at each sufficiently small fixed nonzero total flux, without asserting an exact limiting entropy.

1. Current controlled by the zero-flux energy

In the uniform gauge, $\theta = \Phi/N$,

$$H_g(\Phi) = \cos \theta H_0 + \sin \theta \mathcal{J} + (1 - \cos \theta) \mathcal{R}. \quad (\text{S194})$$

We first prove the size-dependent form bound

$$|\langle f, \mathcal{J}f \rangle_\mu| \leq 204N \langle f, H_0 f \rangle_\mu. \quad (\text{S195})$$

Its factor N is canceled by the bond phase Φ/N .

For distinct sites i, j , let L_{ij} be the positive right-hop generator: its off-diagonal entry is minus the rate for $(x_i, x_j) \mapsto (x_i - 1, x_j + 1)$, with rates $1/2$ on $(1, 0), (0, -1)$ and one on $(1, -1), (0, 0)$. Its diagonal is the outgoing rate, so $L_{ij}1 = 0$. Outgoing minus incoming rate is $d_{ij}(x) = (x_i - x_j)/2$. The physical pair operators are

$$h_{ij} = L_{ij} + L_{ij}^\dagger - d_{ij}, \quad j_{ij} = i(L_{ij} - L_{ij}^\dagger), \quad H_0 = \sum_i h_{i,i+1}, \quad \mathcal{J} = \sum_i j_{i,i+1}. \quad (\text{S196})$$

Here $h_{ij} = h_{ji} \geq 0$, $j_{ij} = -j_{ji}$ and $j_{ij}1 = -id_{ij}$. The diagonal subtraction in h_{ij} is necessary locally, although its ring sum telescopes.

At pair charge ± 1 , $h_{ij} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. At charge zero, in order $(ud, 00, du)$,

$$h_{ij} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}. \quad (\text{S197})$$

Thus its positive gap is one and its kernel projection P_{ij} averages within pair charges. The site swap U_{ij} is the identity on this kernel, and $\|j_{ij}\| \leq \sqrt{2}$.

For three distinct sites put $T_{ijk} = j_{ij} + j_{jk} + j_{ki}$. The divergence identity makes T_{ijk} annihilate every uniform total-triple-charge vector. Moreover $U_{ij}T_{ijk}U_{ij} = -T_{ijk}$, so $P_{ij}T_{ijk}P_{ij} = 0$. Let e_{ij} be the rate-one swap and two-conversion generator of Appendix L. Pairwise $e_{ij} \leq 3h_{ij}$: at charge zero use $|a - c|^2 \leq 2|a - b|^2 + 2|b - c|^2$ for the missing exchange; the other pair blocks are immediate. Equation (S160) at three sites therefore gives

$$h_{ij} + h_{jk} + h_{ki} \geq \frac{1}{2}(I - P_{\text{tot}}), \quad (\text{S198})$$

where P_{tot} averages within total triple charge. For $g = f - P_{\text{tot}}f$, $Q = I - P_{ij}$, use $\|T_{ijk}\| \leq 3\sqrt{2}$, the pair gap and Eq. (S198) to obtain

$$\begin{aligned} |\langle f, T_{ijk}f \rangle| &\leq 2\|T_{ijk}\| \|Qg\| \|g\| \\ &\leq 12\sqrt{\langle f, h_{ij}f \rangle \langle f, (h_{ij} + h_{jk} + h_{ki})f \rangle}. \end{aligned} \quad (\text{S199})$$

Condition on the exterior word and triple charge, then sum with Cauchy–Schwarz. The same inequality holds in the full canonical measure.

Set $j_{ii} = h_{ii} = 0$. Antisymmetry gives the exact identity $\mathcal{J} = N^{-1} \sum_{i,k} T_{i,i+1,k}$; the two long-edge sums cancel. For $E = \langle f, H_0f \rangle$ and $B = \langle f, H_{\text{comp}}f \rangle$, with $H_{\text{comp}} = N^{-1} \sum_{i < j} h_{ij}$, Eq. (S199) yields

$$|\langle f, \mathcal{J}f \rangle| \leq 12\sqrt{E(E + 4B)}. \quad (\text{S200})$$

The short edge occurs at most N times; each long-edge sum is $2 \sum_{i < j} h_{ij}$. This accounts for every multiplicity. Pairwise $h_{ij} \leq e_{ij}$, and Eq. (S164) gives

$$H_{\text{comp}} \leq \mathcal{Q}_{\text{comp}} \leq 8N^2 \mathcal{Q}_{\text{loc}} \leq 64N^2 H_{\text{unit}} \leq 64N^2 H_0,$$

where H_{unit} has four strengths all equal to one; the penultimate inequality is Eq. (S159) at $\epsilon = 1$. Then $12\sqrt{1 + 256N^2} \leq 204N$ proves Eq. (S195). For $N = 2$ its two directed currents cancel exactly.

2. A centered-activity form bound, including endpoint charges

We prove a slightly more general estimate. Let $\mathbf{H}_s$ be the zero-flux homogeneous four-channel Hamiltonian with fixed strengths $s_U, s_D, s_F, s_G > 0$. For a real two-site function F with $K = \|F\|_\infty$, define $V_N = \sum_i F(x_i, x_{i+1})$. Then

$$|\mathbb{E}_\mu[(V_N - \mathbb{E}_\mu V_N)|f|^2]| \leq C_F [N^2 \langle f, \mathbf{H}_s f \rangle_\mu + \|f\|_\mu^2], \quad (\text{S201})$$

uniformly in N, M . The constant depends on K and the smallest strength, not the charge. We will use $F = r$ from Eq. (S171). Only this form estimate is generalized; the physical entropy conclusion uses the gradient rates in Eq. (S166).

First consider n sites of fixed charge q and let $\mathcal{S} = \sum_i x_i^2$, $u_n(q) = \mathbb{E}\mathcal{S}/n$. For $q \geq 0$ the count variable $k = n_d$ has weights $n!/[k!(k+q)!z!]$, $z = n - 2k - q$. The count chain Eqs. (S162)–(S163) has gap at least one. Since $\mathcal{S} = q + 2k$, detailed balance gives

$$\text{Var}(\mathcal{S}) \leq 4\mathbb{E}\lambda_k = 4\mathbb{E}\mu_k \leq n. \quad (\text{S202})$$

The last step uses $k(k+q) \leq n^2/4$. Charge conjugation and singleton count laws cover negative charges and endpoints.

Two labeled sites have equal probabilities of 00 and ud . Hence $\mathbb{E}[(n - \mathcal{S})(n - \mathcal{S} - 1)] = (\mathbb{E}\mathcal{S}^2 - q^2)/4$. For $\rho = q/n$ this gives

$$P(u_n) = \frac{1 - u_n}{n} - \frac{3 \text{Var}(\mathcal{S})}{4n^2}, \quad P(u) = \frac{3}{4}u^2 - 2u + 1 + \frac{1}{4}\rho^2. \quad (\text{S203})$$

The root in $[0, 1]$ is $u(\rho) = (4 - \sqrt{4 - 3\rho^2})/3$, and $|P'| \geq 1/2$ there. Thus $|u_n(q) - u(\rho)| \leq 7/(2n)$, even near saturation.

Let $p(\rho) = ((u + \rho)/2, 1 - u, (u - \rho)/2)$ in order $(1, 0, -1)$, and $g(\rho) = \sum_{a,b} p_a(\rho)p_b(\rho)F(a, b)$. The empirical proportions have total absolute coefficient two in $\mathcal{S}/n$. Sampling without replacement contributes at most $2K/n$ to

the two-site error; their covariances contribute at most $4K/n$ by Eq. (S202). The mean proportion vector differs from $p(\rho)$ in ℓ^1 by at most $7/n$, contributing at most $14K/n$ to its product law. Consequently, for distinct labeled sites,

$$|\mathbb{E}F(x_1, x_2) - g(q/n)| \leq 20K/n, \quad |h_n(q) - ng(q/n)| \leq 21K, \quad (\text{S204})$$

where $h_n(q)$ is the mean open-chain sum of F , and $h_1 = 0$. Also $u' = \rho/\sqrt{4-3\rho^2}$ and $u'' = 4/(4-3\rho^2)^{3/2}$ imply $|g''| \leq 24K$ on $[-1, 1]$. These estimates require neither an interior-density restriction nor a global conditioning denominator.

Split a parent of length $n = a + b$, $a \leq b$, into balanced children. Conditional on its entire histogram, the left charge is sampling a bounded spins without replacement. Its mean is aQ/n for every histogram of total charge Q . The Doob increments are bounded by two, so

$$\mathbb{P}(|\delta| \geq t) \leq 2e^{-t^2/(8a)}, \quad \delta = q - aQ/n. \quad (\text{S205})$$

Averaging over histograms preserves this bound in the canonical ensemble.

Write $p^{(n)}(q)$ for the *exact* canonical single-site law and $b_{a,b}(q, r) = \sum_{s,t} p_s^{(a)}(q) p_t^{(b)}(r) F(s, t)$. For each split set

$$A = h_a(q) + h_b(Q - q) + b_{a,b}(q, Q - q) - h_n(Q). \quad (\text{S206})$$

It is centered at fixed parent charge. Taylor expansion of the smooth g , with its linear term canceled by the weighted child charges, gives $|A| \leq 64K + 24K\delta^2/a$. At the whole-ring root include the other child boundary and subtract its exact canonical expectation; this changes $64K$ to $66K$. Same-block endpoints must not be replaced by independent spins. For a boundary between distinct parents, let B be its refined child-boundary mean minus its parent-boundary mean. Given both parent charges, their interiors are independent canonical words, so B is centered and $|B| \leq 2K$. Equation (S205) implies $\mathbb{E}e^{|A|/(384K)} \leq 3e^{66/384} < 4$ at the root as well. Centering and the second-order exponential remainder give, for all these $X = A, B$,

$$\log \mathbb{E}e^{\pm tX} \leq C_0 K^2 t^2, \quad 0 < t \leq t_* = (768K)^{-1}. \quad (\text{S207})$$

When $K = 0$ Eq. (S201) is immediate.

The needed open-block logarithmic Sobolev inequality is established for bounded-occupation canonical exclusion processes [61]. Take occupations $x_i + 1 \in \{0, 1, 2\}$ with uniform positive one-site weights, conditioned on particle number $n + q$. In the convention summing both directed hop squares and both nearest-neighbor orientations, its Dirichlet form is at most $8/\min s_X$ times that of $\mathbf{H}_s$. Every allowed hop is one of U, D, F, G ; comparing each edge square proves this normalization directly. Therefore

$$\text{Ent}_{\mu_{n,q}} |f|^2 \leq C_{\text{LS}} n^2 \langle f, \mathbf{H}_{s, \text{open}} f \rangle, \quad \text{Ent}_{\mu} Z = \mathbb{E}_{\mu} [Z \log(Z/\mathbb{E}_{\mu} Z)]. \quad (\text{S208})$$

It is uniform in the charge, with singleton sectors trivial. For complex f use the contraction to $|f|$. Tensorization gives the same maximum-length-squared bound for two separately charge-conditioned blocks.

