COTANGENT COHOMOLOGY AND HIGHER-ORDER OBSTRUCTIONS OF MATROIDS

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Abstract. We show that matroid Stanley–Reisner rings can have essential deformation obstructions of arbitrarily high order over every field, even at corank three. For every $m \geq 3$, we construct a connected cosimple matroid with m fine-homogeneous first-order deformation classes whose proper subcollections integrate jointly and whose pairwise primary cup products vanish. These classes define a joint flat family whose $(m-1)$-jet has no lift to order m, even without preserving the grading. For the duals of rank-three truncations of $U_{2,m} \oplus U_{2,n}$, $m, n \geq 3$, every nonzero homogeneous class modulo the primary cup image is realized as a higher-order obstruction, and this quotient has an explicit module presentation with $m+n-2$ minimal homogeneous generators. The construction is based on a computation of the full multigraded second cotangent cohomology T^2 and its primary cup image in terms of flats and parallel classes of the dual matroid. This computation also proves that T^2 vanishes precisely when every connected component has corank at most two, equivalently when the matroid has no $U_{1,4}$-minor, settling the conjecture of Constantinescu, Klein, Nguyen, Singh, and Venturello. The same criterion characterizes formal smoothness of the affine deformation functor.

The main results of this paper were obtained through a generative-AI workflow using OpenAI GPT-5.6 Sol, Anthropic Claude Fable 5, Grok 4.6, OpenAI GPT-6 Astra, Anthropic Claude Fable 5.1, and OpenAI GPT-6.1 Sol. Further details appear in the disclosure at the end of the paper.

1. Introduction

The primary cup product records quadratic compatibility of first-order deformations, but it need not determine whether several classes integrate simultaneously in the fine-multigraded category. For matroid Stanley–Reisner rings this distinction already appears at fixed corank three: arbitrarily many individually integrable directions can have all pairwise cup products zero and nevertheless encounter an essential obstruction at an arbitrarily late order. Our main construction gives the flat family explicitly and proves that its specified jet cannot be lifted, even by nonhomogeneous correction terms. Alongside this construction, we compute every fine-multigraded component of the obstruction module and determine the primary image in every fine degree.

The relevant algebraic input is the relative-cohomology and cup-product description of Altmann and Christophersen [1, Proposition 11, Theorem 13, and Section 6]. For matroids, Brehm and Constantinescu described T^1 [5, Theorem 4.7]; in dual terms its nonzero negative supports have rank one. Constantinescu, Klein, Nguyen, Singh, and Venturello computed T^2 for uniform matroids and established vanishing in corank at most two. They conjectured that these are the only connected matroids with vanishing T^2 [6, Proposition 21 and Conjecture 22]. Thus the full module, its vanishing locus, and the actual lifting obstructions are related but distinct questions.

2020 *Mathematics Subject Classification.* Primary 13D03; Secondary 05B35, 13F55, 05C40, 14B12.

Key words and phrases. matroid, Stanley–Reisner ring, cotangent cohomology, deformation, obstruction, cocircuit, cup product, Tutte polynomial, valuative invariant.

Altmann and Christophersen constructed formal deformation bases for specified combinatorial surfaces and explicit normal forms for polygon cones [2, Theorem 6.4 and Proposition 6.6]. Here the dual two-line matroids keep corank equal to three while the first failed extension of a specified jet has arbitrary order; the argument also excludes nonhomogeneous corrections. Ilten, Meazzini, and Petracci [10, Theorem 4.1] extended link localization to higher cotangent cohomology in characteristic zero and described T^3 and T^4 for flag complexes. Our higher-order result concerns the order of a lifting obstruction taking values in T^2, not a higher cotangent-cohomology index. Its mechanism is a cyclic relation among circuit generators, with an integral flatness proof and a socle calculation excluding every correction of the specified jet.

Theorem 1.1 (Unbounded obstruction order at corank three)**.** *For every $m \geq 3$ there is a connected cosimple matroid M of rank m and corank three with nonzero classes $\alpha_i \in T^1_{-\mathbf{e}_i}(M)$, $0 \leq i < m$, over every field, such that:*

1. *every proper subcollection admits a joint deformation over its polynomial parameter ring;*
2. *all the classes admit a flat joint deformation over $k[t_0, \ldots, t_{m-1}]/(t_0 \cdots t_{m-1})$;*
3. *the resulting deformation modulo $(t_0, \ldots, t_{m-1})^m$ cannot lift modulo $(t_0, \ldots, t_{m-1})^{m+1}$, even in the ungraded category.*

Within the fine-multigraded category, with $\deg t_i = \mathbf{e}_i$, the jet in (3) is uniquely determined by its first-order classes. Its first essential obstruction has order m, although all pairwise primary cup products vanish.

The matroid in Theorem 1.1 is dual to the rank-three truncation of $U_{2,m} \oplus U_{2,3}$. The family is governed by explicit cyclic polynomials and a telescoping identity whose residual is the product of all parameters. Two circuit relations force contradictory corrections of its coefficient, proving nonliftability. Section 18 gives the construction and its proof. Here an essential obstruction of order m means a nonzero obstruction to extending the specified $(m-1)$-jet to order m. The uniqueness statement is graded; we do not identify all ungraded higher jets with the same tangent classes or a full miniversal base.

