

Charge transport, not interaction range, bounds the Lieb–Schultz–Mattis twist cost

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In the Lieb–Schultz–Mattis argument, a slow twist bounds gaps at fractional filling. We show that its energy is set by charge transport, not interaction range. On a U(1)- and translation-invariant ring of L spin- S sites, every fractional-sector gap is at most $8\pi^2 S^2 \mathcal{T}_2 / L^2$, and density couplings of any range add nothing to the transport moment $\mathcal{T}_2$. For the unfrustrated spin-1/2 XY ring with N charges, no symmetry-preserving interaction commuting with the twist can raise the sector gap above its free-fermion value $2|J_\perp| \sin(\pi/L) \sin(\pi N/L)$; diagonal ones lower it unless constant in the sector.

Introduction—Under suitable locality conditions, the Lieb–Schultz–Mattis (LSM) theorem and its extensions forbid a translation-invariant chain with a conserved U(1) charge from having a unique gapped ground state when the charge per unit cell is fractional [1–5]. For half-odd-integer spin chains this proves part of Haldane’s conjecture [2]. In higher dimensions it becomes a filling constraint on gapped phases [6–9], and LSM constraints are now understood as lattice manifestations of quantum anomalies [10–12]. The argument requires no solution of the model. One rotates the spins slowly about the z axis, by an angle that winds once around the ring, and shows that the rotated ground state costs little energy and carries a different crystal momentum.

Long-range interactions make this energy estimate a practical issue: dipolar and van der Waals couplings are native to ultracold polar molecules and Rydberg-dressed atoms [13–18]. The standard estimate prices a term spanning r sites at a twist angle of order r/L , hence at an energy of order $(r/L)^2$. For charge-moving tails decaying as $r^{-\alpha}$ the bound still vanishes when $\alpha > 2$ [19, 20], a condition that dipolar ($\alpha = 3$) and van der Waals ($\alpha = 6$) exchange already meet. Ma further showed that tails moving no charge obey a much weaker condition [19], and complementary long-range results were obtained in Ref. [21]. Earlier, Nomura, Morishige, and Isoyama noted that paired twists leave longitudinal $S^z S^z$ couplings unrestricted [22], and Watanabe used the same twist to prove a lattice version of Bloch’s theorem on ground-state currents [23]. The open issue is therefore not whether the LSM conclusion survives, but how much range-based accounting overcounts. What is missing is quantitative: an intrinsic cost that does not depend on how the Hamiltonian is written as a sum of terms, and the largest finite-size gap compatible with every cancellation.

In this Letter, we answer the first question in general and the second for the spin-1/2 XY ring. The twist multiplies each charge-transfer process by a phase proportional to the change of dipole moment it produces, mod-

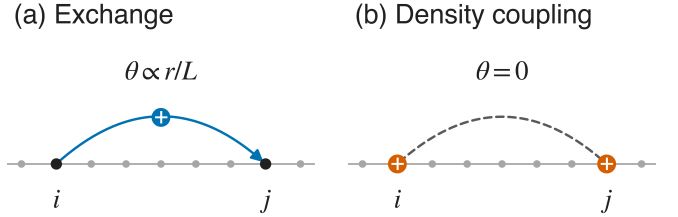


FIG. 1. Range versus transport. (a) An exchange between sites i and j moves a charge a distance r ; the twist rotates the two ends by angles that differ by $2\pi r/L$, so the process acquires a phase $\theta \propto r/L$. (b) A density–density coupling between the same sites moves no charge; its twist phases cancel and $\theta = 0$ at any r . Only the first contributes to the twist energy.

ulo the ring length (Fig. 1). Minimizing over all ways of writing the Hamiltonian turns this phase into a transport moment $\mathcal{T}_2$. It bounds every fractional-sector gap by $8\pi^2 S^2 \mathcal{T}_2 / L^2$ and receives nothing from density couplings of any range or from dipole-conserving correlated hopping. For the unfrustrated XY ring we obtain the optimal bound. No charge- and translation-conserving interaction that commutes with the twist, whatever its strength, range, or body order, can raise the sector gap above its free-fermion value $2v_F \sin(\pi/L) \simeq 2\pi v_F / L$, where v_F is the Fermi velocity of the free chain. Diagonal interactions lower the gap unless they are constant in the sector, and numerical diagonalization of the XXZ ring and of a dipolar density tail shows by how much (Fig. 2). Applied along one direction of a cylinder, the same cost keeps the bound vanishing on thin tori even with dense dipolar exchange.

The cancellation mechanism [19, 22] and the free spectrum of the XY ring [1] are known. What is new is an intrinsic cost, an optimal envelope with a classification of diagonal saturation and explicit off-diagonal saturators, and a direction-resolved application. All bounds are rigorous finite-size statements; the gap bounds constrain, but do not identify, the low-energy phase.

Setup—Consider a ring of L sites, $j = 1, \dots, L$, each carrying spin S , and count charge from the bottom of the local multiplet, $q_j = S_j^z + S$, whose eigenvalues are $0, 1, \dots, 2S$, with total $Q = \sum_j q_j$. The Hamiltonian H_L conserves Q and commutes with T^p , where T translates by one site and p is the number of sites in a unit cell. Nothing is assumed about its range. We write Δ_N for the gap within the sector $Q = N$ and Δ for the gap of the full spectrum; eigenvalues are counted with multiplicity, so a degenerate ground state gives zero gap. The LSM twist

$$U = \exp\left(\frac{2\pi i}{L} \sum_j j q_j\right) \quad (1)$$

rotates the spin at site j by $2\pi j/L$ about the z axis. It is Resta's exponentiated position operator, used to define polarization in periodic systems [24], so the phase it imparts to a process measures the polarization change that the process produces. Its key property is

$$TUT^{-1} = U e^{-2\pi i Q/L} : \quad (2)$$

in the sector $Q = N$ the twist changes the eigenvalue of T^p by $e^{-2\pi i p N/L}$. At fractional filling, $pN/L \notin \mathbb{Z}$, a twisted ground state is therefore orthogonal to the original, and the only question is its energy.

Twist cost of a single term—Write $H_L = \sum_a B_a$, each term B Hermitian, supported on a set X of sites, and conserving the charge on X ; averaging over local charge rotations brings any decomposition to this form without increasing norms, using only global charge conservation (Supplemental Material [25]). Label the charge configurations on X by λ , with twist phase

$$\theta_\lambda = \frac{2\pi}{L} \sum_{j \in X} j \lambda_j \pmod{2\pi}, \quad (3)$$

which is $2\pi/L$ times the dipole moment of λ . The conjugation $U^\dagger B U$ multiplies each matrix element of B between configurations λ and μ by $e^{i(\theta_\mu - \theta_\lambda)}$, so the twist sees a term only through how much its processes change the dipole moment, modulo L . The exact measure is the *twist-phase spread* $\eta(B) \in [0, \pi]$: link the configurations that B connects and, for each linked family, take the radius of the shortest arc of the circle containing its phases; $\eta(B)$ is the largest such radius. It is the smallest angle by which a rotation generated by the on-site charges must act to reproduce the twist on B [25], and it vanishes exactly when $[B, U] = 0$. For a spin-1/2 exchange across a ring distance $r \leq L/2$, the two linked configurations differ in phase by $2\pi r/L$, so $\eta = \pi r/L$.

A computable bound uses the weighted charges $\sum_{j \in X} c_j S_j^z$ that B conserves, $\mathcal{K}_X(B) = \{c \in \mathbb{R}^X : [B, \sum_{j \in X} c_j S_j^z] = 0\}$. An exchange between two sites conserves the sum of their charges but not their difference, whereas a density coupling conserves both. Subtracting conserved weights from the site positions leaves

the displacement that the term actually produces, measured by the *transport distance*

$$\tau(B) = \inf_{n \in \mathbb{Z}^X, c \in \mathcal{K}_X(B)} \sum_{j \in X} |j + n_j L - c_j|, \quad (4)$$

where the integers n_j let each site be counted across the seam of the ring. Then $\eta(B) \leq \min\{2\pi S \tau(B)/L, \pi\}$. If every process of B transfers charge along a multiple of one primitive integer vector d , and x lists the positions of the sites of X ,

$$\tau(B) = \frac{\text{dist}(d \cdot x, LZ)}{\|d\|_\infty} : \quad (5)$$

a term pays only for the amount by which its dipole displacement $d \cdot x$ misses a whole number of turns around the ring. An exchange across a distance r pays r ; a density coupling of any span pays nothing.

Gap bound—A Hamiltonian can be decomposed in many ways, and its cost is that of the cheapest one. We define the transport moment and its finite-angle companion,

$$\begin{aligned} \mathcal{T}_2(H_L) &= \inf \sum_a \tau(B_a)^2 \|B_a\|, \\ \mathcal{B}(H_L) &= \inf \sum_a \sin^2(\min\{\eta(B_a), \frac{\pi}{2}\}) \|B_a\|, \end{aligned} \quad (6)$$

the infima running over finite decompositions into Hermitian terms that conserve the charge on their own supports. Both are seminorms of the operator H_L , not of a presentation of it. The infimum matters: for the spin-1/2 XY ring with exchange $J_\perp$ [Eq. (9) below], the bond decomposition attains $\mathcal{T}_2 = L|J_\perp|/2$, whereas the same Hamiltonian written as one ring-wide term costs at least $|J_\perp|[L^2/4]^2$. Our first result is that in every sector of fractional filling, $pN/L \notin \mathbb{Z}$,

$$\Delta_N(H_L) \leq 2\mathcal{B}(H_L) \leq \frac{8\pi^2 S^2}{L^2} \mathcal{T}_2(H_L). \quad (7)$$

Any family with $\mathcal{T}_2 = o(L^2)$ thus has vanishing gaps in every sequence of fractional sectors. The bound needs neither a unique ground state nor knowledge of which sector contains it, so it applies directly to simulations at fixed particle number or magnetization. It also bounds the full gap: a degenerate ground state gives $\Delta = 0$, and a unique one in a fractional sector gives $\Delta \leq \Delta_N$. With global SU(2) symmetry and $pS \notin \mathbb{Z}$, uniqueness forces a singlet at fractional filling, so Eq. (7) bounds Δ itself.

Mechanism—Two estimates combine. First, on a term B the twist acts as a rotation $e^{-iA} B e^{iA}$ generated by on-site charges, with $\|A\| = \eta(B)$ at best. The explicit generator $A = \frac{2\pi}{L} \sum_{j \in X} (j + n_j L - c_j) S_j^z$ has norm $\frac{2\pi S}{L} \sum_{j \in X} |j + n_j L - c_j|$, which approaches $2\pi S \tau(B)/L$ at the infimum in Eq. (4); subtracting the conserved weights c is the step that converts geometry into transport. Second, twisting in both senses cancels the linear response:

for self-adjoint A and B ,

$$\left\| \frac{1}{2}(e^{-iA}Be^{iA} + e^{iA}Be^{-iA}) - B \right\| \leq 2 \sin^2(\min\{\|A\|, \frac{\pi}{2}\}) \|B\|. \quad (8)$$

The cost is quadratic in the angle, which is why a slow twist is cheap, but the inequality holds at finite angle and its constant cannot be improved. Summing over terms bounds the mean excitation energy of $U^{\pm 1}\Psi$ above a sector ground state Ψ by $2\mathcal{B}$. If Ψ is unique it is a translation eigenvector, and Eq. (2) makes both twisted states orthogonal to it at fractional filling. The lower of the two gives Eq. (7) by the variational principle.

Interactions that cost nothing—The free class is large. Take a spin-1/2 ring with one site per cell, real $J_{\perp}$, and $L \geq 6$,

$$H_L = J_{\perp} \sum_j (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + V + Z. \quad (9)$$

Here V is any Hermitian translation-invariant operator diagonal in the S^z basis, for instance

$$V = \sum_j \sum_{r=1}^{\lfloor (L-1)/2 \rfloor} V_r S_j^z S_{j+r}^z - h \sum_j S_j^z, \quad (10)$$

with no decay condition on V_r : dipolar, van der Waals, and unscreened Coulomb tails are all allowed. Being diagonal, V conserves every on-site charge, so $\tau(V) = 0$ and V drops out of Eq. (7) entirely. The term Z is any Hermitian translation-invariant sum of off-diagonal spin monomials, each conserving the dipole moment modulo L . An example, for $2 \leq r \leq \lfloor L/2 \rfloor$, is the counterflow hopping

$$Z_{j,r} = u_r S_j^+ S_{j+1}^- S_{j+r}^- S_{j+r+1}^+ + \text{H.c.}, \quad (11)$$

in which one charge hops right across one bond while another hops left across a distant bond. Two charges move but the dipole moment does not, so $\tau(Z_{j,r}) = 0$ by Eq. (5), even when a bond straddles the seam. Processes of this kind generate the dynamics of dipole-conserving models [26].