Recursively split the ring into contiguous balanced intervals, retaining singleton leaves. Let $\mathcal{P}_k$ be the depth- k partition and $M_k = \mathbb{E}[V_N \mid \text{charges of } \mathcal{P}_k]$. At depth $D = \lceil \log_2 N \rceil$ every site is resolved, so $V_N - \mathbb{E}V_N = \sum_{k=0}^{D-1} (M_{k+1} - M_k)$ exactly. For $k \geq 1$, M_k is the sum of open-block means h and all between-block means b . Thus each difference is the sum of its parent A terms and boundary B terms. At $k = 0$ there is just the centered root term

$$h_a(q) + h_b(M - q) + b_{a,b}(q, M - q) + b_{b,a}(M - q, q) - \mathbb{E}V_N.$$

Both physical boundaries remain when only two parents are present. No disjoint-block or power-of-two approximation is made.

Parent lengths at level k are $\ell_k = \lceil N/2^k \rceil$ or $\ell_k - 1$, and there are at most $4N/\ell_k$ terms. Conditional on the exterior and parent charges, the entropy inequality, Eqs. (S207) and (S208), imply

$$|\mathbb{E}[X|f|^2]| \leq \frac{C\ell_k^2}{t} \mathcal{E}_{\text{parents}}(f) + Ct \|f\|_{\text{conditional}}^2. \quad (\text{S209})$$

Within a level each parent energy appears once in A and twice in B ; positivity of discarded bonds bounds their total by three global energies. Choose $t = t_*(\ell_k/N)^{3/2}$. Summing levels gives

$$CN^{3/2} \sum_k \sqrt{\ell_k} \langle f, \mathbf{H}_s f \rangle + C \sum_k \sqrt{\ell_k/N} \|f\|^2 \leq C[N^2 \langle f, \mathbf{H}_s f \rangle + \|f\|^2],$$

because $\sum_k \sqrt{\ell_k/N}$ is bounded independently of N . This proves Eq. (S201).

3. The actual lowest sector state

Apply Eq. (S201) with $F = r$ and call its constant C_A . The zero-flux gap is $g_N \geq g_0/N^2$, $g_0 = 1/128$. The self-adjoint sandwich $(I + N^2 H_0)^{-1/2} \mathbf{r} (I + N^2 H_0)^{-1/2}$ has norm at most C_A . Apply it to the constant function, and compare inverse eigenvalues on the orthogonal complement, to obtain

$$A_{\mathcal{R}} = \langle \mathbf{r}, H_0^+ \mathbf{r} \rangle \leq C_R N^2, \quad \|\xi\| \leq \sqrt{C_R/g_0} N^2, \quad C_R = (1 + g_0^{-1}) C_A^2. \quad (\text{S210})$$

Indeed $\langle \mathbf{r}, (I + N^2 H_0)^{-1} \mathbf{r} \rangle \leq C_A^2$ and $\lambda^{-1} \leq N^2(1 + g_0^{-1})/(1 + N^2 \lambda)$ on this complement.

Let $\alpha = 1 - \cos(\Phi/N)$, $\bar{\mathcal{R}} = \mathbb{E}_\mu \mathcal{R}$, $\Pi = I - |\Omega\rangle\langle\Omega|$, and let E be the actual lowest sector energy. Positivity and the uniform trial vector give $0 \leq E \leq \alpha \bar{\mathcal{R}}$. Equations (S195) and (S201) imply

$$B_E := \Pi(H_g(\Phi) - E)\Pi \geq [\cos \theta - 204N|\sin \theta| - C_A \alpha N^2] H_0 - C_A \alpha I.$$

On $\text{ran } \Pi$ this is at least $H_0/2$ whenever

$$204|\Phi| + [1 + C_A(1 + g_0^{-1})]\Phi^2/2 \leq 1/2. \quad (\text{S211})$$

No lowest vector can be orthogonal to Ω . Two independent lowest vectors would have such a linear combination, so the actual level is simple.

Write $\psi = a(\Omega + w)$, $w \perp \Omega$, $a > 0$. Since $H_g(\Phi)\Omega = \alpha \mathcal{R}\Omega$, the exact Schur equations are

$$w = -\alpha B_E^{-1} \mathbf{r} \Omega, \quad \delta E := \alpha \bar{\mathcal{R}} - E = \alpha^2 \langle \mathbf{r} \Omega, B_E^{-1} \mathbf{r} \Omega \rangle \leq 2\alpha^2 A_{\mathcal{R}} \leq C\Phi^4/N^2. \quad (\text{S212})$$

The common shift $\alpha \bar{\mathcal{R}} = O(\Phi^2/N)$ is retained and subtracted, rather than treated as a perturbation against an N^{-2} gap. Also $\langle \psi, H_0 \psi \rangle \leq 4\alpha^2 A_{\mathcal{R}}$.

On $\text{ran } \Pi$ set $\mathcal{T} = H_0^{-1/2} \Pi(H_g(\Phi) - H_0 - E) \Pi H_0^{-1/2}$. The same form bounds and Eq. (S212) give

$$\|\mathcal{T}\| \leq 204|\Phi| + [1 + C_A(1 + g_0^{-1})]\Phi^2/2 + C_R \Phi^4/(2g_0). \quad (\text{S213})$$

Choose a smaller size-independent Φ_0 making this at most $1/2$. The resolvent identity then gives

$$\|w + \alpha \xi \Omega\| \leq \alpha g_N^{-1/2} \frac{\|\mathcal{T}\|}{1 - \|\mathcal{T}\|} \sqrt{A_{\mathcal{R}}} \leq C|\Phi|^3.$$

Moreover $\|w\| \leq C\Phi^2$. Normalization $a = (1 + \|w\|^2)^{-1/2}$ and $|\alpha - \Phi^2/(2N^2)| \leq \Phi^4/(24N^4)$ prove Eq. (S192).

4. Bounded dipoles and entanglement inside charge blocks

Use the centered regional ramps of Eq. (S182), and set $A_N = \mathcal{D}_A/N^{3/2}$, $B_N = \mathcal{D}_B/N^{3/2}$. At neutrality, Eqs. (S181) and (S185) give

$$\left\langle \frac{\xi}{N^2}, A_N B_N \right\rangle_\mu = -\frac{d_N L_A L_B}{N^7} \longrightarrow \frac{a^3(1-a)^3}{216} > 0, \quad l/N \rightarrow a. \quad (\text{S214})$$

The operator norm of these untruncated dipoles grows with N . We control their tails before using the uniform state remainder.

Every fixed even moment of A_N and B_N is uniformly bounded. To see this directly, write $A = \sum_i a_i x_i$ with $\sum_i a_i = 0$, $\max |a_i| \leq C/\sqrt{N}$, $\sum_i a_i^2 \leq C$. A move of the complete-graph process in Eq. (S160) changes A by at most $C/\sqrt{N}$; its total outgoing rate is at most CN . Thus the pointwise squared-increment rate of A is bounded. For each fixed integer $p \geq 1$, $|(A + \delta)^p - A^p| \leq C_p \delta^2(1 + |A|^{2p-2})$. Poincare then implies

$$\mathbb{E} A^{2p} \leq (\mathbb{E} A^p)^2 + C_p \mathbb{E}(1 + |A|^{2p-2}).$$

Since $\mathbb{E} A = 0$, induction and Lyapunov's moment inequality bound all fixed even moments, without assuming independent spins after conditioning.

Let $A_K = \text{clip}(A_N, -K, K)$ and similarly B_K . Clipping preserves signs and reduces absolute values, so

$$\|A_N B_N - A_K B_K\|_2^2 \leq K^{-2} \left[\sqrt{\mathbb{E} A_N^8 \mathbb{E} B_N^4} + \sqrt{\mathbb{E} A_N^4 \mathbb{E} B_N^8} \right] \leq C/K^2.$$

Equation (S210) therefore lets one choose a fixed large K so that $O_A = A_K/K$, $O_B = B_K/K$ have norm at most one and $\langle \xi/N^2, O_A O_B \rangle \geq c_* > 0$, uniformly on the stated cuts for all sufficiently large N . The zero-flux expectation of $O_A O_B$ vanishes. Equation (S192) gives

$$\bar{c}(\Phi) := \langle O_A O_B \rangle_\Phi = -\Phi^2 \langle \xi/N^2, O_A O_B \rangle_\mu + O(|\Phi|^3), \quad |\bar{c}(\Phi)| \geq c' \Phi^2 \quad (\text{S215})$$

in a smaller size-independent window.

The map $j \mapsto (l-1-j) \bmod N$ reverses both regions separately and preserves their magnetizations. Its permutation unitary maps $H_g(\Phi)$ to $H_g(-\Phi)$, as does complex conjugation. Their antiunitary product, together with simplicity and the positive overlap with real Ω , makes the ground probabilities reflection invariant. The odd clipped dipoles consequently have zero conditional mean in *every* retained regional charge outcome. Thus $\bar{c}$ is the charge-weighted mean of connected conditional correlations.

For completeness, a dimension-independent pure-state estimate converts these correlations into entropy. For Hermitian norm-one local observables let c_q be their connected correlation in a conditional pure state, and let $1-w$ be its largest Schmidt probability. Covariance Cauchy–Schwarz gives $|c_q| \leq 1$. The trace-norm distance to the top-Schmidt product state is $2\sqrt{w}$; the joint term and two marginal terms give $|c_q| \leq 6\sqrt{w}$. If $w < 1/2$, the entropy is at least $h_2(w) \geq h_2(c_q^2/36)$; if $w \geq 1/2$, it is at least $-\log(1-w) \geq \log 2$, also sufficient. Hence

$$S(\rho_{A,q}) \geq \frac{c_q^2}{18} \log \frac{6}{|c_q|}. \quad (\text{S216})$$

This is a pure-state geometric bound, not a mixed-state witness. The continuous function $f(t) = t^2 \log(6/t)$ is increasing and convex on $[0, 1]$, since $f''(t) = 2 \log(6/t) - 3 > 0$. Jensen, the triangle inequality and the exact conditional-mean symmetry give

$$\mathcal{E}_{\text{ch}}(\Phi) \geq \frac{f(\sum_q p_q |c_q|)}{18} \geq \frac{f(|\bar{c}(\Phi)|)}{18}. \quad (\text{S217})$$

Combining with Eq. (S215) proves Eq. (S193). The bound concerns the actual lowest neutral state at the specified rates. Its proof uses neither a trial-state entropy substitution nor a continuity estimate growing with Hilbert-space dimension. Fixed-flux entropy for general reverse-conversion rates remains a separate problem.

Appendix O: Unequal exchange strengths: entanglement and the sector gap

The fixed-flux effect does not require equal exchange strengths for up and down spins. Keep the four normalized vectors of Appendix M, with $\theta = \Phi/N$, and define

$$H_t(\Phi) = \sum_i [(1+t)|U_i\rangle\langle U_i| + (1-t)|D_i\rangle\langle D_i| + 2|F_i\rangle\langle F_i| + 2|G_i\rangle\langle G_i|], \quad |t| < 1. \quad (\text{S218})$$

Here t is an exchange imbalance, not a radial bond weight. The zero-flux canonical measure $\mu_{N,M}$ and uniform state $\Omega_{N,M}$ are independent of t . This is a known gradient rate family [29]. Its stochastic gradient condition is distinct from the product-rate class considered in Ref. [62]: in capacity-two rate notation, our rates are $p_{21} = (1+t)/2$, $p_{10} = (1-t)/2$, and $p_{11} = p_{20} = 1$. The product-rate form requires $p_{10}p_{21} = p_{11}p_{20}$, whereas here these products are $(1-t^2)/4$ and one. We do not use that scaling-limit theorem.