Theorem 18.7 strengthens this construction to every nonzero homogeneous class of the complete primary quotient $T^2(M_{m,n})/\Omega$, including degrees with positive coordinates. Here $M_{m,n}$ is dual to $\mathrm{Tr}_3(U_{2,m} \oplus U_{2,n})$, whose two disjoint lines L, H have sizes m, n; put $R = k[M_{m,n}]$, and let Ω be the image of $T^1 \otimes_R T^1 \to T^2$. A prescribed class whose negative support has size r occurs as the only nonzero order-r obstruction coefficient of a specified joint jet in r parameters. All its selected tangent classes have zero primary products, and every proper subcollection integrates formally. For purely negative degrees the stronger polynomial integration holds (Theorem 18.4); the purely negative quotient has exponential dimension. Thus these dimensions count actual higher obstruction values, not merely an ambient obstruction space. Classical link reduction computes the positive-supported component, but constructing a deformation of the original ring requires the variable-weight cyclic family and Artinian relation-lifting argument below. The realizations are degree by degree, not simultaneous realizations of sums.

For $M_{m,n}$ the quotient has a more economical algebraic description than its negative dimension suggests. Theorem 18.8 gives an explicit fine-graded R-module presentation with exactly $m + n - 2$ minimal generators, concentrated in the two degrees $-\mathbf{1}_L$ and $-\mathbf{1}_H$. It also determines the radical annihilator: every irreducible component of the support of this quotient has codimension two in $\operatorname{Spec} R$. The decisive step is a Taylor coboundary identity describing multiplication between negative degrees; the graph dimensions alone do not determine it. The coordinate modules in the presentation also record the positive-support restriction maps.

Our structural starting point reduces the relative-cohomology calculation to connected components of explicit graphs. Positive support is removed by passing to a link; duality then organizes the remaining relative complex by hyperplanes. This reduction is the common source of the formula, the vanishing classification, and the cup-product coordinates below.

Let N have rank s, let $\varnothing \neq B \subseteq E(N)$, and let H be a hyperplane disjoint from B. Define $\mathcal{G}_B(H)$ to be the rank-$(s-2)$ flats $G \subset H$ such that $\mathrm{rk}_N(G \cup B) = s$. Make this set the vertex set of a graph $\Gamma_B(H)$ by joining distinct G, G' when

$$\mathrm{rk}_N\big((G \cap G') \cup B\big) = s.$$

Empty graphs contribute zero to reduced zeroth cohomology.

Theorem 1.2 (Complete hyperplane-graph formula)**.** *Write a fine degree uniquely as* $\mathbf{c} = \mathbf{a} - \mathbf{b}$*, where* $\mathbf{a}, \mathbf{b} \in \mathbb{N}^{E(M)}$ *have disjoint supports. A component* $T^2_{\mathbf{c}}(M)$ *vanishes unless* $\mathbf{b} = \mathbf{1}_B$ *is nonzero and squarefree,* $A = \operatorname{supp} \mathbf{a}$ *is independent, and* B *consists of actual vertices of* $L = \mathrm{lk}_M A$*. Under these conditions, set* $N = L^*$*. Then, over every field,*

$$T^2_{\mathbf{c}}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } N \\ H \cap B = \varnothing}} \widetilde{H}^0(\Gamma_B(H); k).$$

The analogous formula for T^1*, which recovers in dual form the description of Brehm and Constantinescu* [5]*, is given in Theorem 3.1.*

In fact these graphs are empty, discrete, or connected according as the negative support has dual rank at most one, two, or at least three (Lemma 17.1). Hence only rank-two supports contribute to T^2, and its dimensions are flat counts rather than connectivity computations. The dual rank of the negative support is unchanged by passing to the relevant link. This gives the complete fine Hilbert series over every field (Theorem 17.2).

When $B = \{p, q\}$, the graph $\Gamma_B(H)$ is discrete. This special case detects vanishing.

Theorem 1.3 (Vanishing and the excluded minor)**.** *For a finite matroid* M*, the following are equivalent:*

(1) $T^2(M) = 0$*;*
(2) $T^2_{-\mathbf{e}_p - \mathbf{e}_q}(M) = 0$ *for all distinct* $p, q \in E(M)$*;*
(3) *every connected component of* M *has corank at most two;*
(4) *every cocircuit of* M *has size at most three;*
(5) M *has no minor isomorphic to* $U_{1,4}$*.*

We next identify actual quadratic obstructions rather than merely nonzero ambient obstruction spaces. The singleton part of the graph model identifies $T^1_{-\mathbf{e}_p}(M)$ with functions on certain dual hyperplanes, modulo constants. Substitution in the Altmann–Christophersen cup formula gives explicit block coordinates and a universal surjectivity statement.

Theorem 1.4 (Universal pure-pair cup product)**.** *For every matroid* M*, every pair of distinct actual vertices* p, q*, and every field* k*, the cup product*

$$T^1_{-\mathbf{e}_p}(M) \otimes_k T^1_{-\mathbf{e}_q}(M) \longrightarrow T^2_{-\mathbf{e}_p - \mathbf{e}_q}(M)$$

is surjective.

The proof constructs every coordinate delta vector as a decomposable cup product. It uses only coefficients $0, \pm 1$ in the monomial presentation, and therefore works in every characteristic. Iterated link localization transports the statement to arbitrary positive support.