Only the exchange costs anything. Every dipole-conserving transfer annihilates the one-particle sector, and matching one-particle matrix elements against all admissible decompositions gives $\mathcal{T}_2 = L|J_{\perp}|/2$ exactly; the bond decomposition also gives $\mathcal{B} \leq (L|J_{\perp}|/2) \sin^2(\pi/L)$ [25]. Hence, for either sign of $J_{\perp}$,

$$\Delta_N(H_L) \leq L|J_{\perp}| \sin^2 \frac{\pi}{L} \leq \frac{\pi^2 |J_{\perp}|}{L}, \quad N/L \notin \mathbb{Z}, \quad (12)$$

independent of the strength, sign, range, and decay of everything in V and Z . The same argument covers a p -site unit cell, $p \mid L$, with bond exchange J_a and Dzyaloshinskii-Moriya coupling D_a [27]: $\Delta_N \leq$

$(L/p) \sin^2(\pi/L) \sum_{a=0}^{p-1} \sqrt{J_a^2 + D_a^2}$ for $pN/L \notin \mathbb{Z}$ [25]. At $p = 2$ and half filling the hypothesis fails, as it must, since dimerization can open a gap.

Optimal envelope—The transport bound prices each of the L bonds at its norm $|J_{\perp}|/2$, as if all bonds could be satisfied at once. In a Fermi sea they cannot. Let K be the exchange term of Eq. (9), now for any $L \geq 3$, and consider $H = K + H'$ with any Hermitian H' obeying $[H', Q] = [H', T] = [H', U] = 0$; no strength, locality, or body-order restriction is imposed, and V and Z above are special cases. Take the unfrustrated sign, $J_{\perp} < 0$, or $J_{\perp} > 0$ with L even, where a staggered rotation maps one sign onto the other. By the Jordan-Wigner transformation K describes free fermions with dispersion $-|J_{\perp}| \cos k$, whose sector ground energy and gap are $-C_N$ and g_N [1], with

$$C_N = |J_{\perp}| \frac{\sin k_F}{\sin(\pi/L)}, \quad g_N = 2|J_{\perp}| \sin \frac{\pi}{L} \sin k_F, \quad (13)$$

and $k_F = \pi N/L$. Since $C_N < L|J_{\perp}|/2$ for $L \geq 3$, the kinetic energy of the Fermi sea lies strictly inside the bondwise budget. Equivalently $g_N = 2v_F \sin(\pi/L) \simeq 2\pi v_F/L$, where $v_F = |J_{\perp}| \sin k_F$ is the Fermi velocity: g_N is the energy of the softest particle-hole pair. In the free chain the twisted ground state is exactly this pair, because U shifts every occupied momentum by $2\pi/L$. Our second result is that, for every $0 < N < L$,

$$\Delta_N(K + H') \leq g_N, \quad \sup_{H'} \Delta_N(K + H') = g_N, \quad (14)$$

the supremum running over all admitted H' at fixed L and N .

The proof is short. If the sector ground state is unique, choose it as a translation eigenvector ψ . Both twists are orthogonal to ψ and leave H' invariant, while $U^{-1}KU + UKU^{-1} = 2 \cos(2\pi/L)K$. The variational principle then gives

$$\Delta_N \leq \frac{e_+ + e_-}{2} = 2 \sin^2 \frac{\pi}{L} \langle -K \rangle_{\psi} \leq 2 \sin^2 \frac{\pi}{L} C_N = g_N, \quad (15)$$

where $e_{\pm}$ are the excitation energies of $U^{\pm 1}\psi$. A degenerate sector ground state gives $\Delta_N = 0$, and the free chain, $H' = 0$, attains the bound. The paired-twist identity itself is standard [5, 22]; the gain over Eq. (12) comes from keeping the exact minimum of K rather than summing bond norms. At half filling the envelope is $2|J_{\perp}| \sin(\pi/L)$, lowering the coefficient π^2 of Eq. (12) to 2π . At fixed particle or hole number $\tilde{N} = \min(N, L-N)$ it is at most $2\pi^2 |J_{\perp}| \tilde{N}/L^2$, an $O(L^{-2})$ bound where bondwise accounting gives $O(L^{-1})$.

Equality is rigid for diagonal interactions. Saturation forces ψ to be the free ground state ϕ_N , which has nonzero amplitude on every configuration of the sector, so a diagonal H' having ϕ_N as an eigenvector must be constant on the sector. A diagonal interaction therefore

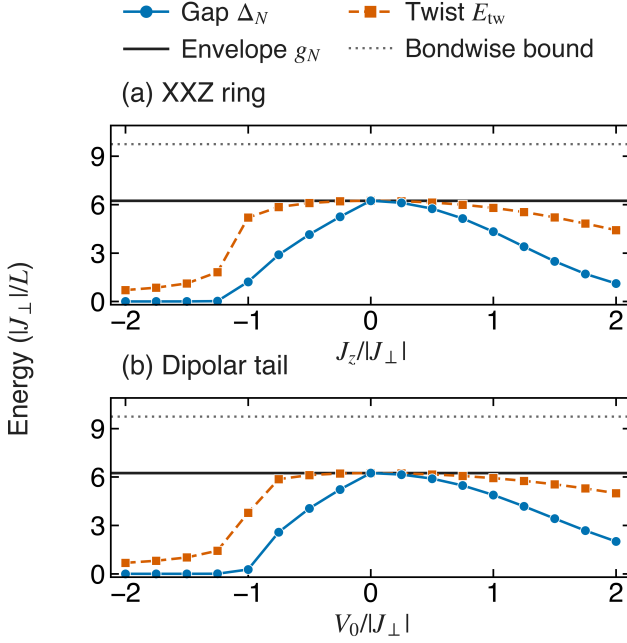


FIG. 2. Natural interactions below the optimal envelope, for $L = 16$, $N = 8$, and $J_{\perp} < 0$. The additions to K are (a) $J_z \sum_j S_j^z S_{j+1}^z$ and (b) $V_0 \sum_{i < j} S_i^z S_j^z / d_L(i, j)^3$, with d_L the shortest ring distance and each pair counted once. Energies are scaled by $L/|J_{\perp}|$. Circles: sector gap Δ_N ; squares: paired-twist cost E_{tw} ; solid line: envelope g_N ; dotted line: bondwise bound of Eq. (12). The separations $g_N - E_{\text{tw}}$ and $E_{\text{tw}} - \Delta_N$ are the kinetic deficit and trial-state slack, respectively. Both models reach the envelope only at $u = 0$. Lines guide the eye; Table S1 lists the values.

attains g_N if and only if it is constant in the N sector, and otherwise strictly lowers the finite-size gap. For example, the Ising coupling $J_z \sum_j S_j^z S_{j+1}^z$ that turns the XY ring into the XXZ chain is diagonal and not constant for $2 \leq N \leq L - 2$; any $J_z \neq 0$, of either sign, pushes Δ_N strictly below g_N at every finite L .

Figure 2 shows, at $L = 16$ and half filling, how far two natural interactions fall below these bounds. For even L the staggered rotation maps the first model onto the antiferromagnetic XXZ chain with anisotropy $u = J_z/|J_{\perp}|$, so $u = 1$ and $u = -1$ are the antiferromagnetic and ferromagnetic Heisenberg points. Two distinct sources of slack appear. A diagonal interaction reduces the kinetic energy $\langle -K \rangle_{\psi}$ below its free value C_N , which pulls the paired-twist cost $E_{\text{tw}} = (e_+ + e_-)/2$ of Eq. (15) below g_N . The twisted states, moreover, need not be the lowest excitations, which leaves Δ_N below E_{tw} . An interaction can thus cost nothing under the twist and still reshape the spectrum strongly.

Off-diagonal interactions can saturate without being constant. For $J_{\perp} < 0$ and $2 \leq N \leq L - 2$, a weighted positive-parent construction [28] yields a nonzero $H'_* \geq 0$ with $H'_* \phi_N = 0$ and the three required commutators. For

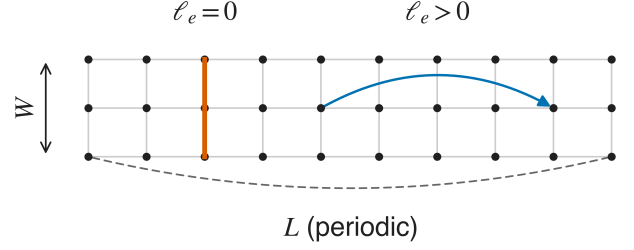


FIG. 3. Cylinder geometry: L slices along the periodic direction, each of W sites. The longitudinal twist prices a hopping bond only by its wrapped longitudinal displacement ℓ_e , so a bond inside one slice ($\ell_e = 0$) costs nothing, whatever its strength.

every $\gamma \geq 0$ and $0 \leq m < L$,

$$\begin{aligned} \Delta_N(K + \gamma H'_*) &= g_N, \\ (K + \gamma H'_*) U^m \phi_N &= -C_N \cos \frac{2\pi m}{L} U^m \phi_N, \end{aligned} \quad (16)$$

and these L states are mutually orthogonal even where their momenta coincide. The operator H'_* is generally nonlocal and depends on all spectator charges; it proves that saturation is possible, not that a natural phase realizes it. The Supplemental Material [25] gives the proofs, a fidelity version of the diagonal rigidity, and bounds on several low levels in dilute sectors.

Cylinders and thin tori—Because the twist acts along one direction, its cost is directional. Consider L slices arranged along a periodic direction, each a copy of a graph with W sites carrying integer charges, and twist only the periodic coordinate (Fig. 3). Let the longitudinal dynamics be unit-charge hopping $t_e |1, 0\rangle \langle 0, 1| + \text{H.c.}$ on bonds e , with ℓ_e the wrapped longitudinal displacement of e , and let every other charge-conserving term commute with its own longitudinal twist, as any term inside one slice does, whatever its strength and transverse range. For p -periodic bonds and amplitudes with $p \mid L$, every sector with $pN/L \notin \mathbb{Z}$ obeys

$$\Delta_N(H_L) \leq 2 \sum_e |t_e| \sin^2 \frac{\pi \ell_e}{L}. \quad (17)$$

For uniform hopping t from each site to its copy in the next slice, this is at most $2\pi^2 W |t|/L$, vanishing when $W = o(L)$: the anisotropic regime of Affleck's higher-dimensional argument [6].

Long-range hopping enters only through the total amplitude that moves charge a given longitudinal distance; transverse displacement and intra-slice couplings do not enter. The sharpest test is geometrically dense. On a translation-invariant ($p = 1$) $L \times W$ torus with $W \leq \lfloor L/2 \rfloor$, let every longitudinal hopping be bounded by $J(r_e^2 + s_e^2)^{-\alpha/2}$, with r_e and s_e its longitudinal and transverse distances. Summing the Euclidean tail before ap-

plying Eq. (17) gives, at dipolar decay $\alpha = 3$,

$$\Delta_N(H_L) \leq \frac{2\pi^2 JW^2}{L} \left[3 + \log \frac{\lfloor L/2 \rfloor}{W} \right]. \quad (18)$$

Even when every pair of transverse channels exchanges through a dipolar tail, the sector gap must close whenever $W^2[1 + \log(L/W)] = o(L)$; a van der Waals tail gives $O(W/L)$. At fixed width this yields a ladder corollary: for even L and an odd number of legs, a spin-1/2 SU(2)-invariant ladder with leg exchanges J_a has $\Delta \leq \pi^2 \sum_a |J_a|/L$ for arbitrary SU(2)-invariant rung dynamics. Fixed-width ladder results [5, 29] and bounds on tori [8] are known; new here are the direction-resolved cost and the width-decay crossover it produces.

Conclusion—The energy of the LSM twist is set by the dipole displacement that an interaction produces, modulo the ring length, not by its range. The resulting transport moment bounds every fractional-sector gap and is never weaker than the geometric estimate: for supports of diameter at most R , local strength at most J , and $L > 3R$, it reproduces the finite-range theorem with $\Delta_N \leq 8\pi^2 S^2 \kappa_R^2 J/L$, $\kappa_R = \lfloor (R+1)^2/4 \rfloor$, and for charge-moving power laws it recovers the threshold $\alpha > 2$ [19]. Where transport and range differ, the gain is real: for a spin-1/2 density tail V/r^α , pricing each term by its span adds $\frac{\pi^2 |V|}{2L} \sum_r r^{2-\alpha}$, of order $\log L/L$ at dipolar decay, whereas the true contribution is zero. For the unfrustrated XY ring, the exact kinetic minimum turns the bound into an optimal envelope that interactions commuting with the twist cannot exceed, nonconstant diagonal ones strictly undercut, and engineered off-diagonal ones can saturate; natural XXZ and dipolar rings lie well below it (Fig. 2).