For every $a_0 \in (0, 1/2)$ there are $t_*, \Phi_*, c > 0$ and N_0 , independent of N and the cut length l , such that

$$\mathcal{E}_{\text{ch}}(t, \Phi) \geq c\Phi^4 \log(1/|\Phi|), \quad \begin{aligned} M &= 0, & N &\geq N_0, & a_0 &\leq l/N \leq 1 - a_0, \\ |t| &\leq t_*, & 0 &< |\Phi| &\leq \Phi_*. \end{aligned} \quad (\text{S219})$$

These are existence bounds, not an optimized numerical parameter window. The gradient relation between exchange and conversion strengths remains essential; the all-unit-strength model is not a member of this family.

Write $\mathbf{H}_t = H_t(0)$, $H_0 = \mathbf{H}_0$, and let $\mathcal{R}_t$ be its diagonal escape-rate function. All function norms below are in $L^2(\mu_{N,M})$; multiplying a function by Ω produces its physical state vector. Define $a_t = \mathcal{R}_t - \mathbb{E}\mathcal{R}_t$ and $\xi_t = \mathbf{H}_t^+ a_t$, with inverse on the constant-function orthogonal complement. Uniformly for $|t| \leq 1/2$ and all N, M , the actual lowest fixed-sector branch is simple in a common small total-flux window and satisfies

$$\left\| \psi_{N,M,t}(\Phi) - \Omega + \frac{\Phi^2}{2N^2} \xi_t \Omega \right\| \leq C|\Phi|^3. \quad (\text{S220})$$

Singleton sectors are trivial. In the same window every nonsingleton sector has an excitation spacing

$$\frac{1}{512N^2} \leq \gamma_{N,M,t}(\Phi) := E_{1,N,M,t}(\Phi) - E_{0,N,M,t}(\Phi) \leq \frac{C_{\text{gap}}}{N^2}. \quad (\text{S221})$$

This is an internal sector gap, not the full Hamiltonian gap. The entropy bound uses the possibly smaller interval $|t| \leq t_* \leq 1/2$.

1. Uniform control of the actual lowest state

For an ordered pair, the current divergence is $g_t(x_i) - g_t(x_j)$, where

$$g_t(x) = \frac{1}{2}(x + tx^2), \quad x \in \{-1, 0, 1\}. \quad (\text{S222})$$

The ring sum telescopes, so $\mathcal{J}_t \Omega = 0$. Pair kernels and their averaging projections are independent of t . Put $m = 1 - |t|$ and $m_+ = 1 + |t|$. Pair energies lie between m and m_+ times their $t = 0$ values; their nonzero gap is at least m , and the pair-current norm is at most $\sqrt{2}$. Site exchange preserves the pair energy and reverses its current.

For three sites the oriented triangular current again annihilates the uniform total-charge subspace. Its compression to the pair kernel vanishes by exchange antisymmetry. The proof of Eqs. (S199)–(S200), now with triple gap at least $m/2$, gives

$$|\langle f, \mathcal{J}_t f \rangle| \leq \frac{12}{m} \sqrt{\mathcal{E}_t(f)[\mathcal{E}_t(f) + 4B_t(f)]}, \quad B_t(f) \leq 64 \frac{m_+}{m} N^2 \mathcal{E}_t(f), \quad (\text{S223})$$

where $\mathcal{E}_t(f) = \langle f, \mathbf{H}_t f \rangle$ and B_t is the complete-pair form divided by N . For $|t| \leq 1/2$, $m \geq 1/2$ and $m_+/m \leq 3$; thus

$$|\langle f, \mathcal{J}_t f \rangle| \leq 672N \mathcal{E}_t(f). \quad (\text{S224})$$

Indeed $1 + 768N^2 \leq 784N^2$. For $N = 2$ the two ordered currents cancel.

Write $\mathbf{H}_t = H_0 + tK$, with $K = H_U - H_D$, where H_U and H_D are the zero-flux sums of the respective normalized projectors. Then $-H_0 \leq K \leq H_0$. In particular its zero-flux sector gap is at least $g_{\min}/N^2$, $g_{\min} = 1/256$, on $|t| \leq 1/2$. Also $\mathcal{R}_t = \mathcal{R}_0 + tD$, where D is a sum of two-site functions of supremum norm $1/2$. The local function for $\mathcal{R}_t$ has norm at most two. The general activity estimate (S201) therefore has a common constant C_A on this interval. Its sandwich argument gives

$$\langle a_t, \mathbf{H}_t^+ a_t \rangle \leq C_R N^2, \quad C_R = (1 + g_{\min}^{-1}) C_A^2. \quad (\text{S225})$$

The canonical measure is fixed, so these constants are uniform even near fully polarized sectors, as in Appendix N.

Let E_t be the actual lowest sector energy, $\alpha = 1 - \cos \theta$, and Π project off Ω . The exact magnetic decomposition is $H_t(\Phi) = \cos \theta \mathbf{H}_t + \sin \theta \mathcal{J}_t + \alpha \mathcal{R}_t$. Positivity and the uniform trial state give $0 \leq E_t \leq \alpha \mathbb{E}\mathcal{R}_t$. Using the centered activity, the compression satisfies

$$\Pi[H_t(\Phi) - E_t]\Pi \geq [\cos \theta - 672N|\sin \theta| - C_A \alpha N^2] \mathbf{H}_t - C_A \alpha I \geq \frac{1}{2} \mathbf{H}_t \quad (\text{S226})$$

if $672|\Phi| + [1 + C_A(1 + g_{\min}^{-1})]\Phi^2/2 \leq 1/2$. No lowest vector is orthogonal to Ω , proving simplicity. The Schur equations retain the common shift and give

$$0 \leq \delta E_t := \alpha \mathbb{E}\mathcal{R}_t - E_t \leq 2\alpha^2 \langle a_t, \mathbf{H}_t^+ a_t \rangle \leq \frac{C_R \Phi^4}{2N^2}. \quad (\text{S227})$$

The normalized perturbation of the compressed resolvent has norm at most

$$672|\Phi| + [1 + C_A(1 + g_{\min}^{-1})]\Phi^2/2 + C_R \Phi^4/(2g_{\min}). \quad (\text{S228})$$

Choose a common smaller window making this at most $1/2$. The inverse gap contributes $N/\sqrt{g_{\min}}$, while the source inverse-energy norm is at most $\sqrt{C_R}N$. Their product cancels the N^{-2} in α . The resolvent identity then gives the $O(|\Phi|^3)$ remainder, and normalization contributes only $O(\Phi^4)$, proving Eq. (S220). This repeats the explicit actual-state argument of Appendix N with uniform constants, not ordinary finite-dimensional continuity in t .

2. A charge-even response survives the imbalance

In the neutral sector let C be charge conjugation. Then $CH_0C = H_0$, $CKC = -K$, a_0 is even and $d_0 = D - \mathbb{E}D = D$ is odd. On the complement of constants define

$$B = H_0^{-1/2}KH_0^{-1/2}, \quad u = H_0^{-1/2}a_0, \quad v = H_0^{-1/2}d_0. \quad (\text{S229})$$

Here B is self-adjoint, $\|B\| \leq 1$, and charge odd. The common activity bounds give $\|u\| \leq AN$, $\|v\| \leq VN$ with constants independent of N . Since $\xi_t = H_0^{-1/2}(I + tB)^{-1}(u + tv)$, its charge-even projection $\Pi_+ = (I + C)/2$ obeys the exact identity

$$\Pi_+(\xi_t - \xi_0) = t^2 H_0^{-1/2}(I - t^2 B^2)^{-1}(B^2 u - Bv). \quad (\text{S230})$$

Thus, with $g_0 = 1/128$,

$$\frac{\|\Pi_+(\xi_t - \xi_0)\|}{N^2} \leq K_* \frac{t^2}{1 - t^2}, \quad K_* = (A + V)/\sqrt{g_0}. \quad (\text{S231})$$

Only the even part is protected to second order; the full response vector can change at first order.

Take the same norm-one clipped regional dipoles O_A, O_B from Appendix N. Their product $W = O_A O_B$ is charge even and $\langle \xi_0/N^2, W \rangle \geq c_0 > 0$ uniformly on the stated cuts at large N . Equation (S231) gives $\langle \xi_t/N^2, W \rangle \geq c_0 - K_* t^2/(1 - t^2)$. Choose $t_* \leq 1/2$ and $t_*^2 \leq 3c_0/(8K_*)$; this lower bound is at least $c_0/2$. Equation (S220) now yields

$$\langle O_A O_B \rangle_{t, \Phi} = -\Phi^2 \langle \xi_t/N^2, W \rangle + O(|\Phi|^3), \quad |\langle O_A O_B \rangle_{t, \Phi}| \geq c_0 \Phi^2/4 \quad (\text{S232})$$

in a smaller size-independent window.

Reflection preserving both regions reverses each U and D exchange separately and interchanges F and G . Thus it still maps Φ to $-\Phi$ when U and D have different strengths: equality of the two conversion strengths is sufficient. Complex conjugation does the same. Their antiunitary product and branch simplicity set each odd dipole's conditional mean to zero in every regional charge outcome. Applying Eqs. (S216)–(S217) to Eq. (S232) proves Eq. (S219). No growing-dimension continuity bound for entropy is used.

3. The internal excitation spacing stays diffusive

For the lower bound in Eq. (S221), choose the codimension-one space orthogonal to Ω in the max-min principle. Equation (S226) and the zero-flux gap give $1/(512N^2)$.

For the upper bound, let $F(x) = \sum_j e^{2\pi i j/N} x_j$. In every nonsingleton canonical sector its mean is zero and $\mathbb{E}|F|^2 = N^2 \text{Var}(x_1)/(N - 1) > 0$. The reference operator, not the imbalanced one, satisfies $H_0 F = [1 - \cos(2\pi/N)]F$. Since $H_t \leq (1 + |t|)H_0$, every f in $\text{span}\{1, F\}$ has $\mathcal{E}_t(f) \leq 3\pi^2 \|f\|^2/N^2$. The centered activity bound and Eq. (S227) give

$$\begin{aligned} \langle f, [H_t(\Phi) - E_t]f \rangle &\leq [1 + 672|\Phi| + C_A \Phi^2/2] \mathcal{E}_t(f) \\ &\quad + [C_A \Phi^2/2 + C_R \Phi^4/2] \|f\|^2/N^2. \end{aligned} \quad (\text{S233})$$

The two-dimensional min-max principle proves the upper bound with a constant independent of N, M, t . The common $O(\Phi^2/N)$ energy shift has been subtracted before this comparison. Thus finite charge-internal entanglement can coexist with an unchanged diffusive exponent of the same-sector excitation spacing, even after charge conjugation is broken.

Appendix P: Exact neutral weak-flux coefficient

For the homogeneous gradient Hamiltonian (S166), we now evaluate the coefficient in Eq. (S168) at $M = 0$. Let $m = N - l$ and $a_N = l/N \in K$, where K is a fixed compact subset of $(0, 1)$. All constants below can depend on K , but not on N . The weak-flux expansion is taken at each finite N , before this size limit. No interchange of these limits is used.

The physical response is particularly simple after the magnetizations of the two regions have been resolved. With the centered linear ramps defined before Eq. (S182), we prove

$$\left\| Q\xi - \frac{6}{4N-3} \mathcal{D}_A \mathcal{D}_B \right\|_\mu \leq CN^{19/10}. \quad (\text{S234})$$

The leading term has norm of order N^2 , so the relative error vanishes. It is a single product of dipoles within each charge block, not an exact finite-flux Schmidt-rank statement. The norm averages over charge outcomes; it does not assert uniform control on rare outcomes.