Theorem 1.5 (Degree-zero obstructions in every rank)**.** *Let M be connected with* $\operatorname{rk} M \geq 2$ *and* $\operatorname{corank} M \geq 3$*. There are distinct $e, p, q \in E(M)$ for which*

$$T^1_{\mathbf{e}_e-\mathbf{e}_p}(M) \otimes_k T^1_{\mathbf{e}_e-\mathbf{e}_q}(M) \twoheadrightarrow T^2_{2\mathbf{e}_e-\mathbf{e}_p-\mathbf{e}_q}(M) \neq 0.$$

All three fine degrees have standard total degree zero. Over every field, some standard-degree-zero first-order deformation fails to lift from $k[\epsilon]/(\epsilon^2)$ to $k[\epsilon]/(\epsilon^3)$.

No characteristic restriction is needed. The primary obstruction map is quadratic with the cup product as polarization, and it doubles fine degrees. Every nonzero fine component of T^1 has nonzero negative support, so the doubled degree fails the squarefree support criterion and the self-obstruction of every fine-homogeneous class vanishes. The mixed cup term is therefore the entire primary obstruction, in every characteristic, including two (Lemma 8.6). Combining this with the vanishing classification yields a complete smoothness criterion.

Corollary 1.6 (Smoothness dichotomy)**.** *For every field k and every matroid M, the affine deformation functor of $k[M]$ is formally smooth if and only if every connected component of M has corank at most two, if and only if M has no $U_{1,4}$-minor.*

Formal smoothness is in principle sensitive to the ground field; for matroids, Corollary 1.6 decides it by one excluded minor, uniformly in the characteristic.

These primary results coexist with the higher obstructions of Theorem 1.1. Every fine-homogeneous class has an all-order formal representative. If two classes have disjoint negative supports and positive supports avoiding their union, their sum admits an all-order formal arc whenever their cup product vanishes (Theorem 8.11). This one-parameter assertion is distinct from joint integration over independent parameters. The cyclic family shows that pairwise vanishing does not guarantee extension of a specified joint jet. Theorem 17.5 determines the primary image in every fine degree: it fills supports meeting exactly two dual parallel classes, and misses every support meeting at least three, regardless of the positive exponents. The lifting argument uses the circuit description of T^1 from [5] to show that first cohomology is generated in purely negative degrees. Corollary 17.6 then gives the degreewise quotient by the primary image and its characteristic-independent dimensions. For a connected matroid, the standard-degree-preserving functor is formally smooth precisely when $\operatorname{rk} M \leq 1$ or $\operatorname{corank} M \leq 2$ (Corollary 8.9).

The proof of Theorem 1.3 is constructive. A connected matroid of corank at least three has a cocircuit D of size at least four. If A is a basis of its complementary hyperplane, every pair $p, q \in D$ gives

$$\dim_k T^2_{\mathbf{1}_A-\mathbf{e}_p-\mathbf{e}_q}(M) = |D| - 3.$$

The certificate (A, D, p, q) is verifiable by rank-oracle queries. The resulting finite family of labeled fine components reconstructs all cocircuits of size at least four. Its total dimension $\Theta(M)$ has sharp uniform-matroid bounds and is not determined by the Tutte polynomial.

Beyond the scalar witness Θ, the pair layer has a second natural quantitative invariant,

$$\mathfrak{q}(M) = \sum_{\{p,q\} \subseteq E(M)} \dim_k T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M).$$

The coordinate-delta construction gives an integral splitting of the direct sum of pure-pair cup maps. Thus $\mathfrak{q}(M)$ measures the dimension of their target quotient, not the entire quadratic obstruction map. It is additive under direct sums and has a sharp rank-two extremal theory distinct from that of Θ, including unique equality profiles and an exact gap to the next profile (Section 11).

The remaining results ask how much combinatorial information these obstruction spaces retain. In rank three, a coefficient extraction gives $\dim_k T^2(M)_0$ from line sizes and point-pencil partitions. The labeled nonvanishing pattern reconstructs every simple rank-three matroid on at least seven points, and the near-pencil uniquely minimizes the degree-zero dimension on $n \geq 8$ points. A different question concerns valuations: the weighted cocircuit profile underlying Θ is a linear specialization of the Derksen–Fink $\mathcal{G}$-invariant, although it is not determined by the Tutte polynomial. The global pair rank $\mathfrak{q}$ is neither valuative nor determined by the Tutte polynomial. These distinctions concern different parts of the same cotangent-cohomology module; none follows from its vanishing locus alone.

The paper is organized as follows. Section 2 records the matroid and cotangent-cohomology input, and Section 3 proves the graph formula. Sections 4 and 5 derive the pair slice and vanishing classification. Sections 6 and 7 compute two positive-support strands. Section 8 proves cup surjectivity and the quadratic obstruction results. Sections 9–16 develop the constructive witnesses, the two rank-two extremal theories, rank-three formulas and reconstruction, the near-pencil theorem, and valuative distinctions. Section 17 reduces T^2 to rank-two flat counts and determines its entire primary image. Section 18 constructs the cyclic families and proves unbounded essential obstruction order; Sections 18.1 and 18.2 realize homogeneous quotient classes, and Section 18.3 gives the quotient-module presentation. Section 19 records finite checks, and Section 20 lists further problems. For the higher-order theorem, the inputs are the support and graph formulas, the singleton description of T^1, and the primary-image calculation; the intervening extremal and valuative results are independent applications.