These are upper bounds. A vanishing bound forbids a uniform gap but does not fix the true low-energy scale, and the density tails that drop out of the estimate may well select the phase. The cylinder bounds require W to grow more slowly than L and leave the isotropic two-dimensional problem open. For quantum simulators, a long-range coupling should be judged by how its processes move charge. Rydberg-dressed extended Bose-Hubbard chains [18] and dipolar exchange in molecular lattices and tweezer arrays [15–17] combine free density tails with costly exchange, although we do not claim that they realize the hypotheses used here. The many-body charge-transport index [30] is a related object; the present cost acts directly on a finite-volume Hamiltonian and assumes no spectral gap. Natural next questions are whether an optimal envelope exists beyond the XY kinetic term, how close the gaps of natural local models come to it, and whether a direction-resolved cost can control isotropic two-dimensional systems.

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Data availability—All plotted values, model definitions, and numerical procedures are supplied in the Supplemental Material [25].

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Supplemental Material for “Charge transport, not interaction range, bounds the Lieb–Schultz–Mattis twist cost”

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In this Supplemental Material, we provide detailed derivations of the following topics: the reduction of the twist cost of a single interaction term to an exact charge-algebra rotation, the sharp finite-angle inequality and the orthogonality supplied by the translation phase, the seminorm and duality properties of the Hamiltonian transport costs together with the exact evaluation of the XY ring, the transfer-matrix certificates and the closed formula for correlated hopping, the recovery of the finite-range theorem and of the known power-law decay thresholds, the direction-resolved bounds on cylinders and ladders, the spin-1/2 ring whose diagonal tails and polarization-neutral hoppings cost exactly nothing, and the optimal XY envelope with its equality cases, low-level staircase, and saturating positive parents, together with the procedures and data of the finite-size numerical illustration.

I. STATEMENT OF RESULTS

We follow the notation and hypotheses of the main text throughout, and work at fixed ring length $L \geq 2$. Operator norms are norms on the full Hilbert space; a supported decomposition is finite, with each term Hermitian and acting trivially outside its support, and repeated support labels are permitted.

Besides the twist-phase spread $\eta(B)$ and the transport distance $\tau(B)$, the proofs below use the finite-angle cost each one assigns to a single term,

$$s(B) = \sin\left(\min\left\{\eta(B), \frac{\pi}{2}\right\}\right), \quad s_\tau(B) = \sin\left(\min\left\{\frac{2\pi S}{L}\tau(B), \frac{\pi}{2}\right\}\right), \quad (\text{S1})$$

so that $\mathcal{B}(H_L) = \inf \sum_a s(B_a)^2 \|B_a\|$ is the quantity of the main text, and the transport proxy defines its companion

$$\mathcal{B}_\tau(H_L) = \inf_{H_L = \sum_a B_a} \sum_a s_\tau(B_a)^2 \|B_a\|, \quad (\text{S2})$$

the infimum running, as for $\mathcal{B}$ and $\mathcal{T}_2$, over finite supported Hermitian decompositions whose terms conserve the charge on their own supports.

The results. We prove the following four statements.

Result A (bound for a given decomposition). Let H_L satisfy the symmetry assumptions of the main text, take any decomposition $H_L = \sum_X \Phi_X$ into Hermitian supported terms — with no symmetry assumed of the individual Φ_X — and let B_X be its charge average, Eq. (S5). Then for every charge N with pN/L not an integer,

$$\Delta_N(H_L) \leq 2 \sum_X s(B_X)^2 \|B_X\| \leq 2 \sum_X s_\tau(B_X)^2 \|B_X\| \leq \frac{8\pi^2 S^2}{L^2} \sum_X \tau(B_X)^2 \|B_X\|. \quad (\text{S3})$$

The first expression uses the exact minimum charge-algebra logarithm but remains an energy upper bound; the second replaces it by a transport proxy, and the third bounds the sine quadratically. None is generally the actual twist energy.

Result B (bound for the Hamiltonian). Under the same hypotheses,

$$\Delta_N(H_L) \leq 2\mathcal{B}(H_L) \leq 2\mathcal{B}_\tau(H_L) \leq \frac{8\pi^2 S^2}{L^2} \mathcal{T}_2(H_L). \quad (\text{S4})$$

This is the central bound of the main text, and it depends on no choice of presentation.

Result C (the full gap). If the global ground energy is degenerate then $\Delta(H_L) = 0$. If the global ground state is unique and carries a charge N with pN/L not an integer, then $\Delta(H_L) \leq \Delta_N(H_L)$ and Eq. (S4) bounds the full gap.

Result D (half-odd-integer spin). If H_L commutes with the global $\text{SU}(2)$ action and pS is not an integer, the full-gap bound of Result C holds. Half-odd-integer spin with one-site translation is the special case $p = 1$.

II. TERMWISE CHARGE CONSERVATION FROM THE GLOBAL SYMMETRY

The main text assumes only that the *Hamiltonian* conserves charge. Every later step, however, works with terms that conserve the charge on their own supports. This section closes that gap: any decomposition can be replaced by one with termwise conservation, keeping the sum, the supports, and the Hermiticity, and without increasing any norm. Nothing is assumed about the interaction beyond the global symmetry.

Average each term over the rotations generated by the charge on its own support,

$$B_X = \frac{1}{2\pi} \int_0^{2\pi} e^{i\chi Q_X} \Phi_X e^{-i\chi Q_X} d\chi, \quad Q_X = \sum_{j \in X} q_j. \quad (\text{S5})$$

This is the projection onto the part of Φ_X that commutes with Q_X ; the integral runs over a compact group, so it is a unital completely positive map and is contractive. The contraction also follows without that structure: every conjugate $e^{i\chi Q_X} \Phi_X e^{-i\chi Q_X}$ has the same norm as Φ_X , so the triangle inequality for the integral gives $\|B_X\| \leq \|\Phi_X\|$.

The averages still sum to H_L . Because Φ_X acts trivially outside X , the charge outside X commutes with it and

$$e^{i\chi Q_X} \Phi_X e^{-i\chi Q_X} = e^{i\chi Q} \Phi_X e^{-i\chi Q}, \quad (\text{S6})$$

so the local average is the global one in disguise. Summing over X and using $[H_L, Q] = 0$,

$$\sum_X B_X = \frac{1}{2\pi} \int_0^{2\pi} e^{i\chi Q} H_L e^{-i\chi Q} d\chi = H_L. \quad (\text{S7})$$

Hermiticity and support are manifestly preserved, and expanding Φ_X into eigenoperators of ad_{Q_X} shows that the average keeps only the piece of charge transfer zero, so $[B_X, Q_X] = 0$. Termwise conservation is therefore a consequence of the global hypothesis, not an extra condition on the interaction.

III. LOCAL REDUCTION OF THE GLOBAL TWIST

The twist

$$U = \exp\left(\frac{2\pi i}{L} \sum_j j q_j\right) \quad (\text{S8})$$

is a global object, but a term feels it only through the sites it touches. This section makes that restriction explicit and shows that the effective local generator can be shifted by any weighted charge the term conserves — which is the freedom the transport distance later exploits.

Let B be supported on X with $[B, Q_X] = 0$. Restricted to X , conjugation by U is generated by

$$G_X = \frac{2\pi}{L} \sum_{j \in X} j q_j. \quad (\text{S9})$$

Because each q_j has integer spectrum, adding L to any coefficient changes e^{iG_X} by $e^{2\pi i q_j} = \mathbb{1}$; hence for every integer vector $n \in \mathbb{Z}^X$,

$$e^{iG_X} = \exp\left[\frac{2\pi i}{L} \sum_{j \in X} (j + n_j L) q_j\right]. \quad (\text{S10})$$

Each site may therefore be counted around the seam of the ring at will, which is why nothing below depends on where the labelling starts.

The second freedom is the conserved weights. Let

$$\mathcal{K}_X(B) = \left\{ c \in \mathbb{R}^X : [B, \sum_{j \in X} c_j S_j^z] = 0 \right\} \quad (\text{S11})$$

be the weighted charges the term leaves invariant. For $c \in \mathcal{K}_X(B)$ the operator $C = \sum_j c_j S_j^z$ commutes with B and with every q_j , so it may be subtracted from the generator without changing the conjugation:

$$e^{-iG_X} B e^{iG_X} = e^{-iA} B e^{iA}, \quad A = A_X(n, c) = \frac{2\pi}{L} \sum_{j \in X} (j + n_j L - c_j) S_j^z. \quad (\text{S12})$$

The replacement of q_j by S_j^z shifts the generator by a multiple of the identity, which drops out of any conjugation. Equation (S12) is the central step: the twist acting on B is a rotation whose angle is not the span of X but whatever remains after the conserved weights are removed.

The norm of that generator is exact rather than estimated. For real coefficients a_j the commuting operator $\sum_j a_j S_j^z$ has eigenvalues $\sum_j a_j m_j$ with each $m_j \in \{-S, \dots, S\}$ selectable independently, and choosing the extremal sign at every site gives

$$\left\| \sum_j a_j S_j^z \right\| = S \sum_j |a_j|, \quad \text{hence} \quad \|A_X(n, c)\| = \frac{2\pi S}{L} \sum_{j \in X} |j + n_j L - c_j|. \quad (\text{S13})$$

No minimizing pair (n, c) need exist: every estimate below holds for each admissible pair and therefore passes to the infimum defining $\tau(B)$.

IV. EXACT TERM COST FROM THE JOINT-CHARGE PHASE GRAPH

Section III produced *some* local rotation reproducing the twist. The cost of a term is set by the *smallest* one, and this section identifies it exactly. The answer is a property of the term alone: the spread of twist phases across the charge configurations the term connects. The construction uses only that the on-site charges are commuting operators with integer spectrum, so it covers arbitrary finite-dimensional sites and degenerate charge eigenspaces.

For each assignment $\lambda = (\lambda_j)_{j \in X}$ of an integer charge to every site of X , let $\mathcal{P}_\lambda$ be the joint spectral projector and θ_λ the twist phase it carries,

$$\mathcal{P}_\lambda = \prod_{j \in X} \mathbb{1}_{\{\lambda_j\}}(q_j), \quad \theta_\lambda = \frac{2\pi}{L} \sum_{j \in X} j \lambda_j \pmod{2\pi}, \quad (\text{S14})$$

so that the local twist is the diagonal unitary $V_X = \sum_\lambda e^{i\theta_\lambda} \mathcal{P}_\lambda$. Join λ to μ whenever $\mathcal{P}_\lambda B \mathcal{P}_\mu \neq 0$, that is, whenever the term actually connects the two configurations. For each connected component ω of the resulting graph let r_ω be the radius of the shortest arc of the circle containing its phases,

$$r_\omega = \inf_{\xi \in \mathbb{R}} \max_{\lambda \in \omega} d_{\mathbb{T}}(\theta_\lambda, \xi), \quad \eta(B) = \max_\omega r_\omega \in [0, \pi], \quad (\text{S15})$$

with $d_{\mathbb{T}}$ the distance along the circle. Using spectral projectors rather than individual basis vectors makes $\eta(B)$ invariant under any relabelling inside a degenerate charge eigenspace.

The spread as the minimal rotation. Let $\mathcal{A}_X$ be the abelian algebra generated by the charges $\{q_j\}_{j \in X}$, whose self-adjoint elements are exactly $A = \sum_\lambda a_\lambda \mathcal{P}_\lambda$ with real a_λ . Then

$$\eta(B) = \min \{ \|A\| : A = A^\dagger \in \mathcal{A}_X, e^{-iA} B e^{iA} = V_X^{-1} B V_X \}. \quad (\text{S16})$$

The spread $\eta(B)$ is thus the smallest angle through which a rotation built from the on-site charges must act in order to reproduce the twist on B .

Proof. For the upper bound, pick for each component ω a centre ξ_ω of a shortest containing arc, and real lifts β_λ of θ_λ with $|\beta_\lambda - \xi_\omega| \leq r_\omega$. Set $A = \sum_\lambda (\beta_\lambda - \xi_{\omega(\lambda)}) \mathcal{P}_\lambda$, so $\|A\| \leq \eta(B)$. The unitary $D = \sum_\lambda e^{-i\xi_{\omega(\lambda)}} \mathcal{P}_\lambda$ is constant on every component, hence commutes with B , and $e^{iA} = D V_X$; the two conjugations therefore agree.

Conversely, suppose $A \in \mathcal{A}_X$ is self-adjoint with $e^{-iA} B e^{iA} = V_X^{-1} B V_X$. Then $\Omega = e^{iA} V_X^{-1}$ commutes with B . Since Ω is a scalar on each joint charge subspace, every nonzero block $\mathcal{P}_\lambda B \mathcal{P}_\mu$ forces those two scalars to agree, so Ω is constant on each component. The coefficients a_λ are therefore real lifts of that component's twist phases up to one common circular shift, whence $\max_{\lambda \in \omega} |a_\lambda| \geq r_\omega$ and $\|A\| \geq \eta(B)$. $\square$

Combining Eq. (S16) with the paired-twist inequality of Sec. V gives the exact term estimate

$$\left\| \frac{1}{2} (V_X^{-1} B V_X + V_X B V_X^{-1}) - B \right\| \leq 2s(B)^2 \|B\|, \quad (\text{S17})$$

the reverse conjugation being covered because A , V_X and Ω all lie in the same abelian algebra and Ω commutes with B .