1. A uniform canonical counting estimate

The following strengthening of Eq. (S183) will control finite charge steps of a block mean. For each $\rho_0 < 1$ there are smooth even functions c_1, c_2 such that, uniformly for $|q/L| \leq \rho_0$,

$$e_2(L, q) = u(q/L) + \frac{c_1(q/L)}{L} + \frac{c_2(q/L)}{L^2} + O_{\rho_0}(L^{-3}), \quad u(\rho) = \frac{4 - \sqrt{4 - 3\rho^2}}{3}. \quad (\text{S235})$$

Here and below only feasible integer charges enter canonical quantities.

To prove the uniform remainder explicitly, use the tilted law $p_t(x) = e^{tx}/(1 + 2 \cosh t)$ on $x = -1, 0, 1$, with mean ρ . On the compact density interval, $t = t(\rho)$ is smooth and bounded, and the variance v is bounded below by a positive constant $v_{\min}$. Set $\varphi_t(h) = \mathbb{E}_t e^{ih(X-\rho)}$, $\psi_t(h) = \mathbb{E}_t X^2 e^{ih(X-\rho)}$, and $\chi_t = \psi_t/\varphi_t$ near zero. The latter is uniformly analytic. Conditioning removes the tilt, giving the Fourier ratio in Eq. (S184).

For an explicit definition of the coefficients, let b be a formal variable, let Y be a centered Gaussian of variance $1/v$, and let κ_r be the centered one-spin cumulants. Define

$$\begin{aligned} A(b, y) &= \exp \left[\sum_{r=3}^{12} \frac{\kappa_r (iy)^r b^{r-2}}{r!} \right], \\ D_j &= \mathbb{E}_Y [b^{2j}] A(b, Y), \\ N_j &= \mathbb{E}_Y [b^{2j}] \left\{ A(b, Y) \sum_{r=0}^{10} \frac{\chi_t^{(r)}(0) (bY)^r}{r!} \right\}, \quad j = 0, 1, 2. \end{aligned} \quad (\text{S236})$$

Square brackets extract a formal coefficient. Since $D_0 = 1$ and $N_0 = u$, set $c_1 = N_1 - uD_1$ and $c_2 = N_2 - N_1D_1 + u(D_1^2 - D_2)$. These are finite Gaussian moments of smooth functions of ρ .

Choose a fixed small $h_0 > 0$ such that $|\varphi_t(h)| \leq e^{-ch^2}$ for $|h| \leq h_0$ uniformly in t . Put $h = y/\sqrt{L}$. On $|y| \leq \sqrt{B \log L}$ expand the cumulant exponential through degree twelve and the marked factor through degree ten, retaining powers through b^5 with $b = L^{-1/2}$. Uniform analytic Taylor bounds give an integrand remainder bounded by $Cb^6(1 + |y|^J)e^{-v_{\min}y^2/4}$ for a fixed J . Its integral is $O(L^{-3})$ without an extra logarithmic loss. Odd powers integrate to zero on the symmetric interval. For sufficiently large fixed B , extending the Gaussian moments to the full line, and the region $\sqrt{B \log L} < |y| \leq h_0\sqrt{L}$, cost $O(L^{-3})$. On $h_0 \leq |h| \leq \pi$, positivity of the three probabilities and aperiodicity imply $|\varphi_t(h)| \leq 1 - \delta$ uniformly. The far tails are therefore exponentially small. The denominator and numerator integrals, respectively, are

$$\begin{aligned} &\sqrt{\frac{2\pi}{vL}} [1 + D_1/L + D_2/L^2 + O(L^{-3})], \\ &\sqrt{\frac{2\pi}{vL}} [u + N_1/L + N_2/L^2 + O(L^{-3})]. \end{aligned} \quad (\text{S237})$$

The denominator is at least $c/\sqrt{L}$ for large L . Uniform division proves Eq. (S235). Charge inversion sends t, h to $-t, -h$ without changing the square observable, so c_1, c_2 are even. Enlarging the constant covers the remaining finite L . No derivative of a remainder is needed.

2. Direct control of correlated block charges

We first specify the comparison normalization. With μ the uniform fixed-charge measure, let

$$\mathcal{D}_N(f) = \frac{1}{N} \sum_{i < j} \langle f, h_{ij} f \rangle_\mu, \quad h_{ij} = |U_{ij}\rangle\langle U_{ij}| + |D_{ij}\rangle\langle D_{ij}| + 2|F_{ij}\rangle\langle F_{ij}| + 2|G_{ij}\rangle\langle G_{ij}|, \quad (\text{S238})$$

at zero flux. Each pair and outside-word context has hopping conductance $1/2$ and conversion conductance one, followed by the factor $1/N$. The comparison form $\mathcal{Q}_{\text{comp}}$ of Appendix L applies each swap S_{ij} and each conversion F_{ij}, G_{ij} with rate $1/N$. A rate- r involution contributes $(r/2)\mathbb{E}_\mu|f - f \circ T|^2$. The conditional-permutation and count-chain proof there gives $\text{Var}_\mu f \leq 2\mathcal{E}_{\text{swap}}(f) + \mathcal{E}_{\text{react}}(f) \leq 2\mathcal{Q}_{\text{comp}}(f)$. Up/flat and down/flat swaps cost twice their hopping form. Implement an opposite-spin swap by $ud \leftrightarrow 00 \leftrightarrow du$, using $|f_{ud} - f_{du}|^2 \leq 2|f_{ud} - f_{00}|^2 + 2|f_{00} - f_{du}|^2$. Each conversion edge in this context is used once. Thus $\mathcal{E}_{\text{swap}} \leq 2\mathcal{D}_N$ and $\mathcal{E}_{\text{react}} \leq \mathcal{D}_N$, proving the uniform, non-sharp bound

$$\text{Var}_\mu f \leq 6\mathcal{D}_N(f). \quad (\text{S239})$$

Apply this inequality to the correlated block-mean sum g from Appendix M. For $L \in [\ell, 2\ell]$ the block mean is $h_L(q) = (1 + c')Le_2(L, q) - q^2/L$, where $|c' - 3| \leq C/\ell$. In a fixed central charge interval, $4u(\rho) - \rho^2 = 8/3 + O(\rho^4)$. Smoothness and evenness in Eq. (S235) therefore imply, for $|s| \leq 2$ and both charges in a smaller central interval,

$$|h_L(q + s) - h_L(q)| \leq C \left[\frac{(|q| + 2)^3}{L^3} + \frac{|q| + 2}{\ell L} + \frac{|q| + 2}{L^2} + \frac{1}{L^2} \right]. \quad (\text{S240})$$

The last term bounds two separate remainder values; it is not obtained by differentiating an O term. The c_2 difference is smaller.

For $L \leq N/2$, the marginal density of a block relative to independent uniform ternary spins is bounded uniformly:

$$\frac{\mathbb{P}_{\text{iid}}(Q_{N-L} = -q)}{\mathbb{P}_{\text{iid}}(Q_N = 0)} \leq C \sqrt{\frac{N}{N-L}} \leq C. \quad (\text{S241})$$

Fourier inversion bounds the largest trinomial local probability by $C/\sqrt{N-L}$, and its neutral value below by $c/\sqrt{N}$. Consequently $\mathbb{E}_\mu Q_L^6 \leq CL^3$, $\mathbb{E}_\mu Q_L^2 \leq CL$, and noncentral charges have probability at most Ce^{-cL} . There $|h_L| \leq CL$. Squaring Eq. (S240) and averaging thus gives

$$\mathbb{E}_\mu [\mathbf{1}_{\text{feasible}} |h_L(Q_L + s) - h_L(Q_L)|^2] \leq C\ell^{-3}. \quad (\text{S242})$$

Every complete-pair move changes at most two block charges, each by at most two. There are $O(N^2)$ pairs, each of rate $O(1/N)$. Equations (S239) and (S242) give $\text{Var}_\mu g \leq 6\mathcal{D}_N(g) \leq CN/\ell^3$. This proves the improved estimate (S190) without assuming independence after conditioning. The open-block estimate (S189), translated reconstruction (S187), and choice $\ell = \lfloor N^{2/5} \rfloor$ then give Eq. (S191).

3. The surviving collective channel and its coefficient

Use the cycle Green kernel (S180) in $\mathcal{F}_N$. The projection $\mathbf{Q}$ kills same-region terms and constants. For $i \in A, j \in B$, one has $0 < j - i < N$; the cross kernel is quadratic in $j - i$. Separate centering removes its constant and single-region pieces. Each ordered cross sum leaves $-2\mathcal{D}_A\mathcal{D}_B$. Both orders therefore give the identity of functions

$$\mathbf{Q}\mathcal{F}_N = -4\mathcal{D}_A\mathcal{D}_B. \quad (\text{S243})$$

Insert this into Eq. (S186) and use $\|H_0^+ \mathfrak{z}\|_\mu \leq CN^{19/10}$ to prove Eq. (S234). This controls the actual interacting resolvent, not a replacement by independent diffusive modes.

For completeness retain the exact finite-population normalization:

$$\begin{aligned} p_q^0 &= \frac{T(l, q)T(m, -q)}{T(N, 0)}, & \sigma_L(q) &= \frac{L}{L-1} [e_2(L, q) - (q/L)^2], \\ \mathcal{V}_{N,0,l} &= \sum_q p_q^0 \sigma_l(q) \sigma_m(-q), & \|\mathcal{D}_A\mathcal{D}_B\|_\mu^2 &= \mathcal{V}_{N,0,l} L_A L_B. \end{aligned} \quad (\text{S244})$$

The factors $L/(L-1)$ are essential for zero-sum ramp weights. Conditional dipole means vanish, including in polarized blocks. The canonical expansion and central-charge concentration give $\mathcal{V}_{N,0,l} = 4/9 + O_K(N^{-1})$: each regional variance is $2/3 + O(1/L + q^2/L^2)$ in the central interval, the charge second moment is $O(N)$, and the remaining tail is exponential.

Define the coefficient of the explicit Green term by

$$\eta_{N,l}^G = \frac{9\mathcal{V}_{N,0,l}L_AL_B}{(4N-3)^2N^4}. \quad (\text{S245})$$

Its tangent norm divided by N^2 is bounded. Equation (S234) and the norm-square identity (S174) therefore imply $|\eta_{N,0,l} - \eta_{N,l}^G| \leq C_K N^{-1/10}$. Substitute $L_A = l(l^2 - 1)/12$, $L_B = m(m^2 - 1)/12$ to obtain Eq. (S170). If only $a_N \rightarrow a$ is specified, the coefficient converges to $a^3(1-a)^3/576$; its error relative to this fixed a also depends on the rate of $a_N - a$. Writing the entropy law with $\Phi^4 \log(1/|\Phi|)$ instead multiplies the coefficient by four.

Quadratic amplitudes, linear-ramp Schmidt modes and the opening of one leading channel already occur near the XXZ ferromagnetic point [39]. The result here is the explicit spin-one flux-response coefficient and its controlled canonical remainder. It is restricted to the neutral homogeneous gradient point: neither the unequal-hopping extension nor generic four-channel rates inherit this exact coefficient. An exact finite-flux entropy limit is not established.