2. Preliminaries

2.1. **Matroids and links.** Let M be a finite matroid on ground set E. We write $(u)_+ = \max\{u, 0\}$ and let $\kappa(\Gamma)$ denote the number of connected components of a graph Γ. For the empty graph, both κ and reduced zeroth cohomology are zero. We identify M with its independence complex. Its rank and corank are

$$\operatorname{rk} M = \max\{|I| : I \in M\}, \qquad \operatorname{corank} M = |E| - \operatorname{rk} M.$$

The circuits are the minimal nonfaces. We write $\mathcal{L}(M)$ for the lattice of flats of M. The dual matroid is denoted M^*; the cocircuits of M are the circuits of M^*, equivalently the complements of the hyperplanes of M.

For $A \in M$, the simplicial link is

$$\operatorname{lk}_M A = \{F \subseteq E \setminus A : A \cup F \in M\}.$$

It is the independence complex of M/A after deleting contraction loops. We write $V(\Delta)$ for the actual vertex set of a simplicial complex and always use this convention for links. If A is a basis of a hyperplane H and $D = E \setminus H$, then $\operatorname{lk}_M A$ is the zero-dimensional complex Z_D on the vertex set D.

Fix a basis B of M. For $e \notin B$, write $C(e; B)$ for the fundamental circuit in $B \cup \{e\}$. The fundamental graph $G_B(M)$ is bipartite on $B \sqcup (E \setminus B)$, with b adjacent to e precisely when $b \in C(e; B)$. We use the standard facts

$$M \text{ connected} \iff G_B(M) \text{ connected}$$

and, for $A_0 \subseteq B$ and every nonloop e of M/A_0,

$$C_{M/A_0}(e; B \setminus A_0) = C_M(e; B) \setminus A_0.$$

2.2. **Cotangent cohomology and the support criterion.** Let $S = k[x_e : e \in E]$ and let $I_M \subset S$ be the Stanley–Reisner ideal generated by the squarefree circuit monomials. Put $R = k[M] = S/I_M$ and $T^i(M) = \operatorname{Ext}^i_R(\mathbb{L}_{R/k}, R)$, where $\mathbb{L}_{R/k}$ is the cotangent complex. The fine $\mathbb{Z}^E$-grading induces

$$T^i(M) = \bigoplus_{\mathbf{c}\in\mathbb{Z}^E} T^i_{\mathbf{c}}(M).$$

For a scalar subscript, $T^i(M)_d$ denotes the standard total-degree component $\bigoplus_{\sum_e c_e = d} T^i_{\mathbf{c}}(M)$. Write each degree uniquely as $\mathbf{c} = \mathbf{a} - \mathbf{b}$, with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^E$ and disjoint supports A, B. The Altmann–Christophersen support criterion gives

$$T^i_{\mathbf{a}-\mathbf{b}}(M) = 0 \quad (i = 1, 2)$$

unless $\mathbf{b}$ is nonzero and squarefree, $A \in M$, and B is contained in the actual vertex set of $\operatorname{lk}_M A$. When these conditions hold,

$$T^i_{\mathbf{a}-\mathbf{1}_B}(M) \cong T^i_{-\mathbf{1}_B}(\operatorname{lk}_M A), \qquad i = 1, 2. \tag{1}$$

The isomorphism depends on the support A, not on the positive exponents.

For a simplicial complex Δ and nonempty B, define

$$N_B(\Delta) = \{F \in \Delta : F \cap B = \varnothing,\ F \cup B \notin \Delta\}$$

and

$$\widetilde{N}_B(\Delta) = \{F \in N_B(\Delta) : F \cup B' \notin \Delta \text{ for some } B' \subsetneq B\}.$$

If $|B| > 1$, then [1, Theorem 9]

$$T^i_{-\mathbf{1}_B}(\Delta) \cong H^{i-1}(|N_B(\Delta)'|, |\widetilde{N}_B(\Delta)'|; k), \qquad i = 1, 2. \tag{2}$$

Here a family of faces is ordered by inclusion, and the prime denotes its order complex. We also write ΔP for the order complex of a finite poset P. Reduced cohomology is augmented, so $\widetilde{H}^{-1}(\varnothing; k) = k$ and $\widetilde{H}^0(\varnothing; k) = 0$. The original formulation in [1] is over $\mathbb{C}$; the same combinatorial cochain presentation with coefficients in an arbitrary field is recorded in [6, Sections 2.2–2.3]. The singleton formula uses the reduced version of the same complex; we record its matroid form in Theorem 3.1.

Lemma 2.1 (Localization and cup products)**.** *Let R be a finitely presented k-algebra and $U \subset R$ a multiplicative set. For $i = 1, 2$, flat localization gives natural isomorphisms*

$$U^{-1}T^i(R/k; R) \cong T^i(U^{-1}R/k; U^{-1}R),$$

compatible with cup products. For a vertex v of a simplicial complex Δ,

$$k[\Delta]_{x_v} \cong k[\operatorname{lk}_\Delta v][x_v^{\pm 1}]$$

after deleting ghost vertices, and the corresponding positive cotangent cohomology is the Laurent extension of that of the link. If a fine degree $\mathbf{a}-\mathbf{1}_B$ satisfies the support criterion and $v \in \operatorname{supp}\mathbf{a}$, localization followed by setting $x_v = 1$ identifies this component with the component of the link obtained by deleting its v-coordinate. Iteration over $\operatorname{supp}\mathbf{a}$ *gives* (1)*.*

Proof. Choose a finite-free model for the cotangent complex through the degrees needed for T^1 and T^2. Flat localization commutes with its dual and with cohomology. The cup product is induced by composition of cochain-level lifts, so it also commutes with localization.