Transport as an upper bound on the spread. Every admissible pair (n, c) in Sec. III supplies an element of $\mathcal{A}_X$ reproducing the twist, so Eqs. (S13) and (S16) give

$$\eta(B) \leq \min \left\{ \frac{2\pi S}{L} \tau(B), \pi \right\}, \quad \text{hence} \quad s(B) \leq s_\tau(B). \quad (\text{S18})$$

For general integer charges the same construction, after centring each q_j , gives

$$\eta(B) \leq \frac{\pi}{L} \inf_{n,c} \sum_{j \in X} w_j |j + n_j L - c_j|, \quad w_j = \max \text{spec}(q_j) - \min \text{spec}(q_j), \quad (\text{S19})$$

where $\sum_j c_j q_j$ commutes with B ; the weight w_j is the range of charge available at site j .

Criterion for zero cost. The graph gives the exact criterion

$$\eta(B) = 0 \iff [B, V_X] = 0, \quad (\text{S20})$$

since either statement says the twist phase is constant on every connected component. For a spin-1/2 term whose processes all carry the same transfer vector d , the components are pairs and

$$\eta(B) = \frac{\pi}{L} \text{dist} \left(\sum_j d_j j, L\mathbb{Z} \right), \quad (\text{S21})$$

which agrees with the transport formula of Sec. VIII. At higher spin the two can differ, and the phase spread is the sharper of the two. Take $L = 6$, two spin-1 sites at positions 0 and 3, and

$$B = | +1, -1 \rangle \langle -1, +1 | + | -1, +1 \rangle \langle +1, -1 |. \quad (\text{S22})$$

The actual transfer is $(2, -2)$, accumulating a phase 2π , so $\eta(B) = 0$ and the term is free. The transport distance, built from the real span of the transfer rows, sees only the primitive direction $(1, -1)$ and returns $\tau(B) = 3$. The phase spread therefore detects an exact cancellation that its computable proxy misses.

V. THE PAIRED-TWIST INEQUALITY

The remaining ingredient is quantitative: how much energy a rotation of a given angle can extract from a term. Twisting in both senses and averaging removes the linear response, leaving a cost second order in the angle — the familiar reason a slow twist is cheap — and the following inequality holds that statement at finite angle instead of expanding it.

For self-adjoint A and B ,

$$\left\| \frac{1}{2} (e^{-iA} B e^{iA} + e^{iA} B e^{-iA}) - B \right\| \leq 2 \sin^2 \left(\min \left\{ \|A\|, \frac{\pi}{2} \right\} \right) \|B\|. \quad (\text{S23})$$

Proof. Write $\mathcal{C}(B) = e^{-iA} B e^{iA}$ for the conjugation, which is an isometry. The paired average factorizes as

$$\frac{\mathcal{C} + \mathcal{C}^{-1}}{2} - I = \frac{1}{2} \mathcal{C}^{-1} (\mathcal{C} - I)^2, \quad (\text{S24})$$

which is the algebraic statement that the linear term cancels. Put $a = \|A\|$ and $V = e^{-iA}$. For $a \leq \pi/2$, functional calculus and $|e^{-it} - \cos a| \leq \sin a$ for $|t| \leq a$ give $\|V - \cos(a)I\| \leq \sin a$, hence

$$\|(\mathcal{C} - I)(B)\| = \|[V, B]\| \leq 2 \sin(a) \|B\|, \quad (\text{S25})$$

where subtracting the scalar $\cos(a)I$ is legitimate because it commutes with B . For $a \geq \pi/2$ the trivial bound $\|\mathcal{C} - I\| \leq 2$ applies. Substituting into Eq. (S24) gives Eq. (S23). $\square$

The coefficient cannot be improved. With $A_a = a \text{diag}(1, -1)$ and $B = \sigma_x$, direct computation gives

$$\left\| \frac{1}{2} (e^{-iA_a} B e^{iA_a} + e^{iA_a} B e^{-iA_a}) - B \right\| = 1 - \cos(2a) = 2 \sin^2 a, \quad 0 \leq a \leq \frac{\pi}{2}, \quad (\text{S26})$$

and such a two-level block can be embedded between product states of equal total S^z , so termwise charge conservation alone does not improve it.

Chaining Eq. (S23) through the exact logarithm and then through transport gives the term-by-term estimate that Result A sums,

$$\left\| \frac{1}{2} (U^{-1} B_X U + U B_X U^{-1}) - B_X \right\| \leq 2s(B_X)^2 \|B_X\| \leq 2s_\tau(B_X)^2 \|B_X\| \leq \frac{8\pi^2 S^2}{L^2} \tau(B_X)^2 \|B_X\|, \quad (\text{S27})$$

the last step being $\sin^2(\min\{x, \pi/2\}) \leq x^2$ with $x = 2\pi S \tau(B_X)/L$.

VI. TRANSLATION PHASE, ORTHOGONALITY, AND THE VARIATIONAL BOUND

A cheap state proves nothing unless it is orthogonal to the ground state. Orthogonality here is exact and comes from a phase the twist picks up under translation, not from any property of the ground state itself.

Reindexing Eq. (S8) using $Tq_j T^{-1} = q_{j+1}$ and $e^{2\pi i q_j} = \mathbb{1}$ gives

$$TUT^{-1} = Ue^{-2\pi i Q/L}, \quad \text{and iterating,} \quad T^p U T^{-p} = Ue^{-2\pi i p Q/L}. \quad (\text{S28})$$

Within a sector of charge N the twist thus multiplies the translation eigenvalue by $e^{-2\pi i p N/L}$, which is not 1 precisely when pN/L is not an integer — the fractional-filling condition, arriving here as an exact algebraic statement rather than an estimate.

Fix such a sector. Both H_L and T^p preserve it, and U commutes with Q , so $U^{\pm 1}\Psi$ stays in the sector for any Ψ in it. If the sector ground state Ψ is unique it is a T^p eigenvector, and by Eq. (S28) the states $U^{\pm 1}\Psi$ carry a different T^p eigenvalue; they are therefore exactly orthogonal to Ψ . Writing

$$e_\sigma = \langle U^\sigma \Psi, H_L U^\sigma \Psi \rangle - E_{0,N}, \quad \sigma = \pm 1, \quad (\text{S29})$$

both are nonnegative, and taking the ground-state expectation of Eq. (S27) summed over X gives

$$\frac{e_+ + e_-}{2} \leq 2 \sum_X s(B_X)^2 \|B_X\|. \quad (\text{S30})$$

At least one sign lies below the right-hand side, and that twisted state is a normalized trial state orthogonal to Ψ inside the sector; the variational principle restricted to the sector then gives Result A. If the sector ground energy is degenerate the conclusion is automatic, since $\Delta_N = 0$. A fractional sector always has dimension at least two, because the only one-dimensional charge sectors put every site in its lowest or highest charge state, for which pN/L is an integer.

For Result D, global $SU(2)$ invariance and uniqueness force the ground state to be a singlet, so $N = LS$; its translation phase is nontrivial exactly when pS is not an integer. If LS is not an integer a singlet is impossible outright, since Q has integer spectrum, and Result C applies.

VII. HAMILTONIAN TRANSPORT SEMINORMS AND DUALITY CERTIFICATES

Result A holds for every admissible decomposition, and the one-term decomposition H_L itself is always admissible. Taking infima therefore gives Result B immediately. What this section establishes is that the resulting quantities are seminorms, and that each is maximal among seminorms obeying its specified one-term bound. This restricted maximality does not make the resulting energy estimate exact or preclude a sharper spectral argument, such as Sec. XIII.

Seminorm properties. If $H = \sum_a B_a$ and $K = \sum_b C_b$ are admissible, their union is an admissible decomposition of $H + K$, so all three quantities are subadditive:

$$\mathcal{B}(H + K) \leq \mathcal{B}(H) + \mathcal{B}(K), \quad \mathcal{B}_\tau(H + K) \leq \mathcal{B}_\tau(H) + \mathcal{B}_\tau(K), \quad \mathcal{T}_2(H + K) \leq \mathcal{T}_2(H) + \mathcal{T}_2(K). \quad (\text{S31})$$

For a nonzero real μ the conserved weights and the phase graph are unchanged, $\mathcal{K}_X(\mu B) = \mathcal{K}_X(B)$ and $\Gamma_{\mu B} = \Gamma_B$, while $\|\mu B\| = |\mu| \|B\|$; scaling decompositions both ways gives exact homogeneity,

$$\mathcal{B}(\mu H) = |\mu| \mathcal{B}(H), \quad \mathcal{B}_\tau(\mu H) = |\mu| \mathcal{B}_\tau(H), \quad \mathcal{T}_2(\mu H) = |\mu| \mathcal{T}_2(H). \quad (\text{S32})$$

They are seminorms rather than norms: for example, every product-charge diagonal operator has zero cost in all three. This common nonzero kernel does not assert that their kernels coincide for general higher-spin terms. They are independent of any chosen presentation of the interaction, but remain finite-volume quantities tied to the fixed lattice and on-site charge.

Maximality among seminorms with the same one-term bound. Let p_* be any seminorm on globally $U(1)$ -invariant Hermitian operators satisfying the one-term bound $p_*(B) \leq s_\tau(B)^2 \|B\|$. For every admissible decomposition, subadditivity gives $p_*(H) \leq \sum_a p_*(B_a) \leq \sum_a s_\tau(B_a)^2 \|B_a\|$, and taking the infimum yields $p_*(H) \leq \mathcal{B}_\tau(H)$. Conversely the one-term decomposition shows $\mathcal{B}_\tau$ itself obeys the one-term bound. Hence $\mathcal{B}_\tau$ is the pointwise largest seminorm with that property, and the same argument applies to $\mathcal{B}$ and $\mathcal{T}_2$ with their own one-term costs.

Lower bounds by duality. Proving that a decomposition is optimal requires a certificate, and duality supplies one. Let $\mathcal{V}$ be the finite-dimensional real vector space of globally $U(1)$ -invariant Hermitian operators, let $w(B, X)$ denote any of $s(B)^2\|B\|$, $s_\tau(B)^2\|B\|$ or $\tau(B)^2\|B\|$, and let p_w be the corresponding seminorm. Then

$$p_w(H) = \max_{\ell \in \mathcal{V}^*} \{ \ell(H) : |\ell(B)| \leq w(B, X) \text{ for every admissible labelled term } (B, X) \}. \quad (\text{S33})$$

A feasible linear functional ℓ therefore certifies a lower bound on the cost, term by term, for every decomposition at once.

Proof. Any feasible ℓ bounds each decomposition termwise, so $\ell(H) \leq p_w(H)$. Conversely, if $p_w(H) > 0$, the functional $f(aH) = ap_w(H)$ on the line $\mathbb{R}H$ is dominated by the sublinear functional p_w ; the real Hahn–Banach theorem extends it to $\ell \in \mathcal{V}^*$ with $\ell(K) \leq p_w(K)$ for all K , and applying this to $-K$ gives $|\ell(K)| \leq p_w(K)$. The one-term decomposition gives $p_w(B) \leq w(B, X)$, so the extension is feasible and attains $\ell(H) = p_w(H)$. If $p_w(H) = 0$ the zero functional suffices. $\square$

Because Eq. (S33) never refers to a minimizing decomposition, a lower certificate does not require the infimum to be attained.

The XY ring, evaluated exactly. Let $L \geq 3$, let every site carry spin $1/2$, and take the hopping Hamiltonian

$$H_{\text{hop}} = \sum_j (t_j S_j^+ S_{j+1}^- + \bar{t}_j S_j^- S_{j+1}^+). \quad (\text{S34})$$

Let Z be Hermitian, globally $U(1)$ invariant, of vanishing cost $\mathcal{T}_2(Z) = 0$, and with no nearest-neighbour off-diagonal matrix elements in the one-magnon sector. Then

$$\mathcal{T}_2(H_{\text{hop}} + Z) = \sum_j |t_j|. \quad (\text{S35})$$

The cost is carried entirely by the hopping amplitudes; whatever else is present contributes nothing.