Appendix Q: Tuning the charge-internal entanglement response

The four-channel comparison admits a parameter at which its actual fixed-flux entanglement response disappears in the thermodynamic limit, although its neutral energy curvature remains nonzero. This section specifies the model, the order of limits and the entropy-tail control. The model belongs to the known capacity-two gradient family [29]; it is not a generic perturbation of the original three-projector Hamiltonian.

1. Model and statements

Keep the normalized hopping vectors of Eq. (S166) and replace the pair terms by the outer products of the unnormalized vectors

$$|f_w\rangle = |ud\rangle - e^{-i\theta}|00\rangle/w, \quad |g_w\rangle = |du\rangle - e^{i\theta}|00\rangle/w, \quad \theta = \Phi/N, \quad w > 0. \quad (\text{S246})$$

Thus $w = 1$ is precisely the gradient comparison already studied, with normalized-projector strengths $(1, 1, 2, 2)$. Write $H_w(\Phi)$ for this Hamiltonian restricted to total magnetization zero, and $H_{w,0} = H_w(0)$. Its zero-flux normalized state is $\Omega_w(x) = \sqrt{\mu_w(x)}$, where

$$\mu_w(x) = \frac{w^{n_0(x)}}{Z_{N,w}(0)}, \quad Z_{L,w}(q) = [z^q](z^{-1} + w + z)^L. \quad (\text{S247})$$

Here w is a probability weight, not the amplitude weight $\sqrt{w}$. Constants below may depend on fixed w , or a fixed compact positive interval of weights. At $w = 2$, direct evaluation gives $h_{ij}(0) = (I - \mathbf{S}_i \cdot \mathbf{S}_j)/2$, the familiar spin-one ferromagnet; this is a benchmark, not a new Hamiltonian.

Let A be a contiguous interval, B its complement, and $a_N = |A|/N$. For a_N in a fixed compact $K \subset (0, 1)$, the finite-size weak-flux coefficient has the explicit limit

$$\eta_{w,N,A} = \left(\frac{w-4}{w+2}\right)^2 \frac{a_N^3(1-a_N)^3}{576} + O_{w,K}(N^{-1/10}). \quad (\text{S248})$$

This statement takes weak flux first, as in Eq. (S7). The coefficient at $w = 4$ is $O_K(N^{-1/5})$, not identically zero at finite N .

More strongly, constants $C, \Phi_0 > 0$ independent of N and the interval length give, for the actual lowest neutral state $\psi_{4,N}(\Phi)$,

$$\begin{aligned} \mathcal{E}_{\text{ch}}(\psi_{4,N}(\Phi)) &\leq C\Phi^4 N^{-1/5} \log \frac{eN}{|\Phi|}, \\ |S_A(\psi_{4,N}(\Phi)) - S_A(\Omega_4)| &\leq C\Phi^2 N^{-1/10} \log \frac{eN}{|\Phi|}, \quad 0 < |\Phi| \leq \Phi_0. \end{aligned} \quad (\text{S249})$$

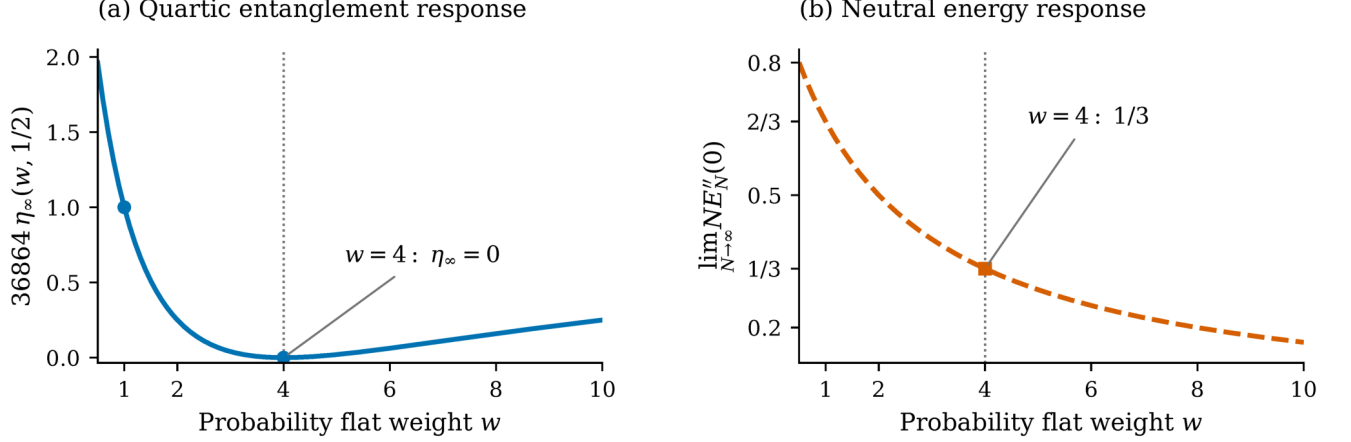


FIG. 4. **A vanishing entanglement coefficient with nonzero neutral energy curvature.** The homogeneous four-channel gradient model of Eq. (S246), with equal contiguous halves. (a) The limiting quartic coefficient is $\eta_\infty(w, 1/2) = [(w-4)/(w+2)]^2/36864$. Weak total flux is taken at each finite N before $N \rightarrow \infty$. (b) The neutral energy response is $\lim_{N \rightarrow \infty} NE_N''(0) = 2/(w+2)$, where E_N is the lowest neutral-sector energy, not the full ground energy. The curves directly evaluate these analytic formulas; symbols mark exact special-weight values, not measured data. Here w weights probabilities, not amplitudes. At $w = 4$ the finite- N coefficient need not vanish. Suppression of the actual fixed-small-flux entropy requires the separate uniform bound in Eq. (S249), not panel (a) alone.

Both responses vanish at every fixed sufficiently small nonzero total flux. The zero-flux baseline entropy need not vanish. For every fixed $w \neq 4$ and macroscopic contiguous cuts, there are instead positive $c_{w,K}, \Phi_{w,K}, N_{w,K}$ such that

$$\mathcal{E}_{\text{ch}}(\psi_{w,N}(\Phi)) \geq c_{w,K} \Phi^4 \log(1/|\Phi|), \quad N \geq N_{w,K}, \quad 0 < |\Phi| \leq \Phi_{w,K}. \quad (\text{S250})$$

This is a conditional-entropy lower bound, not a sign assertion for the offcritical total-entropy increment.

2. Weighted coefficient and its vanishing leading channel

Multiplication by Ω_w identifies $H_{w,0}$ with the positive reversible generator $\mathcal{L}_w$ in $L^2(\mu_w)$. Hopping rates are $1/2$, $ud, du \rightarrow 00$ have rate one, and the reverse rates are w^{-2} . The signed outgoing transfer rate is $(x-y)/2$. Hence the ring current J_w annihilates Ω_w . The quantum diagonal is the actual activity

$$R_w = \sum_i r_w(x_i, x_{i+1}), \quad r_w = \begin{pmatrix} 0 & 1/2 & 1 \\ 1/2 & 2/w^2 & 1/2 \\ 1 & 1/2 & 0 \end{pmatrix}, \quad (\text{S251})$$

in the order $(-1, 0, 1)$. Put $r = R_w - \mathbb{E}R_w$, $s = \sum_i x_i^2 - \mathbb{E}\sum_i x_i^2$, and $\xi = \mathcal{L}_w^+ r$. The projection $\mathbf{Q}_w f = f - \mathbb{E}(f|x_A) - \mathbb{E}(f|x_B) + \mathbb{E}(f|Q_A)$ is orthogonal because the reference law is a product within each regional-charge block. Finite-dimensional Schmidt perturbation therefore gives $\eta_{w,N,A} = \|\mathbf{Q}_w \xi\|_{\mu_w}^2 / (4N^4)$.

For $k = 2\pi j/N \neq 0$, define $F_k = \sum_i e^{iki} x_i$, $c_k = 1 - \cos k$, $G_k = |F_k|^2 - \mathbb{E}|F_k|^2$ and $\mathcal{F} = \sum_{k \neq 0} G_k / c_k$. The linear drift and jump-square identity give $\mathcal{L}_w F_k = c_k F_k$ and $\mathcal{L}_w G_k = 2c_k(G_k - r)$. Parseval then yields

$$\begin{aligned} \mathcal{L}_w \mathcal{F} &= 2Ns - 2(N-1)r, \\ \xi &= -\frac{4-w}{2d_{N,w}} \mathcal{F} + \frac{wN}{d_{N,w}} \mathcal{L}_w^+ z, \quad d_{N,w} = 4(N-1) + w, \quad z = r + (4/w - 1)s. \end{aligned} \quad (\text{S252})$$

The cycle Green kernel is $(N^2 - 1)/6 - d(N - d)$ for d modulo N . Separate centering of its two ordered cross sums gives $\mathbf{Q}_w \mathcal{F} = -4\mathcal{D}_A \mathcal{D}_B$, with the regional ramps of Appendix P. The weighted measure does not alter this pointwise polynomial identity.

Here is the needed bound on the interacting remainder. Yau's canonical logarithmic Sobolev inequality [61], compared termwise to the positive rates $1/2, 1, w^{-2}$, gives every-charge open-block and neutral ring gaps at least c_w/L^2 . Its finite-capacity law has weights $(1, w, 1)$ on occupations $0, 1, 2$; the comparison constant is finite for each fixed $w > 0$. No comparison of globally exponentially different measures is used.

Partition the ring into blocks of length $L \in [\ell, 2\ell]$ and average over all translated partitions. If there are b blocks, let $\alpha = N/(N - b)$, $c_0 = 4/w - 1$ and $c' = c_0/\alpha$. The reconstruction $R_w + c_0 \sum x_i^2 = \alpha \text{Av} \sum_B (R_{w,B}^{\text{open}} + c' \sum_{i \in B} x_i^2)$ is exact. Its block conditional mean is

$$h_L(q) = (1 + c')Le_{2,w}(L, q) - q^2/L, \quad e_{2,w}(L, q) = \frac{Z_{L-1,w}(q-1) + Z_{L-1,w}(q+1)}{Z_{L,w}(q)}. \quad (\text{S253})$$

The tilted one-spin second moment $u_w(\rho)$ solves $(w^2 - 4)u^2 + 8u - 4 - w^2\rho^2 = 0$ at its physical root. In particular $u_w(0) = v = 2/(w + 2)$ and $u_w''(0) = w/2$. On a fixed central density interval the marked Fourier-ratio argument of Appendix P gives, uniformly, $e_{2,w}(L, q) = u_w(q/L) + c_{1,w}(q/L)/L + O_w(L^{-2})$, with smooth even $c_{1,w}$. Indeed the tilted three probabilities remain positive, their variance is bounded below, the centered integrals have Gaussian expansions, and aperiodicity controls the outer tails. The denominator is at least $c_w/\sqrt{L}$ after removing the tilt. No derivative of the remainder is taken. These are established ensemble/block methods [59, 60].