Inverting x_v kills variables which are not vertices of the link and turns the remaining monomial relations into those of $\operatorname{lk}_\Delta v$. The transitivity triangle for the Laurent extension splits into the base-changed cotangent complex of the link and the free summand generated by dx_v. The free summand has no positive cotangent cohomology. If $v \in \operatorname{supp}\mathbf{a}$, increasing its positive

exponent does not change the relative complex in (2); multiplication by x_v is the identity under these identifications. Thus the localization direct limit is already represented by the original fine component. Removing the Laurent monomial by setting $x_v = 1$ deletes that coordinate. The positive-support hypothesis is essential: localization need not preserve a component with zero v-coordinate. □

Lemma 2.2 (Zero-dimensional calculation)**.** *Let Z_D be the zero-dimensional complex on $d = |D|$ vertices. For a nonzero squarefree* $\mathbf{b}$*,*

$$\dim_k T^2_{-\mathbf{b}}(Z_D) = \begin{cases} (d-3)_+, & \operatorname{supp}\mathbf{b} = \{p,q\},\ p \neq q, \\ 0, & \textit{otherwise}. \end{cases}$$

All components with nonempty positive support vanish. Thus $T^2(Z_D) = 0$ if and only if $d \leq 3$.

Proof. For $B = \{p,q\}$, the support is a minimal nonface and $\widetilde{N}_B$ is the discrete set indexed by $D \backslash \{p,q\}$. The relevant group is its reduced zeroth cohomology, of dimension $(d-3)_+$. Singleton support has vanishing first cohomology. A support of size at least three is a nonminimal nonface, so the relative pair agrees and contributes zero. A nonempty positive support has a simplex as its link. □

We also use two standard facts. First, matroids of corank at most two have vanishing T^2 [6, Proposition 21]. Second, for matroids on disjoint ground sets,

$$T^2(M_1 \oplus M_2) \cong \big(T^2(M_1) \otimes_k k[M_2]\big) \oplus \big(k[M_1] \otimes_k T^2(M_2)\big). \tag{3}$$

This follows directly from the cotangent complex of a tensor product over a field. In particular, $T^2(M_1 \oplus M_2) = 0$ exactly when both summands vanish.

3. The complete hyperplane-graph model

Let N be a matroid of rank s on E and let $\varnothing \neq B \subseteq E$. For a hyperplane H of N with $H \cap B = \varnothing$, put

$$\mathcal{G}_B(H) = \{G \in \mathcal{L}(N) : \operatorname{rk}_N G = s-2,\ G \subset H,\ \operatorname{rk}_N(G \cup B) = s\}.$$

If $s < 2$, this set is empty. Define a graph $\Gamma_B(H)$ on $\mathcal{G}_B(H)$ by joining distinct vertices according to

$$G \sim G' \quad \Longleftrightarrow \quad \operatorname{rk}_N((G \cap G') \cup B) = s. \tag{4}$$

An empty graph contributes zero to $\widetilde{H}^0$.

Theorem 3.1 (Universal hyperplane-graph formula)**.** *Let M be a matroid, write $\mathbf{c} = \mathbf{a} - \mathbf{b}$ with disjoint nonnegative parts, and put $A = \operatorname{supp}\mathbf{a}$. If $\mathbf{b} = 0$, if $\mathbf{b}$ is not squarefree, or if $A \notin M$, then $T^i_{\mathbf{c}}(M) = 0$ for $i = 1, 2$. Suppose $\mathbf{b} = \mathbf{1}_B$ is nonzero, set $L = \operatorname{lk}_M A$, and assume that B lies in the actual vertex set of L. Put $N = L^*$.*

If $|B| \geq 2$, then

$$T^1_{\mathbf{c}}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } N \\ H \cap B = \varnothing \\ \mathcal{G}_B(H) = \varnothing}} k, \tag{5}$$

$$T^2_{\mathbf{c}}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } N \\ H \cap B = \varnothing}} \widetilde{H}^0(\Gamma_B(H); k). \tag{6}$$

Consequently,

$$\dim_k T^2_{\mathbf{c}}(M) = \sum_{\substack{H \text{ hyperplane of } N \\ H\cap B=\varnothing}} \big(\kappa(\Gamma_B(H)) - 1\big)_+. \tag{7}$$

If $B = \{p\}$ and $\mathscr{H}_p(N)$ is the discrete set of hyperplanes avoiding p, then

$$T^1_{\mathbf{c}}(M) \cong \widetilde{H}^0(\mathscr{H}_p(N); k), \qquad T^2_{\mathbf{c}}(M) = 0. \tag{8}$$

Proof. The support criterion and (1) reduce the problem to $T^i_{-\mathbf{1}_B}(N^*)$. Assume first that $|B| \geq 2$. For $F \subseteq E \setminus B$, set

$$X = E \setminus (F \cup B).$$

Complementation and the dual-rank identity identify the Altmann–Christophersen pair, with its order reversed, with

$$P_B = \{X \subseteq E \setminus B : \operatorname{rk}_N X < s,\ \operatorname{rk}_N(X \cup B) = s\}$$

and

$$Q_B = \{X \in P_B : \operatorname{rk}_N(X \cup \{b\}) < s \text{ for some } b \in B\}.$$