Proof of the lower bound. Write $|j\rangle$ for the state with a single up spin at site j . For an admissible B on X , build a graph G_B on X joining $u \neq v$ when $\langle u|B|v\rangle \neq 0$. Charge conservation prevents one-magnon edges from leaving X , and each edge contributes the transfer row $e_u - e_v$; hence any conserved weight is constant on each nontrivial component ω of G_B , and dropping the remaining constraints gives

$$\tau(B) \geq \sum_{\omega} \delta(\omega), \quad \delta(\omega) = \min_c \sum_{j \in \omega} \text{dist}_L(j, c), \quad (\text{S36})$$

where δ is the circular dispersion about the best centre. Let $m(\omega)$ count nearest-neighbour ring edges with both ends in ω . If ω is a proper subset then $m(\omega) \leq |\omega| - 1$, and packing $|\omega|$ distinct sites as close as possible around a centre gives distances $0, 1, 1, \dots$ for odd $|\omega|$ and $\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{3}{2}, \dots$ for even $|\omega|$, both summing to

$$\delta(\omega) \geq \left\lfloor \frac{|\omega|^2}{4} \right\rfloor \geq |\omega| - 1. \quad (\text{S37})$$

If ω is the whole ring then $m(\omega) = L$ and $\delta(\omega) = \lfloor L^2/4 \rfloor$, whose square is at least L . Either way

$$\sum_{\omega} m(\omega) \leq \sum_{\omega} \delta(\omega)^2 \leq \left(\sum_{\omega} \delta(\omega) \right)^2 \leq \tau(B)^2, \quad (\text{S38})$$

and since every matrix element is bounded by $\|B\|$,

$$\sum_j |\langle j|B|j+1\rangle| \leq \tau(B)^2 \|B\|. \quad (\text{S39})$$

Set $\phi_j = t_j/|t_j|$ when $t_j \neq 0$ and $\phi_j = 0$ otherwise, and define $\ell_t(K) = \sum_j \text{Re}[\bar{\phi}_j \langle j|K|j+1\rangle]$. Equation (S39) makes ℓ_t feasible in Eq. (S33); the hypothesis on Z gives $\ell_t(Z) = 0$, while $\ell_t(H_{\text{hop}}) = \sum_j |t_j|$. The upper bound follows from subadditivity and the bond decomposition, each bond having transport distance one and norm $|t_j|$. $\square$

The same evaluation at finite angle. For $L \geq 6$ the sharper quantity can also be computed. Let A be the one-magnon restriction of an admissible B and let F be the set of ring edges j with $\langle j|B|j+1 \rangle \neq 0$. If F is nonempty, let R_B be the Hermitian signed edge matrix carrying the conjugate phase of $\langle j|B|j+1 \rangle$ on each oriented edge. Compression gives $\|A\| \leq \|B\|$, the two orientations give $\text{Tr}(R_B A) = 2 \sum_j |\langle j|B|j+1 \rangle|$, and Schatten 1- ∞ duality yields

$$\sum_j |\langle j|B|j+1 \rangle| = \frac{1}{2} \text{Tr}(R_B A) \leq g(F) \|B\|, \quad g(F) = \frac{1}{2} \|R_B\|_*. \quad (\text{S40})$$

If F is not the whole ring, its runs of consecutive edges have lengths $r_1, \dots, r_c$; a run of r_i edges is unitarily equivalent, after gauging away the phases along it, to the adjacency matrix of a path on $r_i + 1$ vertices, whose eigenvalues are $2 \cos[k\pi/(r_i + 2)]$. Hence

$$g(F) = \sum_{i=1}^c \sum_{k=1}^{r_i+1} \left| \cos \frac{k\pi}{r_i+2} \right|, \quad D(F) = \sum_{i=1}^c \left\lfloor \frac{(r_i+1)^2}{4} \right\rfloor, \quad (\text{S41})$$

and the gauge freedom is why aligned edge phases do not enter. For the full ring set $D(F) = \lfloor L^2/4 \rfloor$. Distinct runs occupy disjoint vertex sets and merging them cannot lower the summed dispersions, so Eq. (S36) gives $\tau(B) \geq D(F)$, while subadditivity of the nuclear norm over single edges gives $g(F) \leq |F| \leq D(F)$.

Put $x = \pi/L$. If $D(F) < L/2$ then $D(F) \sin^2 x \leq \sin^2(D(F)x)$: the cases $D = 1, 2$ are direct, and for $D \geq 3$ the sine chord gives $\sin(Dx) \geq (2D/\pi) \sin x \geq \sqrt{D} \sin x$. If $D(F) \geq L/2$ the cost is clipped at one, and it remains to check that the left side cannot exceed it. For $L \geq 7$, with $m = |F|$ edges and at most $n \leq L$ incident vertices, the Frobenius bound gives $g(F) \leq \sqrt{mn}/2 \leq L/\sqrt{2}$ and hence $g(F) \sin^2(\pi/L) \leq \pi^2/(\sqrt{2}L) < 1$. For $L = 6$: if $m \leq 4$ then $g(F) \leq m \leq 4$; if $m = 5$ the six-vertex path has $g(F) = \csc(\pi/14) - 1 < 4$; and if $m = 6$, pairing antipodal ring eigenvalues gives $g(F) = 2 \sum_{k=0}^2 |\cos(\theta + k\pi/3)| \leq 4$. Since $\sin^2(\pi/6) = 1/4$, the endpoint holds too. Therefore

$$\sin^2 \frac{\pi}{L} \sum_j |\langle j|B|j+1 \rangle| \leq s_\tau(B)^2 \|B\|, \quad (\text{S42})$$

so $\sin^2(\pi/L) \ell_t$ is feasible in the finite-angle version of Eq. (S33). Its value on $H_{\text{hop}} + Z$ is $\sin^2(\pi/L) \sum_j |t_j|$, and since $\mathcal{B}_\tau(Z) \leq (2\pi S/L)^2 \mathcal{T}_2(Z) = 0$ the bond decomposition matches it. Hence

$$\mathcal{B}_\tau(H_{\text{hop}} + Z) = \sin^2 \frac{\pi}{L} \sum_j |t_j|, \quad L \geq 6. \quad (\text{S43})$$

For the uniform XY ring, $t_j = J/2$ gives $\mathcal{T}_2 = L|J|/2$. Presented instead as a single ring-wide term the same Hamiltonian has $\tau = \lfloor L^2/4 \rfloor$ and costs at least $|J| \lfloor L^2/4 \rfloor^2$. The bond decomposition therefore attains the infimum and removes a genuine inflation that has nothing to do with the physics. More generally, if p divides L , $t_{j+p} = t_j$, and Z also commutes with T^p , then for pN/L not an integer

$$\Delta_N(H_{\text{hop}} + Z) \leq 2 \sin^2 \frac{\pi}{L} \sum_j |t_j| = \frac{2L}{p} \sin^2 \frac{\pi}{L} \sum_{a=0}^{p-1} |t_a|. \quad (\text{S44})$$

For real p -periodic exchange and Dzyaloshinskii-Moriya couplings, $J_j(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + D_j(S_j^x S_{j+1}^y - S_j^y S_{j+1}^x)$ corresponds to $t_j = (J_j + iD_j)/2$, which proves the unit-cell bound of the main text for dimerized chains with Dzyaloshinskii-Moriya coupling. At $p = 2$ and half filling $pN/L = 1$, so the hypothesis fails and the corollary correctly makes no claim about the dimerization gap.

VIII. TRANSFER MATRICES AND THE TRANSPORT DISTANCE

This section turns $\tau(B)$ from a definition into something one evaluates, gives its closed form for the processes that occur in practice, and shows why the infimum over decompositions in Eq. (S2) cannot be dispensed with.

Let $|a\rangle$ be the product S^z basis and $M(c) = \sum_j c_j S_j^z$. Its commutator has matrix elements

$$\langle a|[B, M(c)]|b\rangle = \langle a|B|b\rangle \sum_j c_j [m_j(b) - m_j(a)], \quad (\text{S45})$$

so $[B, M(c)] = 0$ exactly when c is orthogonal to every charge transfer the term actually performs. Collecting those transfers as the rows of a matrix D_B , and writing $x = (j)_{j \in X}$ for the site positions,

$$\mathcal{K}_X(B) = \ker_{\mathbb{R}} D_B, \quad \tau(B) = \inf_{n \in \mathbb{Z}^X} \text{dist}_1(x + Ln, \ker D_B). \quad (\text{S46})$$

The transport distance is thus the ℓ^1 distance from the site positions to the space of conserved weights, minimized over ways of counting sites around the seam. Since exact zeros of D_B can be read off a term by inspection, this is a certificate rather than a numerical search.

Lower bounds come from duality. For a subspace K of a finite-dimensional real space with the ℓ^1 norm, the dual of the quotient is the annihilator $K^\perp$ with the restricted ℓ^∞ norm, so

$$\text{dist}_1(v, K) = \max_{y \in K^\perp, \|y\|_\infty \leq 1} y \cdot v, \quad (\text{S47})$$

and $(\ker D_B)^\perp = \text{ran } D_B^T$ identifies the witnesses as combinations of the transfer rows themselves.

Closed form for a fixed transfer pattern. Suppose the nonzero integer rows of D_B span a one-dimensional real space with primitive generator $d \in \mathbb{Z}^X$, unique up to sign with $\gcd_j |d_j| = 1$. Then

$$\tau(B) = \frac{\text{dist}(d \cdot x, L\mathbb{Z})}{\|d\|_\infty}. \quad (\text{S48})$$

The term costs what its charge displacement $d \cdot x$ falls short of a whole number of turns around the ring.

Proof. Here $\ker D_B = \ker d$, and Eq. (S47) gives $\text{dist}_1(v, \ker d) = |d \cdot v| / \|d\|_\infty$ for each fixed lift $v = x + Ln$. Primitivity and Bézout's identity make $\{d \cdot n : n \in \mathbb{Z}^X\} = \mathbb{Z}$, so minimizing over n moves $d \cdot x$ to the nearest multiple of L . Shifting a representative by L changes $d \cdot x$ by a multiple of L , and flipping the sign of d changes nothing. $\square$

For a spin-1/2 term with fixed transfer, the transfer vector $d \in \{-1, 0, 1\}^X$ is automatically primitive and the Hermitian conjugate supplies $-d$, so

$$\tau(B) = \text{dist}\left(\sum_{j \in X} d_j j, L\mathbb{Z}\right), \quad (\text{S49})$$

which vanishes exactly when the wrapped polarization displacement is a multiple of the circumference. This covers ordinary dipole-neutral processes and processes that wrap the seam alike. At higher spin an actual transfer may be a non-primitive multiple of d , and its finite twist phase can then cancel even when Eq. (S48) is positive — Eq. (S22) is such a case. The formula concerns the real span of the transfer rows, which is what defines τ , not the integer lattice they generate.

Non-additivity, and the need for the decomposition infimum. The cost of a single term is not additive, so a Hamiltonian must be allowed to choose how it is written. Take $L = 6$, sites $x = (0, 1, 3, 4)$, and two Hermitian spin-1/2 fixed-transfer terms with

$$d_1 = (-1, -1, 1, 1), \quad d_2 = (-1, 1, 1, -1). \quad (\text{S50})$$

Their displacements are 6 and 0, so by Eq. (S48) each term is free. The transfer kernel of the *sum*, however, is $\{(a, b, a, b)\}$, and for every integer lift $v = x + 6n$,

$$\text{dist}_1(v, \ker D) = |v_0 - v_2| + |v_1 - v_3| \geq 3 + 3 = 6, \quad (\text{S51})$$

with equality attained — the sum, taken as one term, costs 6. Keeping the two summands as separate admissible terms instead proves that all three Hamiltonian costs of the sum vanish. A specified term with a specified lift is certified by Eq. (S46); optimizing over all decompositions is a separate algorithmic question, and any explicit decomposition already gives a rigorous upper bound on the seminorm and hence on the gap.

Finally, if the transfer rows are the edge vectors $e_i - e_j$ of a graph, a weight lies in the kernel exactly when it is constant on each component. Integer lifts and constants then separate componentwise, and minimizing their ℓ^1 costs gives the sum of circular dispersions used in Eq. (S36).

IX. REDUCTION TO FINITE-RANGE GEOMETRY

Replacing the conserved weights in the definition (4) of the transport distance by constant vectors — the crudest admissible choice — turns transport back into geometry and reproduces the standard finite-range theorem with an

explicit constant. This section carries out that reduction, and so shows that Result B is never weaker than the familiar estimate.

For a set of sites X define its circular dispersion

$$\delta(X) = \min_c \sum_{j \in X} \text{dist}_L(j, c), \quad (\text{S52})$$

the smallest total distance from the sites of X to a common centre on the ring; taking c constant in the definition of τ gives $\tau(B) \leq \delta(X)$ whenever B is supported on X .

Suppose every support has cycle diameter at most R and the local strength is bounded,

$$\sup_x \sum_{X \ni x} \|\Phi_X\| \leq J. \quad (\text{S53})$$

If $L > 2R$, anchor X at one of its sites and give every other site its unique principal displacement in $[-R, R]$; distinct sites take distinct values, so the worst case uses every nonzero integer there and $\delta(X) \leq R(R+1)$. If $L > 3R$ the support lifts to an integer interval of diameter at most R , and among at most one site per position the sum of absolute deviations about a median is largest when every position is occupied,

$$\max_{Y \subset \{0, \dots, R\}} \min_c \sum_{j \in Y} |j - c| = \left\lfloor \frac{(R+1)^2}{4} \right\rfloor = \kappa_R, \quad (\text{S54})$$

with equality for the full consecutive set. Summing Eq. (S53) over sites gives $\sum_X |X| \|\Phi_X\| \leq LJ$, and since every term has $|X| \geq 1$ and the charge average contracts norms, $\sum_X \|B_X\| \leq LJ$. Combining with Result A, for every N with pN/L not an integer,

$$\Delta_N(H_L) \leq \frac{8\pi^2 S^2 R^2 (R+1)^2 J}{L} \quad (L > 2R), \quad \Delta_N(H_L) \leq \frac{8\pi^2 S^2 \kappa_R^2 J}{L} \quad (L > 3R). \quad (\text{S55})$$

Both are the familiar $O(1/L)$, now with the geometric constant displayed. Result B can be sharper when additional conserved weights reduce the quotient distance; additional weights alone need not improve the constant-vector choice strictly.