For precision about correlations between blocks, use iid spins with law $(1, w, 1)/(w + 2)$ and denote their sum probabilities by $P_n(q)$. Fourier inversion gives $\sup_q P_n(q) \leq C_w/\sqrt{n}$ and $P_N(0) \geq c_w/\sqrt{N}$. Thus a block of length at most $N/2$ has canonical marginal bounded by a constant times its iid law. Since $(1 + c_0)u_w''(0)/2 = 1$ and $c' - c_0 = O(\ell^{-1})$, the centered block mean $f_L = h_L - \mathbb{E}_{\text{iid}} h_L$ obeys

$$\mathbb{E}_{\text{iid}} f_L^2 \leq C_w \ell^{-2}, \quad \mathbb{E}_{\text{iid}} |f_L| Q_L^2 \leq C_w. \quad (\text{S254})$$

The nonconstant central terms are bounded by $C_w[q^4/L^3 + q^2/(\ell L) + q^2/L^2 + L^{-1}]$; moments through order eight and exponential noncentral tails prove these inequalities.

For two distinct blocks with remaining length $M = N - L - L' \geq N/2$, their joint density relative to the two iid charge laws is $K(q+r) = P_M(-q-r)/P_N(0)$. Extend P_M by its real even Fourier interpolation. Then $\sup_t |K^{(4)}(t)| \leq C_w/N^2$, because the numerator integral is bounded by $C_w M^{-5/2}$. Evenness and zero iid means let us replace the kernel inside $\mathbb{E} f_L(q) f_{L'}(r) K(q+r)$ by $[K(q+r) + K(q-r)]/2 - K(q) - K(r) + K(0)$. Its absolute value is at most $C_w q^2 r^2 / N^2$. Consequently the cross covariance is $O_w(N^{-2})$, and the variance of the sum of block means is at most $C_w(b/\ell^2 + b^2/N^2) \leq C_w N/\ell^3$. This does not assume that canonical block charges are independent.

The centered internal-block part has inverse energy at most $C_w N \ell^2$: condition on all block charges, use the open gaps, and note that the variance of each unconditioned local block sum is $O_w(L)$. The latter follows from bounded overlapping terms and $O_w(N^{-1})$ covariance of disjoint fixed-size local terms. The conditional-mean part has inverse energy at most $C_w N^3/\ell^3$ by the ring gap. Taking $\ell = \lfloor N^{2/5} \rfloor$, inverse-form convexity under translations gives

$$\langle z, \mathcal{L}_w^+ z \rangle \leq C_w N^{9/5}, \quad \|\mathcal{L}_w^+ z\|^2 \leq C_w N^{19/5}. \quad (\text{S255})$$

The second inequality uses the same operator's gap, not monotonicity of squared inverses. Small sizes are absorbed into the constants.

Let $L_A = l(l^2 - 1)/12$, $L_B = m(m^2 - 1)/12$ and $\mathcal{V}_w = \sum_q p_q^0 \sigma_l(q) \sigma_m(-q)$, where $p_q^0 = Z_{l,w}(q) Z_{m,w}(-q) / Z_{N,w}(0)$ and $\sigma_L(q) = L[e_{2,w}(L, q) - (q/L)^2]/(L - 1)$. Exact exchangeability gives $\|\mathcal{D}_A \mathcal{D}_B\|^2 = \mathcal{V}_w L_A L_B$ and canonical concentration gives $\mathcal{V}_w = v^2 + O_{w,K}(N^{-1})$. Equation (S252) and projection contraction imply $\|\mathcal{Q}_w \xi - 2(4 - w) \mathcal{D}_A \mathcal{D}_B / d_{N,w}\| \leq C_w N^{19/10}$. Squaring the normalized tangent proves Eq. (S248). At $w = 4$ the Green term is absent, so Eq. (S255) directly gives $\eta_{4,N,A} \leq C_K N^{-1/5}$.

3. Uniform current and centered-activity forms

We next justify that the flux window need not shrink with size. For a pair, whole-site swap U_{ij} fixes the kernel of h_{ij} , whose positive gaps are $1, 1, 1 + 2/w^2$. Hence $(I - U_{ij})^*(I - U_{ij}) \leq 4h_{ij}$ and $\|h_{ij}\| \leq 1 + 2/w^2$. A string V of neighboring swaps moves an arbitrary pair together. In $V - I = \sum_s U_r \cdots U_{s+1}(U_s - I)$ all remaining factors are left unitaries. Cauchy-Schwarz gives

$$h_{ij} \leq 2h_{i,i+1} + 8(1 + 2/w^2)(j - i - 1) \sum_{b=i+1}^{j-1} h_{b,b+1}, \quad H_{\text{comp}} = N^{-1} \sum_{i < j} h_{ij} \leq C_w N^2 H_{w,0}. \quad (\text{S256})$$

This uses unitary site permutations, not measure-preserving unit-transfer prefixes. For the oriented pair current j_{ij} , the triangle sum $T_{ijk} = j_{ij} + j_{jk} + j_{ki}$ annihilates every fixed-charge triangle ground vector. The positive triangle gap $\delta_3(w) > 0$ follows from its finite connected charge graphs. Swapping i, j reverses T_{ijk} and fixes the pair kernel, so its pair-kernel sandwich is zero. Projecting off the pair and triangle kernels bounds its quadratic form by $C_w \sqrt{E_{ij} E_{jk}}$. The exact identity $J_w = N^{-1} \sum_{i,k} T_{i,i+1,k}$, Cauchy-Schwarz and Eq. (S256) therefore give

$$|\langle f, J_w f \rangle| \leq C_w N \langle f, H_{w,0} f \rangle. \quad (\text{S257})$$

At $N = 2$ the two ordered currents cancel.

The other bound needed is, for any fixed bounded pair observable F ,

$$|\langle f, (V - \mathbb{E}V)f \rangle| \leq C_{w,F} [N^2 \langle f, H_{w,0} f \rangle + \|f\|^2], \quad V = \sum_i F(x_i, x_{i+1}). \quad (\text{S258})$$

We give the endpoint-uniform argument. On n sites of charge $Q \geq 0$, let $k = n_d$, $z = n - Q - 2k$. Histogram weights are proportional to $n!w^z/[k!(k+Q)!z!]$. The birth/death rates $\lambda_k = z(z-1)/(w^2n)$, $\mu_k = k(k+Q)/n$ have derivative killing $[(4z-6)/w^2 + 2k+Q+1]/n \geq \delta_w = \min(1, w^{-2})$ on each edge. The maximum principle for the differentiated eigenvalue equation gives gap at least δ_w , hence $\text{Var} \sum x_i^2 \leq n/\delta_w$. Charge inversion and deterministic saturated histograms cover other endpoints. Weighted pair balance gives

$$(w^2 - 4)u_n^2 + 8u_n - 4 - w^2\rho^2 = -(w^2 - 4) \text{Var}(\sum x_i^2)/n^2 - 4(1 - u_n)/n, \quad \rho = Q/n. \quad (\text{S259})$$

The polynomial derivative is at least $2 \min(w^2, 4)$ on $[0, 1]$. Its smooth physical root has bounded first two density derivatives through $[-1, 1]$. Two-site sampling without replacement thus gives $|\mathbb{E}_{n,Q} F - g_w(Q/n)| \leq C_w \|F\|_\infty / n$ and $|h_n(Q) - ng_w(Q/n)| \leq C_w \|F\|_\infty$, with bounded g''_w .

Split balanced contiguous intervals recursively down to singletons. Conditional expectations of V given all block charges telescope exactly. A parent split contributes $A = h_a(q) + h_b(Q-q) + b_{a,b}(q, Q-q) - h_{a+b}(Q)$, where b is the mean of the boundary pair under the two fixed-charge child laws. It is centered at fixed parent charge. A boundary between distinct parents contributes the refined child-boundary mean minus its parent-boundary mean, denoted B ; it is centered at both parent charges and bounded by $2\|F\|_\infty$. At the root retain both physical ring boundaries and subtract the exact ring mean, not an open-chain mean.

Given any parent histogram, $\delta = q - aQ/(a+b)$ is a bounded-spin sample without replacement and obeys $\Pr(|\delta| > t) \leq 2e^{-t^2/(8a)}$. Mixing histograms preserves this. Linear Taylor terms cancel, so $|A| \leq C_w \|F\|_\infty (1 + \delta^2/a)$. Centering gives $\log \mathbb{E} e^{\pm sX} \leq Cs^2$ for $X = A, B$ and $|s| \leq s_*$, uniformly in every parent charge. Conditional entropy inequality and the all-charge open-block LSI then give $|\mathbb{E} X|f|^2| \leq C\ell^2 \mathcal{E}_{\text{support}}(f)/s + Cs\mathbb{E}|f|^2$ at parent scale ℓ . For B , tensorize the two parent laws with their individual charges fixed. At a level there are $O(N/\ell)$ terms, but each parent energy occurs only once for its split and twice for neighboring boundaries. Set $s = s_*(\ell/N)^{3/2}$. Summing scales costs at most $CN^{3/2} \sum \sqrt{\ell} \mathcal{E}(f) \leq CN^2 \mathcal{E}(f)$ in energy and $C \sum \sqrt{\ell/N} \|f\|^2 \leq C\|f\|^2$ in normalization. Singletons and the root only change constants. This proves Eq. (S258), including saturated charges, without assuming independence after global conditioning. Modulus contraction extends the reversible form argument to complex f .

4. Actual lowest state and a size-independent flux window

Put $\alpha = 1 - \cos \theta$. The exact magnetic decomposition is

$$H_w(\Phi) = \cos \theta H_{w,0} + \sin \theta J_w + \alpha R_w, \quad H_w(\Phi) \Omega_w = \alpha R_w \Omega_w. \quad (\text{S260})$$

Let E be the actual lowest neutral energy, $P = |\Omega_w\rangle\langle\Omega_w|$ and $Q = I - P$. Positivity and the reference trial state give $0 \leq E \leq \alpha \mathbb{E} R_w$. Center the diagonal before applying Eqs. (S257)–(S258) and use $H_{w,0}|Q \geq g_0(w)/N^2$. For sufficiently small $|\Phi| \leq \Phi_0(w)$, independent of N ,

$$B_E = Q[H_w(\Phi) - E]Q \geq \frac{1}{2} H_{w,0}|Q. \quad (\text{S261})$$

For example it suffices to take Φ_0 so that $C_J|\Phi| + [1 + C_R(1 + g_0^{-1})]\Phi^2/2 \leq 1/2$. No lowest vector is orthogonal to Ω_w ; two independent lowest vectors would give such a combination. Thus the actual lowest state is simple.

For $\psi = a(\Omega_w + v)$, $v \perp \Omega_w$, the exact Schur equations give

$$v = -\alpha B_E^{-1} r \Omega_w, \quad 0 \leq \alpha \mathbb{E} R_w - E \leq 2\alpha^2 A_R, \quad \langle \psi, H_{w,0} \psi \rangle \leq 4\alpha^2 A_R, \quad A_R = \langle r, \mathcal{L}_w^+ r \rangle. \quad (\text{S262})$$

Polarizing the centered-activity form with $\Omega_w + tf$ and optimizing t gives $|\langle r\Omega_w, f \rangle| \leq C_w N \sqrt{\langle f, H_{w,0} f \rangle}$; hence $A_R \leq C_w N^2$. The common $O(\Phi^2/N)$ trial energy has been retained and subtracted, not estimated as an uncentered perturbation. At $w = 4$, $z = r$ and Eq. (S255) improves this to

$$\langle \psi_{4,N}, H_{4,0} \psi_{4,N} \rangle \leq C \Phi^4 N^{-11/5}, \quad 1 - |\langle \Omega_4, \psi_{4,N} \rangle|^2 \leq C \Phi^4 N^{-1/5}. \quad (\text{S263})$$

This direct estimate has no size-independent cubic error floor.