Indeed, if $B' \subsetneq B$ and $D = B \setminus B'$, then

$$E \setminus (F \cup B') = X \cup D.$$

The condition $\operatorname{rk}_N(X \cup D) < s$ implies the singleton condition for every $b \in D$; conversely a singleton condition is obtained from $B' = B \setminus \{b\}$. Thus

$$T^i_{-\mathbf{1}_B}(N^*) \cong H^{i-1}(|P'_B|, |Q'_B|; k), \qquad i = 1, 2. \tag{9}$$

Define

$$c(X) = \operatorname{cl}_N(X) \setminus B.$$

Because $X \subseteq c(X) \subseteq \operatorname{cl}_N X$, the sets X and $c(X)$ have the same closure. Hence c preserves every rank condition defining P_B and Q_B; it is an ascending idempotent on P_B and restricts to one on Q_B. The closure-operator lemma [3, Corollary 10.12, p. 1852] gives homotopy equivalences on both spaces. Applying the five lemma to the two long exact sequences of pairs yields

$$H^\bullet(|P'_B|, |Q'_B|; k) \cong H^\bullet(|c(P_B)'|, |c(Q_B)'|; k). \tag{10}$$

If $X \in P_B \setminus Q_B$, then $\operatorname{rk}_N(X \cup \{b\}) = s$ for every $b \in B$. A single element raises rank by at most one, so $\operatorname{rk}_N X = s - 1$, and no element of B lies in $\operatorname{cl}_N X$. Conversely, every hyperplane disjoint from B has these properties. Since c preserves Q_B,

$$c(P_B) \setminus c(Q_B) = \{H : H \text{ is a hyperplane of } N,\ H \cap B = \varnothing\}. \tag{11}$$

Fix such an H and set

$$\mathcal{F}_B(H) = \{F \in \mathcal{L}(N) : F \subsetneq H,\ \operatorname{rk}_N(F \cup B) = s\}.$$

The elements of $c(P_B)$ strictly below H are exactly the members of $\mathcal{F}_B(H)$. Every relative chain contains a unique hyperplane from (11), and that hyperplane is its maximum. Deleting it produces a chain in $c(Q_B)$, while deleting another vertex remains in the same block. Thus the relative cochain complex splits, already at chain level, into the blocks

$$\big(\operatorname{cone} \Delta\mathcal{F}_B(H), \Delta\mathcal{F}_B(H)\big).$$

It follows that

$$T^1_{-\mathbf{1}_B}(N^*) \cong \bigoplus_{H\cap B=\varnothing} \widetilde{H}^{-1}(\Delta\mathcal{F}_B(H); k), \tag{12}$$

$$T^2_{-\mathbf{1}_B}(N^*) \cong \bigoplus_{H\cap B=\varnothing} \widetilde{H}^0(\Delta\mathcal{F}_B(H);k). \tag{13}$$

For $G \in \mathcal{G}_B(H)$, let

$$\mathcal{U}_G = \Delta\{F \in \mathcal{F}_B(H) : F \subseteq G\}.$$

Each $\mathcal{U}_G$ is a cone with apex G. These cones cover $\Delta\mathcal{F}_B(H)$: any qualifying flat extends within H to a rank-$(s-2)$ flat, and the spanning condition is preserved under enlargement. For $G_1, \dots, G_m$, put $K = \bigcap_i G_i$. The corresponding intersection is nonempty exactly when $\mathrm{rk}_N(K \cup B) = s$; in that case K is its maximum. Hence all nonempty finite intersections are contractible, and the finite nerve lemma [3, Theorem 10.6(i), p. 1850] applies. The one-skeleton of the nerve is precisely $\Gamma_B(H)$. Reduced H^0 depends only on this one-skeleton, so

$$\widetilde{H}^0(\Delta\mathcal{F}_B(H);k) \cong \widetilde{H}^0(\Gamma_B(H);k).$$

Moreover $\mathcal{F}_B(H)$ is empty exactly when $\mathcal{G}_B(H)$ is empty. Equations (12) and (13) now prove (5)–(7). The low-rank cases are included by the empty-graph convention.

If $B = \{p\}$, adding p raises rank by at most one, so every member of the complementary poset has rank $s-1$ and has closure avoiding p. Closure retracts this poset to the antichain of hyperplanes avoiding p. The reduced singleton form of the Altmann–Christophersen formula gives (8). Finally, (1) transports all formulas back to M. □

Remark 3.2 (Relation to the known T^1 description). For T^1, the formulas (5) and (8) are, after dualization and (1), a reformulation of the combinatorial description of the first cotangent cohomology of matroids obtained by Brehm and Constantinescu [5]. We include the T^1 statements because the proof above yields both cohomological degrees simultaneously and because the singleton identification (8) supplies the coordinates for the cup products of Section 8. The graph $\Gamma_B(H)$ and the resulting T^2 statements are the contributions developed here.

Corollary 3.3 (Complete fine Hilbert series)**.** *For $A \in M$ and $B \subseteq V(\mathrm{lk}_M A)$ with $|B| \geq 2$, write $\Gamma^A_B(H)$ for the graph of* (4) *associated with the dual matroid $(\mathrm{lk}_M A)^*$ and the set B, and define*

$$\omega_M(A,B) = \sum_{\substack{H \text{ hyperplane of } (\mathrm{lk}_M A)^* \\ H\cap B=\varnothing}} \big(\kappa(\Gamma^A_B(H)) - 1\big)_+.$$

Then

$$\mathrm{Hilb}_{T^2(M)}(\mathbf{t}) = \sum_{A\in M} \left(\prod_{a\in A} \frac{t_a}{1-t_a}\right) \left(\sum_{\substack{B\subseteq V(\mathrm{lk}_M A) \\ |B|\geq 2}} \omega_M(A,B) \prod_{b\in B} t_b^{-1}\right). \tag{14}$$

In particular, every multigraded dimension of $T^2(M)$ is independent of $\mathrm{char}\, k$.