X. LONG-RANGE DECAY CLASSES

Specialize to one site per cell, $p = 1$; all bounds hold for every N with N/L not an integer. The point of this section is that the two thresholds known in the literature are the two ways transport can behave as the range grows, and both follow from the same functional.

For one translation orbit of two-site exchange terms at each cycle distance r , with $\|B_{j,r}\| \leq C(1+r)^{-\alpha}$, the exchange moves charge the full distance, $\tau(B_{j,r}) \leq r$, and Result B gives

$$\Delta_N(H_L) \leq \frac{8\pi^2 S^2 C}{L} \sum_{r=1}^{\lfloor L/2 \rfloor} \frac{r^2}{(1+r)^\alpha}. \quad (\text{S56})$$

This is $O(L^{-1})$ for $\alpha > 3$, $O(L^{-1} \log L)$ at $\alpha = 3$, and $O(L^{2-\alpha})$ for $2 < \alpha < 3$; in every case it vanishes for $\alpha > 2$. The two powers of r are the cost of transporting charge across the term.

If instead the transport distance stays bounded, $\tau(B_{j,r}) \leq \tau_0$ uniformly in j , r and L — as for a product of two separately neutral clusters of bounded internal spread — those two powers disappear and

$$\Delta_N(H_L) \leq \frac{8\pi^2 S^2 C \tau_0^2}{L} \sum_{r=1}^{\lfloor L/2 \rfloor} \frac{1}{(1+r)^\alpha}, \quad (\text{S57})$$

which is $O(L^{-1})$ for $\alpha > 1$, $O(L^{-1} \log L)$ at $\alpha = 1$, and $O(L^{-\alpha})$ for $0 < \alpha < 1$, so it vanishes for every $\alpha > 0$. Equations (S56) and (S57) are the one-dimensional Type-I and Type-II thresholds of Ref. [19], recovered here as the two regimes of a single quantity.

More generally, for any sequence of rings and any chosen sequence of fractional sectors,

$$\mathcal{T}_2(H_L) = o(L^2) \implies \Delta_{N_L}(H_L) \rightarrow 0. \quad (\text{S58})$$

XI. DIRECTION-RESOLVED BOUNDS ON CYLINDERS AND LADDERS

Twisting one direction of a cylinder charges an interaction only for the displacement it produces along that direction. This section proves the directional bound, shows that intra-slice dynamics is exactly free, and works out the width-decay crossover it implies.

Take L slices around a periodic direction, each slice a copy of a finite graph with W sites, and let each site carry a finite-dimensional charge with integer spectrum. Twist only the periodic coordinate,

$$U^\parallel = \exp\left(\frac{2\pi i}{L} \sum_x \sum_a x q_{x,a}\right), \quad (\text{S59})$$

where x indexes slices and a indexes sites within a slice. With T advancing x and preserving a , integer charge gives exactly as before

$$T^p U^\parallel T^{-p} = U^\parallel e^{-2\pi i p Q/L}. \quad (\text{S60})$$

Assume p divides L , $[H_L, Q] = 0$ and $[H_L, T^p] = 0$. Now repeat Sec. IV verbatim, with each charge assignment carrying the longitudinal phase

$$\theta_\lambda^\parallel = \frac{2\pi}{L} \sum_{(x,a) \in X} x \lambda_{x,a} \pmod{2\pi}, \quad (\text{S61})$$

which counts only how far along the ring the charge sits, ignoring where it lies within a slice. This defines a directional spread $\eta_\parallel(B)$, a directional term cost $s_\parallel(B) = \sin(\min\{\eta_\parallel(B), \pi/2\})$, and a directional Hamiltonian cost $\mathcal{B}_\parallel$ by the same infimum. Then for every N with pN/L not an integer,

$$\Delta_N(H_L) \leq 2\mathcal{B}_\parallel(H_L). \quad (\text{S62})$$

Proof. Apply the charge average of Sec. II to any supported Hermitian decomposition; it contracts norms and produces $[B_X, Q_X] = 0$, and it commutes with conjugation by the local longitudinal twist since both are generated by the same commuting charges, so a twist-invariant term stays invariant. The exact-logarithm theorem and Eq. (S23) then give

$$\left\| \frac{1}{2} ((U^\parallel)^{-1} B_X U^\parallel + U^\parallel B_X (U^\parallel)^{-1}) - B_X \right\| \leq 2s_\parallel(B_X)^2 \|B_X\|. \quad (\text{S63})$$

By Eq. (S60) each of $(U^\parallel)^{\pm 1} \Psi$ is orthogonal to a unique sector ground state Ψ , though the two twisted states need not be orthogonal to each other. Averaging their excitation energies, applying the sector variational principle, and taking the decomposition infimum gives Eq. (S62); a degenerate sector ground energy makes it automatic. The same proof works for any chosen periodic direction of a finite product torus, one direction at a time; it does not combine several twists into a higher-dimensional flux-insertion theorem. $\square$

Exact vanishing of the intra-slice cost. If B is supported inside the single slice $x = x_0$, its local longitudinal twist is $e^{2\pi i x_0 Q_X/L}$, a function of the charge on X alone, and so commutes with B . Hence

$$\eta_\parallel(B) = 0. \quad (\text{S64})$$

Arbitrary charge-conserving transverse hopping, intra-slice many-body interaction, and intra-slice disorder therefore cost nothing along the ring, whatever their strength, order, or transverse range; so do product-charge diagonal interactions of arbitrary longitudinal range.

Selective hopping. Let every site carry integer levels $0, \dots, M$ with $M \geq 1$. Choose one oriented representative e of each Hermitian hopping bond, running from a site in one slice to a site r_e slices along, and write

$$H_L = \sum_e (t_e |1, 0\rangle \langle 0, 1|_e + \bar{t}_e |0, 1\rangle \langle 1, 0|_e) + V, \quad (\text{S65})$$

where the bonds and amplitudes are p -periodic, $[V, Q] = [V, T^p] = 0$, and V decomposes into charge-conserving terms commuting with their own local longitudinal twists. Writing ℓ_e for the distance from r_e to the nearest multiple of L , the edge term has norm $|t_e|$ and directional spread $\pi \ell_e/L$, so Eq. (S62) gives

$$\Delta_N(H_L) \leq 2 \sum_e |t_e| \sin^2 \frac{\pi \ell_e}{L}, \quad pN/L \notin \mathbb{Z}. \quad (\text{S66})$$

Nothing here depends on transverse separation, on the unused charge levels, or on any admitted term of V , and edges with $\ell_e = 0$ contribute exactly zero. For the fiber-preserving nearest-neighbour subclass with $t_{x,a} = t_{c,a}$, $c = x \bmod p$,

$$\Delta_N(H_L) \leq \frac{2L}{p} \sin^2 \frac{\pi}{L} \sum_a \sum_{c=0}^{p-1} |t_{c,a}|, \quad \text{and for uniform } t, \quad \Delta_N(H_L) \leq 2LW|t| \sin^2 \frac{\pi}{L} \leq \frac{2\pi^2 W|t|}{L}. \quad (\text{S67})$$

The bound vanishes for bounded hopping whenever $W = o(L)$. At isotropic aspect ratio $W \asymp L$ this uniform-twist estimate is only $O(1)$ and makes no two-dimensional claim. At $W = 1$ it returns the multilevel chain, and its independence of M removes the spurious factor M^2 that the charge-width certificate of Eq. (S19) produces in the quadratic relaxation.

Power-law channels. Write $n = \lfloor L/2 \rfloor$ and let

$$A_L(r) = \sum_{e: \ell_e=r} |t_e| \quad (\text{S68})$$

be the total amplitude moving charge exactly r slices. Grouping Eq. (S66) by r gives the exact profile estimate $\Delta_N \leq 2 \sum_r A_L(r) \sin^2(\pi r/L)$, and assuming $A_L(r) \leq c_0 L K_L r^{-\alpha}$ with $\alpha > 2$ and a channel mass K_L , the inequality $\sin x \leq x$ yields

$$\Delta_N(H_L) \leq \frac{2\pi^2 c_0 K_L}{L} \sum_{r=1}^n r^{2-\alpha}, \quad \sum_{r=1}^n r^{2-\alpha} \leq \begin{cases} \zeta(\alpha-2), & \alpha > 3, \\ 1 + \log n, & \alpha = 3, \\ 1 + \frac{n^{3-\alpha} - 1}{3-\alpha}, & 2 < \alpha < 3. \end{cases} \quad (\text{S69})$$

For $K_L = O(L^\beta)$ the three branches scale as $O(L^{\beta-1})$, $O(L^{\beta-1} \log L)$ and $O(L^{\beta+2-\alpha})$, so the bound vanishes throughout

$$\beta < \min\{1, \alpha - 2\}, \quad (\text{S70})$$

with $K_L = o(L/\log L)$ at the dipolar boundary $\alpha = 3$. The hypothesis $K_L = O(W)$ is a statement about the amplitude that actually moves charge between slices; it holds, for instance, when each distance and longitudinal cell carries $O(W)$ moving bonds with bounded prefactors. Bounded degree of the slice graph is not sufficient, since a dense inter-slice graph can carry a larger channel mass — while purely transverse charge-conserving dynamics remains exactly free.

Dense Euclidean thin torus. Take $p = 1$, let the slice be a ring of $W \leq n$ sites, and allow every pair of transverse channels to couple. For one representative of each Hermitian longitudinal bond let $r_e \geq 1$ be the longitudinal and $s_e \geq 0$ the transverse cycle distance, and assume the Euclidean tail

$$|t_e| \leq J(r_e^2 + s_e^2)^{-\alpha/2}, \quad \alpha > 2. \quad (\text{S71})$$

Let $\nu_W(s) = 1$ if W is even and $s = W/2$, and $\nu_W(s) = 2$ otherwise, so that $1 + \sum_s \nu_W(s) = W$ counts the transverse antipode once. When L is even, $r = L/2$ is represented by the half longitudinal orbit and translation covariance forces the corresponding channel matrix to be Hermitian, with self-reversing simultaneous-antipode entries real, so each physical bond is counted exactly once. Counting longitudinal translates and transverse sources,

$$A_{L,W}(r) \leq J L W \left[r^{-\alpha} + \sum_{s=1}^{\lfloor W/2 \rfloor} \nu_W(s) (r^2 + s^2)^{-\alpha/2} \right]. \quad (\text{S72})$$

With

$$c_\alpha = 1 + \frac{\sqrt{\pi} \Gamma((\alpha-1)/2)}{\Gamma(\alpha/2)}, \quad (\text{S73})$$

the transverse summand is decreasing and comparison with its integral over the half-line gives two complementary estimates for the bracket, namely $\leq c_\alpha r^{1-\alpha}$ and $\leq W r^{-\alpha}$. Using the first for $r \leq W$ and the second for $r > W$, then $\sin x \leq x$,

$$\Delta_N(H_L) \leq \frac{2\pi^2 J W}{L} \left[c_\alpha \sum_{r=1}^W r^{3-\alpha} + W \sum_{r=W+1}^n r^{2-\alpha} \right]. \quad (\text{S74})$$

Integral comparison gives the explicit branches

$$\Delta_N(H_L) \leq \begin{cases} \frac{2\pi^2 JW}{L} \left[c_\alpha \zeta(\alpha - 3) + \frac{1}{\alpha - 3} \right], & \alpha > 4, \\ \frac{2\pi^2 JW}{L} [(1 + \pi/2)(1 + \log W) + 1], & \alpha = 4, \\ \frac{2\pi^2 JW^{5-\alpha}}{L} \left[\frac{c_\alpha}{4 - \alpha} + \frac{1}{\alpha - 3} \right], & 3 < \alpha < 4, \\ \frac{2\pi^2 JW^2}{L} \left[3 + \log \frac{n}{W} \right], & \alpha = 3, \\ \frac{2\pi^2 JW^2}{L} n^{3-\alpha} \left[c_\alpha + \frac{1}{3 - \alpha} \right], & 2 < \alpha < 3. \end{cases} \quad (S75)$$

At the dipolar endpoint the constant is not hidden inside an asymptotic statement: $c_3 = 1 + \sqrt{\pi} \Gamma(1)/\Gamma(3/2) = 3$, and $\sum_{r=W+1}^n r^{-1} \leq \log(n/W)$, so the near part contributes $3W$ and the far part at most $W \log(n/W)$ inside the bracket. This is the dipolar thin-torus bound quoted in the main text, and it vanishes whenever $W^2[1 + \log(L/W)] = o(L)$; a charge-moving van der Waals tail gives $O(W/L)$. For $W = O(L^\beta)$ the strict width thresholds are

$$\beta < \begin{cases} (\alpha - 2)/2, & 2 < \alpha \leq 3, \\ 1/(5 - \alpha), & 3 \leq \alpha < 4, \\ 1, & \alpha \geq 4, \end{cases} \quad (S76)$$

with logarithmic refinements at $\alpha = 3, 4$. This is a sufficient anisotropic region, not a converse and not an isotropic two-dimensional theorem. Dipolar spin exchange in polar-molecule arrays supplies the relevant charge-conserving mechanism [15–17], though we do not assert that those experiments realize the periodic thin-torus geometry, filling, or spectral hypotheses assumed here. A coherently driven Rydberg Ising simulator [31], by contrast, does not conserve Rydberg occupation and is not covered merely because its density interaction is van der Waals.