For later use, set $G = H_{w,0}|_Q$ and $K_E = G^{-1/2}(B_E - G)G^{-1/2}$. Centering and the same forms give $\|K_E\| \leq C_w(|\Phi| + \Phi^2 + \Phi^4)$: the final term is $(\alpha \mathbb{E} R_w - E)/g_N \leq C_w \Phi^4$. Thus $\|(I + K_E)^{-1} - I\| \leq C_w |\Phi|$ and $\|(B_E^{-1} - G^{-1})r\Omega_w\| \leq C_w |\Phi| N^2$. Normalization and $\alpha = \Phi^2/(2N^2) + O(\Phi^4/N^4)$ yield

$$\|\psi_{w,N} - \Omega_w + \Phi^2 \xi \Omega_w / (2N^2)\| \leq C_w |\Phi|^3. \quad (\text{S264})$$

The gap lower follows from Eq. (S261). A nonzero-momentum reference density vector is orthogonal to the simple momentum-zero ground and supplies an $O_w(N^{-2})$ spacing upper by the current/activity forms. It is a variational vector, not a flux-Hamiltonian eigenvector. Therefore the neutral internal spacing is $\Theta_w(N^{-2})$ in this window. Finally $J_w \Omega_w = 0$ gives exactly $E''(0) = \mathbb{E} R_w / N^2 = e_{2,w}(N, 0)/(N - 1)$, so

$$NE''(0) \longrightarrow 2/(w + 2) > 0. \quad (\text{S265})$$

This is neutral-sector curvature, not global-ground stiffness; polarized zero-energy states remain available in the full Hamiltonian.

5. Full entropy upper, rather than fidelity alone

In block q write $\sigma_q = C_q C_q^*$, $p_q = \text{Tr } \sigma_q$, $t_q = p_q - \lambda_{\max}(\sigma_q)$ and $W = \sum_q t_q$. The reference has one product vector in every block, so $W \leq 1 - |\langle \Omega_w, \psi \rangle|^2$ for any bipartition. Conditioned linear entropy obeys $W \leq L_{\text{ch}} \leq 2W$. Vidal's pure-state conversion formula [63] gives $P_{\text{Bell}} = \sum_q \min(p_q, 2t_q)$, hence $W \leq P_{\text{Bell}} \leq 2W$, for single-copy regional-charge-preserving LOCC, initially product neutral registers, no shared charge/phase reference and no entangled catalyst. Charge dephasing leaves these neutral-target statistics unchanged, and each conditional conversion is achievable in its charge eigenspace. These upper bounds hold for arbitrary cuts; no multicopy rate follows.

The complete VN upper needs a contiguous region. Every charge sector of its physical open Hamiltonian has canonical LSI constant $C_w L^2$ and minimum nonzero canonical probability at least $e^{-b_w L}$, where $b_w = \log 3 + |\log w|$. Hypercontractivity [64], in the convention $\text{Ent } f^2 \leq C_{\text{LS}} \mathcal{E}(f)$, gives $\|P_t\|_{2 \rightarrow p} \leq 1$ when $p - 1 \leq e^{4t/C_{\text{LS}}}$. Choose $p = 2 + b_w L$, $t = (C_w L^2/4) \log(1 + b_w L)$. Weighted norm comparison then gives $\|P_t\|_{2 \rightarrow \infty} \leq e$. Reversibility implies $P_{2t}(x, x)/\mu(x) \leq e^2$; multiplying by $\mu(x)$ and summing yields the spectral trace bound. Consequently

$$\text{Tr } e^{-T_N H_{A,q}} \leq e^2, \quad T_N = C_w N^2 \log(1 + b_w N), \quad (\text{S266})$$

uniformly in all interval lengths and charges. This uses logarithmic time overhead; it is not a constant- βN^2 heat-trace theorem.

Remove the largest eigenvalue of σ_q , leaving positive $\tau_q \leq \sigma_q$ with trace t_q . Binary entropy concavity and the Gibbs relative-entropy inequality for τ_q/t_q give

$$\mathcal{E}_{\text{ch}} \leq h_2(W) + T_N \sum_q \text{Tr}(H_{A,q} \tau_q) + \sum_q t_q \log \text{Tr } e^{-T_N H_{A,q}} \leq h_2(W) + T_N \langle \psi, H_{w,0} \psi \rangle + 2W. \quad (\text{S267})$$

No commutation between τ_q and $H_{A,q}$ is required. The last energy inequality only discards positive physical bonds, hence requires contiguous A . A disconnected region can have many conserved component charges and a larger zero space; reconnecting it would not justify this energy step. Equations (S263) and (S267) prove the first line of Eq. (S249). Small tail weight alone would not do so.

The total entropy is $H(p) + \mathcal{E}_{\text{ch}}$, while $S_A(\Omega_4) = H(p^0)$. Measurement contraction gives $\text{TV}(p, p^0) \leq \sqrt{1 - |\langle \Omega_4, \psi \rangle|^2}$. There are at most $N + 1$ regional charges. Classical entropy continuity [49] gives $|H(p) - H(p^0)| \leq h_2(\text{TV}) + \text{TV} \log N$, which proves the second line. No exponential-dimension quantum continuity bound is used.

6. Nonzero response away from the cancellation point

For fixed $w \neq 4$ and $a_N \in K$, the tangent projection and finite-population variance above give $\langle \xi/N^2, \mathcal{D}_A \mathcal{D}_B/N^3 \rangle \rightarrow (4-w)a^3(1-a)^3/[72(w+2)^2]$ along $a_N \rightarrow a$. Zero-sum normalized ramps have uniformly bounded moments of every fixed order: condition on a histogram and expose its uniformly random spatial permutation. Coefficients have maximum $C/\sqrt{N}$, giving bounded martingale variance proxy. Mixing histograms preserves these bounds. Large fixed clipping of the two ramps therefore retains a nonzero tangent pairing with bounded odd observables O_A, O_B . Equation (S264) implies $|\langle \psi, O_A O_B \psi \rangle| \geq c_{w,K} \Phi^2$ in a size-independent small-flux window. Simultaneously reversing both intervals is a full-ring reflection preserving each regional charge and sending Φ to $-\Phi$. Together with complex conjugation and simplicity, this makes each odd conditional mean zero. The displayed expectation is thus an average conditional connected correlation.

To obtain the entropy logarithm, not just $W \geq c\Phi^4$, let $d_q = \|\psi_q - \Omega_{w,q}\|^2$, so $\sum_q d_q \leq C_w \Phi^4$. Set $\varepsilon = \Phi^2$ and choose a fixed large K_0 . On the good set $\mathcal{G} = \{q : d_q \leq K_0 \varepsilon^2 p_q^0\}$, the normalized state is within $4\sqrt{K_0}\varepsilon$ of its product reference for small flux, so its tail fraction $s_q = t_q/p_q \leq 16K_0 \varepsilon^2$. On the complement, $\sum p_q^0 \leq C_w/K_0$, and the baseline product expectation of $O_A O_B$ is zero. Thus its correlation contribution is at most $2\sqrt{(\sum_{\mathcal{G}^c} p_q^0)(\sum_{\mathcal{G}^c} d_q)} + \sum_{\mathcal{G}^c} d_q \leq C_w \varepsilon/\sqrt{K_0} + C_w \varepsilon^2$. Choose K_0 large, then flux small, leaving a good-set correlation at least $c\varepsilon$. A conditional pure state's connected correlation of norm-one observables is at most $6\sqrt{s_q}$, by distance to its largest-Schmidt product vector. Cauchy-Schwarz yields $\sum_{\mathcal{G}} p_q s_q \geq c\varepsilon^2$. The conditional entropy is at least $h_2(s_q)$, and on this good set $h_2(s_q) \geq 2s_q \log(1/|\Phi|)$ for sufficiently small flux. Averaging proves Eq. (S250). This explicitly excludes the possibility that the lower bound was inferred only from a tail concentrated in rare charge sectors.

7. Physical interpretation and boundaries

The tilted activity mean is $u_w(\rho) - \rho^2$, whose second derivative at neutrality is $w/2 - 2$. Its zero coincides with the coefficient zero in Eq. (S248). Mobility/activity curvature already has a central role in diffusive current statistics [65]; the cancellation is not a new general fluctuation principle. Likewise, Schmidt support opening near a ferromagnetic point [39] and entanglement of tilted-generator eigenvectors [66, 67] are established settings. The conclusion here is the controlled quantum entropy contrast for actual lowest neutral states at fixed small total flux in this specified family. It does not establish general nongradient or disordered stability, a size-dependent crossover as $w - 4$ shrinks, arbitrary flux, a bulk transition or a global-ground transport coefficient.

Appendix R: Entanglement suppression without translation symmetry

At $w = 4$, actual entanglement suppression survives unequal bonds. In the neutral sector, multiply each four-term operator $h_i(\theta_i)$ of Eq. (S246), including the normalized hopping terms, by a positive strength:

$$H_\lambda(\Phi) = \sum_{i=1}^N \lambda_i h_i(\theta_i), \quad \sum_i \theta_i = \Phi, \quad 0 < \lambda_- \leq \lambda_i \leq \lambda_+ < \infty. \quad (\text{S268})$$

The bounds are independent of N . Each bond's four processes share the same multiplier; their relative strengths are unchanged. The profile may be arbitrary and deterministic. No translation, reflection, smoothness or statistical assumption is made. The reference state is still Ω_4 .

There exist $C, \Phi_0 > 0$, depending only on $\lambda_\pm$, such that for all $N \geq 2$ and $0 < |\Phi| \leq \Phi_0$ the lowest neutral state $\psi_\lambda(\Phi)$ is simple. Its internal spacing is $\Theta(N^{-2})$, uniformly over the profiles. For every contiguous interval A ,

$$\begin{aligned} \mathcal{E}_{\text{ch}}(\psi_\lambda) &\leq C\Phi^4 N^{-1/5} \log \frac{eN}{|\Phi|}, \\ |S_A(\psi_\lambda) - S_A(\Omega_4)| &\leq C\Phi^2 N^{-1/10} \log \frac{eN}{|\Phi|}. \end{aligned} \quad (\text{S269})$$

The total-entropy statement concerns the change from the nonzero reference entropy. Both changes vanish at each fixed sufficiently small total flux. The constants do not deteriorate with the spatial arrangement of the bonds. For arbitrary bipartitions the optimal single-copy charge-preserving Bell success probability and conditional linear entropy are at most $C\Phi^4 N^{-1/5}$. We do not claim a complete VN upper bound for disconnected regions or an offcritical inhomogeneous lower bound.

1. A gauge with the same current on every bond

Choose the unique small solution j of

$$\sum_i \arcsin(j/\lambda_i) = \Phi, \quad \theta_i = \arcsin(j/\lambda_i). \quad (\text{S270})$$

The left side is strictly increasing on $|j| < \lambda_-/2$. For small fixed total flux, $|j|, |\theta_i| \leq C|\Phi|/N$ uniformly. Every phase profile with this total flux is related by onsite charge rotations, which factorize across a spatial cut and preserve regional charge. Thus the entropies and operation class are unchanged by this gauge.