Proof. The support criterion leaves precisely the displayed supports. For fixed A, positive exponents contribute

$$\prod_{a\in A}(t_a + t_a^2 + \cdots) = \prod_{a\in A} \frac{t_a}{1-t_a}.$$

The graph formula supplies the coefficient of every squarefree negative support. Singleton supports contribute nothing to T^2. □

The formula admits a useful closure compression. For a flat K of N and a hyperplane H, define $\Gamma_K(H)$ by replacing B with K in all rank conditions in (4), and put

$$\omega_N(K,H) = \dim_k \widetilde{H}^0(\Gamma_K(H);k).$$

Let

$$\Sigma_{K,H}(\mathbf{z}) = \sum_{\substack{B\subseteq K\setminus H,\ |B|\geq 2\\ \mathrm{cl}_N\, B=K}} \prod_{b\in B} z_b.$$

Corollary 3.4 (Closure-stratified series)**.** *For $M = N^*$,*

$$\sum_{\substack{B\subseteq V(M)\\ |B|\geq 2}} \dim_k T^2_{-\mathbf{1}_B}(M) \prod_{b\in B} z_b = \sum_{K\in\mathcal{L}(N)} \sum_{H \text{ hyperplane}} \omega_N(K,H)\Sigma_{K,H}(\mathbf{z}). \tag{15}$$

If $I \subseteq B$ and $\mathrm{cl}_N\, I = \mathrm{cl}_N\, B$, then

$$\dim_k T^2_{-\mathbf{1}_B}(M) \leq \dim_k T^2_{-\mathbf{1}_I}(M).$$

If B spans N, then $T^2_{-\mathbf{1}_B}(M) = 0$.

Proof. Every graph rank condition depends on B only through $\mathrm{cl}_N\, B$. For fixed K, H, the supports contributing to the block are precisely the subsets of $K \setminus H$ which span K, proving (15). If $I \subseteq B$ has the same closure, the graphs agree and the hyperplanes avoiding B form a subcollection of those avoiding I. If B spans, the bottom flat is a maximum-compatible cone point in every nonempty flat block, so each block is contractible. □

4. The pair slice and cocircuit witnesses

Suppose $B = \{p, q\}$. Distinct rank-$(s-2)$ flats G, G' satisfy $\mathrm{rk}(G \cap G') \leq s-3$, and adjoining two elements raises rank by at most two. Therefore $\Gamma_{\{p,q\}}(H)$ is discrete. We obtain the following exact pair formula.

Corollary 4.1 (Two-element block formula)**.** *Let $M = N^*$ and let p, q be distinct. Then*

$$T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } N\\ H\cap\{p,q\}=\varnothing}} k^{\mathcal{G}_{\{p,q\}}(H)}/k\mathbf{1}, \tag{16}$$

and

$$\dim_k T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M) = \sum_{H\cap\{p,q\}=\varnothing} (|\mathcal{G}_{\{p,q\}}(H)| - 1)_+. \tag{17}$$

If $\{p, q\}$ is dependent in N, this component vanishes.

Proof. Only the last assertion needs comment. If p, q are parallel nonloops, a rank-$(s-2)$ flat together with $\{p, q\}$ cannot span rank s, so all vertex sets are empty. If one is a loop, no hyperplane avoids the pair. □

Proposition 4.2 (Purely negative cocircuit witnesses)**.** *Let D be a cocircuit of M, let $d = |D| \geq 4$, and let $p, q \in D$ be distinct. Then*

$$\dim_k T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M) \geq d-3.$$

Proof. In $N = M^*$, the set D is a circuit. Put $I = D \setminus \{p, q\}$ and extend $I \cup \{p\}$ to a basis $I \cup \{p\} \cup Z$. Let

$$H = \mathrm{cl}_N(I \cup Z), \qquad G_x = \mathrm{cl}_N((I \setminus \{x\}) \cup Z), \quad x \in I.$$

The fundamental circuit of q with respect to this basis is D. Removing p from the basis gives the independent set $I \cup Z$ of rank $\mathrm{rk}\, N - 1$; neither p nor q belongs to its closure, the latter because its fundamental circuit contains p. For $x \in I$, the set $(I \setminus \{x\}) \cup Z$ has rank $\mathrm{rk}\, N - 2$, while adjoining p, q gives a basis: this is fundamental exchange at $x \in D$. Thus G_x is a vertex

of $\Gamma_{\{p,q\}}(H)$. If $x \neq y$, then $x \in G_y$ but $x \notin G_x$, so these vertices are distinct. There are $d-2$ of them, and (17) gives the bound. □

Proposition 4.3 (Exact rank-one-link component)**.** *Let D be a cocircuit of M, let A be a basis of the complementary hyperplane, and put $d = |D|$. For distinct $p, q \in D$,*

$$\dim_k T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M) = (d-3)_+. \tag{18}$$

Proof. The link $\mathrm{lk}_M A$ is the zero-dimensional complex on D. Apply (1) and Lemma 2.2. □

5. Large cocircuits and the vanishing classification

Lemma 5.1 (A bipartite graph lemma)**.** *Let G be a connected bipartite graph with bipartition $L \sqcup R$, where $|L| \geq 2$ and $|R| \geq 3$. There are distinct $x, y \in L$ such that*

$$N_G(x) \cap N_G(y) \neq \varnothing, \qquad |N_G(x) \cup N_G(y)| \geq 3.$$

Proof. Choose three vertices of R, a spanning tree of G, and the minimal subtree T containing the chosen vertices. Every leaf of T is one of the chosen vertices and hence lies in R.