Odd-leg spin-1/2 ladder. Let L be even and W odd, and take

$$H_L = \sum_x \sum_{a=1}^W J_a \mathbf{S}_{x,a} \cdot \mathbf{S}_{x+1,a} + \sum_x h_x^{\text{rung}} + Z, \quad (S77)$$

with real J_a , the h_x^{rung} translates of an arbitrary $\text{SU}(2)$ -invariant Hermitian operator supported in one rung, and Z translation and $\text{SU}(2)$ invariant with a decomposition into terms commuting with their local longitudinal twists. No norm or interaction-order bound is placed on h_x^{rung} or on the admitted terms of Z . Using $q_{x,a} = S_{x,a}^z + 1/2$, every rung term is longitudinally free by Eq. (S64), and the charge-moving part of a longitudinal Heisenberg bond has norm $|J_a|/2$ and spread π/L . If the ground state is degenerate the full gap is zero; otherwise $\text{SU}(2)$ invariance makes it a singlet with $N = LW/2$, and W odd with L even makes N/L non-integer. Hence

$$\Delta(H_L) \leq L \sin^2 \frac{\pi}{L} \sum_{a=1}^W |J_a| \leq \frac{\pi^2}{L} \sum_{a=1}^W |J_a|, \quad (S78)$$

which for uniform longitudinal exchange is $\pi^2 W |J_\parallel|/L$ and vanishes for bounded $J_\parallel$ whenever $W = o(L)$. If instead the longitudinal exchanges are long ranged with $\sum_a |J_{a,r}| \leq CW r^{-\alpha}$ at distance r , the same counting gives

$$\Delta(H_L) \leq L \sum_{r,a} |J_{a,r}| \sin^2 \frac{\pi r}{L} \leq \frac{\pi^2 CW}{L} \sum_{r=1}^n r^{2-\alpha}. \quad (S79)$$

Fixed-width ladder statements are known [5, 29], as are higher-dimensional bounds on tori [8]; what is recorded here is the direction-resolved finite-volume cost, the crossover as the width grows, and exact independence from arbitrary admitted rung dynamics.

XII. THE SPIN-1/2 MODEL WITH ZERO-COST TAILS

Here we derive the bound quoted in the main text for that model, with every finite-size factor explicit, and verify that both classes of free interaction are indeed free.

Write the transverse bond of the model as

$$B_j = J_\perp (S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) = \frac{J_\perp}{2} (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+). \quad (\text{S80})$$

On the span of $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ it is the off-diagonal matrix with entries $J_\perp/2$ and it annihilates the other two product states, so $\|B_j\| = |J_\perp|/2$. For $J_\perp \neq 0$ its transfer graph is the single edge $(j, j+1)$ and $\tau(B_j) = 1$; otherwise $\tau(B_j) = 0$.

Diagonal tails. Any operator diagonal in the product S^z basis commutes with $\sum_j c_j S_j^z$ for *every* real c , so its conserved-weight space is all of $\mathbb{R}^X$ and the infimum in τ is zero. Hence $\tau(V) = 0$ even when the whole diagonal part V is retained as a single ring-wide term, and no decay condition on its couplings is needed anywhere.

Polarization-neutral hoppings. Write $Z = \sum_\gamma B_\gamma$, keeping each Hermitian fixed-transfer spin-1/2 monomial as a separate admissible term. If its transfer vector satisfies $\sum_j d_j^{(\gamma)} j \in L\mathbb{Z}$ then Eq. (S49) gives $\tau(B_\gamma) = 0$. A nontrivial zero-cost spin-1/2 transfer cannot consist of one raising and one lowering operator, since that is a transfer between two distinct ring sites and has nonzero cycle distance; every such B_γ therefore carries at least two raising and two lowering operators and annihilates the one-magnon sector. The displayed decomposition thus proves

$$\mathcal{T}_2(Z) = \mathcal{B}_\tau(Z) = \mathcal{B}(Z) = 0, \quad (\text{S81})$$

even though $\tau(Z)$ computed for the sum as one term need not vanish — the non-additivity of Eq. (S50) made explicit.

For the counterflow term of the main text, use the integer lifts $(j, j+1, j+r, j+r+1)$, which are valid even when either bond crosses the seam. The first monomial has transfer vector $d = (1, -1, -1, 1)$ and its conjugate has $-d$, and

$$d \cdot (j, j+1, j+r, j+r+1) = 0. \quad (\text{S82})$$

The two hops move charge in opposite senses by equal amounts, so the term commutes with the weighted charge whose coefficients are those lifts, and choosing that same pair in the definition of τ gives $\tau(Z_{j,r}) = 0$ exactly. For $u_r \neq 0$ the term has nonzero off-diagonal matrix elements in the product basis; when L is even and $r = L/2$, translation by r exchanges the two monomials, so the orbit sum depends only on $\text{Re } u_r$, and at least one non-cancelling orbit produces a genuine off-diagonal interaction.

The exact cost and the resulting bound. Neither V nor Z has one-magnon nearest-neighbour matrix elements, so with $H_L = H_{\text{hop}} + V + Z$ the hypotheses of Eq. (S35) hold and

$$\mathcal{T}_2(H_L) = \frac{L|J_\perp|}{2}. \quad (\text{S83})$$

Result B evaluated on the bond decomposition already gives the bound of the main text. It can also be read directly off the exact twist response, which is instructive because no inequality is used until the last line. Put $\theta = 2\pi/L$. Diagonality of V and the displacement condition on Z make the uniform-twist phase of every displayed constituent cancel exactly, while the two hopping directions acquire opposite phases, so

$$\frac{U^{-1}H_{\text{hop}}U + UH_{\text{hop}}U^{-1}}{2} = \cos \theta H_{\text{hop}}, \quad [V + Z, U] = 0. \quad (\text{S84})$$

If the fractional-sector ground state Ψ is unique, the excitation energies $e_\pm$ of $U^{\pm 1}\Psi$ are nonnegative and their trial states are orthogonal to Ψ by Sec. VI, so

$$0 \leq \frac{e_+ + e_-}{2} = (\cos \theta - 1) \langle \Psi, H_{\text{hop}} \Psi \rangle \leq (1 - \cos \theta) \|H_{\text{hop}}\| \leq \frac{L|J_\perp|}{2} (1 - \cos \theta). \quad (\text{S85})$$

One twist sign lies below the average, and the sector variational principle, together with the automatic zero gap in the degenerate case, gives

$$\Delta_N(H_L) \leq L|J_\perp| \sin^2 \frac{\pi}{L} \leq \frac{\pi^2 |J_\perp|}{L}, \quad N/L \notin \mathbb{Z}. \quad (\text{S86})$$

The first form is strictly sharper than the quadratic one whenever $J_\perp \neq 0$, and Result C converts it into a full-gap bound when its spectral condition holds.

Comparison with a support-based estimate. For comparison, pricing the diagonal tail by its diameter assigns $r^2 \|V_r S_j^z S_{j+r}^z\| = r^2 |V_r|/4$ to each of the L terms at distance r ; multiplying by the spin-1/2 prefactor $2\pi^2/L^2$ gives an extra contribution

$$\frac{\pi^2}{2L} \sum_{r=1}^{\lfloor (L-1)/2 \rfloor} r^2 |V_r|, \quad (\text{S87})$$

which for $V_r = V/r^\alpha$ is the expression quoted in the main text. The restriction $r \leq \lfloor (L-1)/2 \rfloor$ makes r the unambiguous shortest cycle distance; a general diagonal V may also contain the even-ring antipodal term. The true contribution of all of it is zero.

XIII. OPTIMAL SECTOR ENVELOPE, EQUALITY, AND INTERACTING SATURATION

This section strengthens the exchange-only bound using the exact sector minimum of its kinetic term. It does not change the general-spin or finite-range hypotheses used earlier. Sites may be labelled $0, \dots, L-1$ instead of $1, \dots, L$: this changes U by a sector-scalar phase and hence changes neither a commutator nor an excitation energy. Let

$$K = \frac{J_\perp}{2} \sum_j (S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+), \quad H = K + H', \quad [H', Q] = [H', T] = [H', U] = 0. \quad (\text{S88})$$

Throughout $S = 1/2$, $L \geq 3$, $1 \leq N \leq L-1$, and $J_\perp < 0$ or $J_\perp > 0$ with even L . No norm, decay, locality, or interaction-order bound on H' is imposed. The quantities C_N and g_N are defined in Eq. (13); throughout this section $\vartheta = \pi/L$, so that $k_F = N\vartheta$. All eigenvalues below are in the N sector and count multiplicity; a degenerate ground energy gives $\Delta_N = 0$. A statement about this sector is not automatically a statement about the full gap: the ground-sector condition in Result C remains necessary.

A. Free-chain input and normalization

For negative exchange the Jordan–Wigner transformation gives one-particle energies $-|J_\perp| \cos k$. Moving a fermion through the boundary produces the fixed-sector multiplier $(-1)^{N-1}$: the grid is periodic for odd N and antiperiodic for even N . Thus the occupied ground-state momenta are

$$k_\ell = 2\vartheta [\ell - (N-1)/2], \quad \ell = 0, \dots, N-1. \quad (\text{S89})$$

This is the appropriate parity grid in both cases. It fills every occupied $k, -k$ pair completely, so the many-body ground ϕ_N is unique. The geometric sum gives

$$\sum_{\ell=0}^{N-1} \cos k_\ell = \frac{\sin(N\vartheta)}{\sin \vartheta}, \quad E_0(K) = -C_N. \quad (\text{S90})$$

The highest occupied energy and the lowest vacant energy are separated by

$$|J_\perp| \{ \cos[(N-1)\vartheta] - \cos[(N+1)\vartheta] \} = g_N. \quad (\text{S91})$$

Any excited occupation replaces at least one occupied level by a vacant one and costs at least this difference. This includes $N=1$ and $N=L-1$. For even L the onsite staggered rotation $\exp(i\pi \sum_j j q_j)$ reverses every bond, including the seam; it maps positive exchange to negative exchange. For odd L it does not reverse the seam, which explains the exclusion of that sign choice. These free-chain formulas and the positive sine-product wavefunction used below are recalled inputs, not new spectral results [1, 32].

B. Optimality and diagonal equality

Choose a normalized sector ground vector ψ which is a translation eigenvector. The identity $TUT^{-1} = Ue^{-2\pi i Q/L}$ makes each of $U^{\pm 1}\psi$ orthogonal to ψ for $0 < N < L$. Since H' is unchanged under both twists,

$$\frac{U^{-1}KU + UKU^{-1}}{2} = \cos(2\vartheta)K, \quad \frac{e_+ + e_-}{2} = 2\sin^2 \vartheta [-\langle K \rangle_\psi]. \quad (\text{S92})$$

For a unique sector ground the variational principle bounds Δ_N by the smaller trial energy, hence by their average. The sector operator inequality $K \geq -C_N$ proves $\Delta_N \leq g_N$. For a degenerate sector ground this bound is automatic. The free choice $H' = 0$ attains it, proving the supremum in Eq. (14). The paired-twist mechanism and the absence of restrictions on longitudinal couplings have precedents in Nomura–Morishige–Isoyama [22]; the optimization here retains the exact sector minimum rather than replacing the kinetic expectation by a sum of bond norms.

Since $g_N > 0$, saturation entails a unique sector ground and equality in $\langle K \rangle_\psi \geq -C_N$. Thus ψ is ϕ_N up to phase, and the eigenvalue equation forces $H'\phi_N = w\phi_N$. The nearest-neighbour particle–vacancy exchange graph is connected in every nontrivial sector. Negative hopping makes its ground vector strictly positive by Perron–Frobenius; even- L staggering only changes its phases. Therefore every component of ϕ_N is nonzero. If H' is configuration diagonal, $H'\phi_N = w\phi_N$ forces its diagonal value to be w on every configuration in the sector. Conversely a sector scalar simply shifts the free spectrum. This proves diagonal saturation if and only if sector constancy. Global nonconstancy, for example a function of Q , is immaterial. For off-diagonal H' the eigenvector condition is necessary, not by itself sufficient for saturation.