Write $H_{\lambda,0} = H_\lambda(0)$ and let J_i, R_i denote the single-bond current and quantum diagonal from Appendix Q. The exact decomposition is

$$H_\lambda(\Phi) = \sum_i \lambda_i \cos \theta_i h_i(0) + j \sum_i J_i + \sum_i \alpha_i R_i, \quad \alpha_i = \lambda_i(1 - \cos \theta_i). \quad (\text{S271})$$

The gradient identity gives $\sum_i J_i \Omega_4 = 0$. Uniform phases would generally fail to cancel the source when the bonds are unequal; Eq. (S270) is essential. Put $\beta = \Phi^2/N^2$, $a_i = \alpha_i/\beta$ and $V_a = \sum_i a_i R_i$. The a_i have uniform bounds, including their continuous limits at zero flux. With expectation $\mu = \mu_4$ and $Q = I - |\Omega_4\rangle\langle\Omega_4|$, the exact source is

$$QH_\lambda(\Phi)\Omega_4 = \beta F_a \Omega_4, \quad F_a = V_a - \mu V_a. \quad (\text{S272})$$

2. The weighted activity form

We need the centered bound, uniformly for deterministic $|a_i| \leq K$,

$$|\langle f, F_a f \rangle_\mu| \leq C_K \{N^2 \mathcal{E}_{\text{ref}}(f) + \|f\|_\mu^2\}, \quad \mathcal{E}_{\text{ref}}(f) = \langle f, H_{4,0} f \rangle. \quad (\text{S273})$$

An uncentered extensive potential would not give this estimate. First, for a partition into b disjoint intervals, define $W = \sum_B Q_B^2/|B|$. Conditioning on a histogram makes the spin ordering a uniform permutation with total charge zero. For coefficients c_i , reveal sites in decreasing order of $|c_i|$. The Doob increment of $\sum_i c_i x_i$ is the product of the next spin's centered value and the next coefficient minus the average remaining coefficient. Its conditional mean is zero and its absolute value is at most $4|c_i|$. The conditional exponential bound therefore gives $\mu e^{\sum_i c_i x_i} \leq e^C \sum_i c_i^2$, also after mixing histograms. Gaussian linearization with coefficients $c_i = \sqrt{2s} G_B / \sqrt{|B|}$ then implies $\mu e^{s_0 W} \leq e^{C^b}$ for fixed small $s_0 > 0$. The global canonical LSI and the entropy inequality give

$$\langle f, W f \rangle_\mu \leq C N^2 \mathcal{E}_{\text{ref}}(f) + C b \|f\|_\mu^2. \quad (\text{S274})$$

Modulus contraction handles complex f .

For a length- n open block B with charge q , exchangeability gives the exact weighted conditional mean

$$h_B^a(q) = \frac{\sum_{i \in B, \text{ internal}} a_i}{n-1} h_n(q), \quad h_n(q) = n g_4(q/n) + O(1). \quad (\text{S275})$$

The unweighted mean and its even, bounded- C^2 function g_4 were established in Appendix Q, including saturated charge endpoints. Singletons have no internal term. Use the same balanced binary charge filtration as there, retaining both physical ring boundaries at the root. At a parent split of scale ℓ and charge q , put $\delta = q_{\text{left}} - |B_{\text{left}}|q/|B|$. The centered internal increment is now $b_q \delta + R$, with $|b_q| \leq C_K |q|/\ell$ and $|R| \leq C_K(1 + \delta^2/\ell)$. Unequal child weights prevent cancellation of the linear term. Conditional permutation concentration controls δ with variance proxy $C\ell$; Cauchy-Schwarz consequently gives

$$\log \mathbb{E}[e^{\pm s(b_q \delta + R)} | q] \leq C_K s^2(1 + q^2/\ell), \quad |s| \leq s_*. \quad (\text{S276})$$

The centered between-parent boundary increment is bounded and obeys the same estimate without q^2/ℓ .

Conditional entropy and the all-charge open LSI cost $C\ell^2/s$ in energy per level. Each parent energy occurs once for an internal split and at most twice for its neighboring boundaries. With Eq. (S274), the total level cost is bounded by

$$C(\ell^2/s + sN^2)\mathcal{E}_{\text{ref}}(f) + C s N/\ell \|f\|_\mu^2. \quad (\text{S277})$$

Take $s = s_*(\ell/N)^{3/2}$. The dyadic sums of the energy coefficients are $O(N^2)$ and that of sN/ℓ is $O(1)$, proving Eq. (S273). This keeps the linear fluctuation rather than assuming the homogeneous cancellation survives unequal weights.

3. The critical source remains smaller than the gap scale

Let $G = H_{\lambda,0}|_Q$. The bound needed for the actual state is

$$\mathcal{A}_a = \langle F_a \Omega_4, G^{-1} F_a \Omega_4 \rangle \leq C_K N^{9/5}. \quad (\text{S278})$$

Partition into lengths in $[\ell, 2\ell]$ and average all translations while keeping the original coefficients fixed. If there are b blocks, each physical bond is internal a fraction $1 - b/N$ of the time. Hence $V_a = N/(N - b)$ times the averaged internal sum, exactly. Split each sum into its fixed-charge fluctuation and Eq. (S275).

For the fast part, the neutral finite-point expansion used in Appendix Q bounds covariance of disjoint bounded pair observables by C/N . A weighted length- L sum thus has variance at most $C(L + L^2/N) \leq CL$; averaging its conditional variance only lowers this bound. The conditional open inverse gap costs CL^2 . Summing the conditional dual-energy estimates gives $CN\ell^2$.

For the slow part, use iid block-charge centering: $f_B(q) = h_L(q) - \mathbb{E}_{\text{iid},L} h_L$. At $w = 4$, evenness and $g_4''(0) = 0$ give the scalar estimates $\mathbb{E}_{\text{iid}} f_B^2 \leq C/L^2$ and $\mathbb{E}_{\text{iid}} |f_B| q^2 \leq C$ derived in the preceding appendix. The block multipliers in Eq. (S275) are bounded. For their weighted sum Y , estimate $\text{Var}_\mu Y \leq \mu Y^2$; no zero canonical mean is assumed for the individual iid-centered functions. The canonical one-block density is bounded by a constant times its iid density. For distinct blocks whose remainder has length at least $N/2$, the joint density kernel is $K(q + r)$. Iid centering and evenness cancel its constant, single-variable and lower-order cross pieces. The mixed fourth-difference bound from Appendix Q, with the two q^2 moments above, gives $|\mu f_B f_{B'}| \leq C/N^2$. Consequently

$$\text{Var}_\mu Y \leq \mu Y^2 \leq C(N/\ell^3 + 1/\ell^2) \leq CN/\ell^3. \quad (\text{S279})$$

The neutral inverse gap costs CN^2 by $H_{\lambda,0} \geq \lambda_- H_{4,0}$. Choose $\ell \asymp N^{2/5}$ and explicitly require $N \geq 8\ell$ for this argument: any two blocks then leave at least $N/2$ sites. The finitely many remaining sizes are absorbed uniformly using the same gap comparison and bounded source norm. Thus $\mathcal{A}_a \leq C(N\ell^2 + N^3/\ell^3) \leq CN^{9/5}$. The proof uses neither a translated weight profile nor an assumption of independent canonical block charges.

4. Actual state, entropy tails and internal spacing

The current form from Eq. (S257), multiplied by the single scalar j , costs at most $C|\Phi|H_{\lambda,0}$. The cosine correction costs $C\Phi^2 H_{\lambda,0}/N^2$. Centering the common trial energy $\beta\mu V_a$ and using Eq. (S273) gives, for sufficiently small $|\Phi| \leq \Phi_0$,

$$B_E = Q[H_\lambda(\Phi) - E]Q \geq G/2, \quad \delta E = \beta\mu V_a - E \geq 0. \quad (\text{S280})$$

Here E is the actual lowest neutral energy. No lowest vector can lie in Q ; two independent lowest vectors would have such a combination, so the state is simple. Its exact Schur equations imply

$$\begin{aligned} \psi_\lambda &= c(\Omega_4 - \beta B_E^{-1} F_a \Omega_4), \\ 0 \leq \delta E &\leq 2\beta^2 \mathcal{A}_a, \quad \langle \psi_\lambda, H_{\lambda,0} \psi_\lambda \rangle \leq 4\beta^2 \mathcal{A}_a \leq C\Phi^4 N^{-11/5}, \\ 1 - |\langle \Omega_4, \psi_\lambda \rangle|^2 &\leq C\Phi^4 N^{-1/5}. \end{aligned} \quad (\text{S281})$$

The common $O(\Phi^2/N)$ energy was subtracted, not bounded as a perturbation of the N^{-2} complement gap.

For contiguous A , its physical open Hamiltonian is bounded below by λ_- times the reference open Hamiltonian and above by the full zero-flux Hamiltonian. The all-charge LSI therefore retains the heat-trace bound of Eq. (S266) at $T_N = CN^2 \log(eN)$. The Gibbs Schmidt-tail estimate Eq. (S267) and Eq. (S281) yield the first line of Eq. (S269). Reference conditional states are products; the total conditional tail weight is at most the infidelity. For total VN entropy, write $S_A = H(p_q) + \mathcal{E}_{\text{ch}}$. The charge distribution has at most $N + 1$ outcomes and total variation at most $C\Phi^2 N^{-1/10}$. Classical entropy continuity gives the second line. For arbitrary cuts the same tail weight bounds conditional linear entropy and optimal single-copy Bell probability, as in the preceding appendix. No exponentially large entropy-continuity factor is used.

Complement min-max in Eq. (S280) bounds the internal gap below by c/N^2 . For its upper bound take a normalized real cosine charge mode at zero flux. Termwise comparison gives reference energy $O(N^{-2})$; its imaginary-current expectation is zero and Eq. (S273) controls the centered potential. Its overlap with the actual lowest state is $O(\Phi^2 N^{-1/10})$. Orthogonalizing against that exact eigenstate preserves the Rayleigh numerator and leaves a uniformly positive norm. This gives the upper C/N^2 bound without translation symmetry, momentum orthogonality, or claiming that a Fourier mode is an inhomogeneous eigenstate.

5. Nonzero neutral curvature and physical boundary

Let $\mathcal{R}_N = \sum_i \lambda_i^{-1}$ and $\bar{r}_N = \mu_4 R_i \rightarrow 1/3$. Differentiating Eq. (S270) gives $j'(0) = 1/\mathcal{R}_N$ and $\theta'_i(0) = 1/(\lambda_i \mathcal{R}_N)$. Since $H'_\lambda(0)\Omega_4 = 0$, the eigenvector derivative contributes no second-order term, giving exactly

$$E''_\lambda(0) = \frac{\bar{r}_N}{\mathcal{R}_N}, \quad N E''_\lambda(0) - \frac{N}{3\mathcal{R}_N} \longrightarrow 0. \quad (\text{S282})$$

The convergence is uniform over the bounded profiles. A separate profile limit is unnecessary; a curvature limit exists if its harmonic mean converges. Polarized global zero modes remain available, so this is not a global-ground stiffness. Conductance comparison and resistance scales are familiar in inhomogeneous exclusion processes [68]; they alone do not control the quantum Schmidt tail. The result here removes translation symmetry from the actual-state entropy suppression, while retaining fixed local process ratios. It is not stability under arbitrary four-rate disorder or a general nongradient perturbation.

The homogeneous zero-flux dynamics after the ground-state transform also belongs to Sasada's two-species exclusion family [69]. Its hydrodynamic and spectral-gap results concern classical dynamics; they do not imply the finite-flux quantum entropy bound (S269).