The rigidity is quantitative. Let $\delta_K = \langle K \rangle_\psi + C_N \geq 0$ be the kinetic deficit of the chosen sector ground vector and $F = |\langle \phi_N, \psi \rangle|^2$ its fidelity with the free ground state. Then

$$\Delta_N \leq g_N - 2\sin^2 \vartheta \delta_K, \quad \delta_K \geq g_N(1 - F). \quad (\text{S93})$$

The first inequality is Eq. (S92) with the deficit retained. The free spectral decomposition gives $K + C_N \geq g_N(I - |\phi_N\rangle\langle\phi_N|)$, proving the second. For a degenerate sector ground state $\Delta_N = 0$, and the right side of the first inequality remains nonnegative because both trial energies are. The prefactor $2\sin^2 \vartheta$ vanishes as $L \rightarrow \infty$, so this finite-size estimate implies no size-uniform rigidity. If $J_\perp = 0$, then $H = H'$ commutes with U ; twisting a translation ground vector produces an orthogonal ground vector and ordered gap zero. This case lies outside the equality classification.

C. A quantitative staircase of low levels

Let $q = L/\gcd(L, N)$ and let $r \geq 1$ be an integer with $2r < q$. Then

$$E_r(H) - E_0(H) \leq 2C_N \sin^2(r\vartheta), \quad (\text{S94})$$

$$E_{2r}(H) - E_0(H) \leq \frac{2C_N \sin(r\vartheta) \sin[(r+1)\vartheta]}{\cos \vartheta}. \quad (\text{S95})$$

Neither ground-state uniqueness nor a clustering assumption is needed. To see this, split $K = K_+ + K_+^\dagger$ with $K_+ = (J_\perp/2) \sum_j S_j^+ S_{j+1}^-$ and define the kinetic energy $\mathcal{E} = -\langle K \rangle_\psi$ and the current $\mathcal{J} = \langle i(K_+ - K_+^\dagger) \rangle_\psi$ for the chosen translation ground vector. With the present twist convention,

$$e_{\pm m} = \mathcal{E}[1 - \cos(2m\vartheta)] \pm \mathcal{J} \sin(2m\vartheta). \quad (\text{S96})$$

Nonnegativity of $e_{\pm 1}$ and $K \geq -C_N$ give $0 \leq \mathcal{E} \leq C_N$ and $|\mathcal{J}| \leq \mathcal{E} \tan \vartheta$. At $\mathcal{E} = 0$ these also force $\mathcal{J} = 0$. This is a finite-size current constraint of the same variational type as the Bloch argument [23], not a new current principle.

The $2r + 1$ vectors $U^m \psi$, $-r \leq m \leq r$, have distinct translation eigenvalues because their exponents differ by less than q . They are orthonormal, and their Hamiltonian compression is diagonal by translation symmetry. Choosing the smaller-energy sign for each $m = 1, \dots, r$ gives a trial subspace of dimension $r + 1$. Each selected energy is at most $\mathcal{E}[1 - \cos(2m\vartheta)] \leq 2C_N \sin^2(r\vartheta)$ since $r < L/2$. The min–max principle proves the first bound. Using both signs, each energy is at most

$$2C_N \{\sin^2(m\vartheta) + \tan \vartheta \sin(m\vartheta) \cos(m\vartheta)\} = \frac{2C_N \sin(m\vartheta) \sin[(m+1)\vartheta]}{\cos \vartheta}. \quad (\text{S97})$$

For $0 \leq m \leq r - 1$ the consecutive numerator difference is $\sin[2(m+1)\vartheta] \sin \vartheta > 0$, so the largest needed value occurs at $m = r$. The dimension- $2r + 1$ min–max bound proves the second claim.

At fixed $\bar{N} = \min(N, L - N)$ and bounded $|J_\perp|$, one has $q \geq L/\bar{N}$, and these envelopes are $O(r(r+1)/L^2)$. Choosing diverging $r = o(L)$ therefore gives increasingly many low ordered levels. At fixed rational filling with fixed denominator q , only a bounded number of momentum-distinct twists is supplied; at half filling $q = 2$ there is no positive admissible r . Local-twist theorems giving many low states under different hypotheses already exist [5]. No phase or dynamical exponent is inferred from these upper bounds alone.

D. Nonzero positive parents attaining the envelope

Take $J_\perp < 0$ and $2 \leq N \leq L - 2$. For the ordered occupied positions $x = (x_1 < \dots < x_N)$ the free ground, with a positive normalization, has amplitudes

$$\phi_N(x) = \mathcal{Z}_N^{-1/2} \prod_{a < b} \sin \frac{\pi(x_b - x_a)}{L} > 0. \quad (\text{S98})$$

This follows from the determinant of Eq. (S89); the phase from centering the momentum block cancels its configuration-dependent Vandermonde phase. It is also the known ground-state weight in Ref. [32]. Its translation eigenvalue is one, because the unique ground is positive and translation permutes configurations.

Let $P(x) = \sum_a x_a$ modulo L and choose a symmetric nonnegative translation-invariant graph $a_{xy} = a_{yx}$ with $a_{xx} = 0$, whose nonzero edges satisfy $P(x) = P(y)$ modulo L . Write

$$v(x) = \sum_y a_{xy} \frac{\phi_N(y)}{\phi_N(x)}, \quad H'_* = \text{diag } v - a. \quad (\text{S99})$$

All ratios are finite. This is an application of the weighted positive-parent or ground-state-transform method [28, 32]. Its explicit positive decomposition is

$$H'_* = \sum_{\{x,y\}} a_{xy} |v_{xy}\rangle \langle v_{xy}|, \quad |v_{xy}\rangle = \sqrt{\frac{\phi_N(y)}{\phi_N(x)}} |x\rangle - \sqrt{\frac{\phi_N(x)}{\phi_N(y)}} |y\rangle. \quad (\text{S100})$$

Thus $H'_* \geq 0$ and $H'_* \phi_N = 0$. The graph and amplitudes are translation invariant, so $[H'_*, T] = 0$; equal wrapped polarization on each edge gives $[H'_*, U] = 0$. Extending this block by zero on every other Q sector gives a Hermitian positive operator on the whole spin Hilbert space with all three required commutators.

A nonzero graph exists at every admitted size and filling. Place the occupations 1001 and 0110 on four consecutive sites and keep the same $N - 2$ spectators elsewhere. They have equal polarization and the spectators fit because $0 \leq N - 2 \leq L - 4$. Assign a positive edge weight to this pair and its simultaneous translation orbit, combining repeated edges. This includes $L = 4, N = 2$. Translating both endpoints adds the same N to their polarization modulo L , so every orbit edge stays neutral.

For $\gamma \geq 0$, positivity and $H'_* \phi_N = 0$ leave the unique sector ground at $-C_N$. Min-max gives $E_1(K + \gamma H'_*) \geq E_1(K)$. The singly boosted free ground is a free eigenvector at $-C_N + g_N$; commutation with U makes $H'_* U \phi_N = 0$. This supplies the opposite inequality and proves exact saturation for every γ .

More generally, the shift by U^m moves every occupied momentum by one grid step per twist. A proper nonempty consecutive block in a cyclic grid has no nontrivial shift stabilizer: its unique directed occupied-to-vacant boundary would otherwise move to a second such boundary. Consequently the L shifted occupied sets are different, and their Slater determinants are orthogonal, even if translation momenta or energies coincide. Summing the cosines of the centered block gives $-C_N \cos(2m\vartheta)$, proving Eq. (16) for all $0 \leq m < L$. The distinction between these L states and the q distinct translation residues used for Eq. (S95) is essential. Because U^m factors into onsite unitaries, all these states have the same Schmidt spectrum across any site bipartition.

For nonzero graph weights, $\text{Tr } H'_* > 0$; hence $\gamma > 0$ changes the remaining sector spectrum and is not a scalar shift or an isospectral rewriting. The compensation $v(x)$ generally depends on all spectators, and the zero extension is charge-sector dependent. Neither finite range, bounded body order, a natural experimental realization, nor a generic nonintegrable phase is claimed. The chosen sector need not be the globally minimizing sector of the full Hamiltonian.

XIV. FINITE-SIZE NUMERICAL ILLUSTRATION

Figure 2 illustrates the analytic inequalities (12) and (15); no thermodynamic extrapolation or phase identification is made from these data. We use spin-1/2 operators with S^z eigenvalues $\pm 1/2$, $J_\perp = -1$, periodic boundary conditions, and the sector $N = L/2$. The two diagonal additions are

$$H'_{\text{Xxz}} = J_z \sum_{j=0}^{L-1} S_j^z S_{j+1}^z, \quad H'_{\text{dip}} = V_0 \sum_{0 \leq i < j < L} \frac{S_i^z S_j^z}{\min(j-i, L-j+i)^3}. \quad (\text{S101})$$

Every unordered pair in the dipolar model, including an antipodal pair, is included once. In each model the coupling $u = J_z$ or V_0 runs from -2 to 2 in increments of $1/4$. Both additions commute with Q , T , and U . The plotted gap is the ordered gap within $Q = N$, not a gap above the full-spectrum ground state in an unspecified charge sector.

Matrix construction and solver. Enumerate binary configurations $b = (b_0, \dots, b_{L-1})$ with $\sum_j b_j = N$, and put $s_j(b) = b_j - 1/2$. The diagonal entries are $J_z \sum_j s_j(b) s_{j+1}(b)$ or $V_0 \sum_{i < j} s_i(b) s_j(b) / d_L(i, j)^3$. Whenever neighboring bits differ, the corresponding flipped configuration has matrix element $-1/2$, including across the periodic seam. All other off-diagonal entries vanish. Thus the matrices have dimensions 70, 924, and 12870 for $L = 8, 12, 16$, respectively. For each of the 102 parameter points we compute the six lowest Ritz pairs using the symmetric sparse ARPACK solver in SciPy 1.5.4, with tolerance 10^{-12} , a Krylov space of dimension 50, and at most 30000 iterations. Eigenvalues are sorted with multiplicity and $\Delta_N = E_1 - E_0$. From the normalized lowest vector ψ we evaluate $2 \sin^2(\pi/L) \langle -K \rangle_\psi$ separately. Figure 2 shows $L = 16$; the two smaller sizes serve as additional finite-size checks, not a scaling fit.

Checks and precision. A second run starts the solver from an independently seeded random vector at every point (seeds 1211 and 51019). The largest change in the gap, ground energy, or paired-twist cost is 4.7×10^{-14} in units of $|J_\perp|$. The largest Ritz residual is 5.6×10^{-12} ; eigenvector orthogonality is checked to 10^{-9} . Direct evaluation of both twisted Rayleigh quotients reproduces the paired-twist identity to 1.2×10^{-13} . At $L = 6$, an independent full tensor-product spin construction agrees with the fixed-charge matrices and their low eigenvalues for both models at $u = -1, 0, 1$. A deliberately doubled matrix fails this normalization check. The free-chain value g_N and the entire inequality chain are checked at each size; nonzero diagonal couplings remain strictly below the envelope. These floating-point tests support the illustration; the inequalities and their strictness follow from the analytic proofs above, not from a finite parameter sweep. Table S1 gives the plotted values, rounded to eight decimal places.

TABLE S1. Numerical values underlying Fig. 2. All energies are in units of $|J_\perp|$, with $L = 16$ and $N = 8$. E_{tw} is the paired-twist cost in Eq. (15). The envelope and bondwise bounds are $2 \sin(\pi/16)$ and $16 \sin^2(\pi/16)$, respectively.

u	Δ_N (XXZ)	E_{tw} (XXZ)	Δ_N (dipolar)	E_{tw} (dipolar)
-2.00	0.00001414	0.04407346	0.00000660	0.04257147
-1.75	0.00004214	0.05338881	0.00001879	0.05065158
-1.50	0.00016586	0.06966457	0.00006711	0.06347457
-1.25	0.00130293	0.11340419	0.00036864	0.08937851
-1.00	0.07612047	0.32478066	0.01675848	0.23624615
-0.75	0.18076346	0.36575942	0.16145290	0.36663283
-0.50	0.25949672	0.38102667	0.25340481	0.38230564
-0.25	0.32783310	0.38812332	0.32667031	0.38850562
0.00	0.39018064	0.39018064	0.39018064	0.39018064
0.25	0.38196989	0.38836229	0.38436136	0.38879667
0.50	0.35952182	0.38313906	0.36871942	0.38495976
0.75	0.32135300	0.37458566	0.34212162	0.37890472
1.00	0.27018968	0.36244996	0.30546155	0.37063529
1.25	0.21231019	0.34625571	0.26147814	0.35998367
1.50	0.15554873	0.32574067	0.21406852	0.34670038
1.75	0.10680386	0.30167770	0.16749150	0.33065940
2.00	0.06973107	0.27618765	0.12552848	0.31216468