If some $z \in R \cap V(T)$ has degree at least two in T, choose distinct neighbors $x, y \in L$. Neither is a leaf, so each has a further neighbor in R; the two further neighbors are distinct because T has no cycle. Otherwise every R-vertex of T has degree one. Bipartiteness then forces T to be a star with center $x \in L$, adjacent to all three chosen vertices. Since $|L| \geq 2$ and G is connected, a path from x to another L-vertex begins x, z, y. Thus x, y have a common neighbor, while x already has the three chosen neighbors. □

Proposition 5.2 (Rank-two reduction)**.** *Let M be connected of rank $r \geq 2$ and corank at least three. From any basis B one can find distinct $x, y \in B$ such that, after contracting $A_0 = B \setminus \{x, y\}$ and deleting loops, the resulting rank-two matroid has at least five elements and at least three parallel classes; in particular, it is connected.*

Proof. Apply Lemma 5.1 to the connected fundamental graph $G_B(M)$, with sides B and $E \setminus B$. The chosen x, y have a common nonbasis neighbor and at least three nonbasis neighbors in total. After contracting A_0, all these neighbors remain nonloops, so together with x, y there are at least five elements. If e is a common neighbor, then

$$C_{M/A_0}(e; \{x, y\}) = \{x, y, e\},$$

so x, y, e lie in three different parallel classes. Finally, a loopless rank-two matroid with at least three parallel classes is connected: a disconnected one is a direct sum of two nonempty loopless matroids of rank one and has exactly two parallel classes. □

Theorem 5.3 (Constructive large-cocircuit lemma)**.** *Every connected matroid of corank at least three has a cocircuit of size at least four. If $r = \mathrm{rk}\, M \geq 2$, there is an independent $(r-1)$-set A such that*

$$D = E(M) \setminus \mathrm{cl}_M A$$

is such a cocircuit.

Proof. A connected rank-zero matroid has at most one element, so the corank hypothesis excludes that case. If $r = 1$, connectedness gives $M = U_{1,n}$ with $n \geq 4$, and E itself is a cocircuit. Assume $r \geq 2$ and use Proposition 5.2. Let N be the loopless rank-two contraction, let $m = |E(N)| \geq 5$, and choose a smallest parallel class P. Since there are at least three classes,

$$|E(N) \setminus P| \geq m - \lfloor m/3 \rfloor \geq 4.$$

Choose $v \in P$ and put $A = A_0 \cup \{v\}$. The loops created by contracting A_0 already lie in $\mathrm{cl}_M A_0$, while the other elements in $\mathrm{cl}_M A$ correspond exactly to P. Thus

$$E(M) \setminus \mathrm{cl}_M A = E(N) \setminus P.$$

The closure of A is a hyperplane, so this complement is the required cocircuit. □

Theorem 5.4 (Vanishing classification)**.** *For a finite matroid M, the following are equivalent:*

(1) $T^2(M) = 0$;
(2) $T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M) = 0$ *for all distinct* $p, q \in E(M)$;
(3) *every connected component of M has corank at most two;*
(4) *M is a direct sum of matroids of corank at most two;*
(5) *every cocircuit of M has size at most three;*
(6) *M has no $U_{1,4}$-minor.*

Proof. The first condition implies the second. If the second holds, Proposition 4.2 excludes cocircuits of size at least four, giving (5). If (5) holds and some connected component had corank at least three, Theorem 5.3 would give a contradiction. Thus (3) holds, and (3) is equivalent to (4) by the component decomposition. Condition (4) implies (1) by the known corank-two vanishing and (3). Also, (3) directly implies (5), since cocircuits lie in components and are circuits of duals of rank at most two.

If D is a cocircuit and $H = E \setminus D$, then $M/H \cong U_{1,|D|}$. A cocircuit of size at least four therefore produces a $U_{1,4}$-minor. Conversely, deletion and contraction do not increase corank, and minors of a direct sum are direct sums of minors. Hence a direct sum of corank-at-most-two components has no $U_{1,4}$-minor. This proves the remaining equivalence. □

Corollary 5.5. *The vanishing conjecture in* [6, Conjecture 22] *holds. The class of matroids with $T^2 = 0$ is minor-closed and has the unique excluded minor $U_{1,4}$.*

Remark 5.6. No monotonicity assertion for the module T^2 under arbitrary minors is needed. Only its vanishing class is being identified with a minor-closed matroid class.

6. The rank-one-link strand and the invariant Θ

Assume in this section that $r = \mathrm{rk}\, M \geq 1$. For every independent $(r-1)$-set A, put

$$D_A = E(M) \setminus \mathrm{cl}_M A, \qquad d_A = |D_A|.$$

The set D_A is a cocircuit.

Theorem 6.1 (Complete rank-one-link strand)**.** *Let $\mathbf{c} = \mathbf{a} - \mathbf{b}$ have disjoint nonnegative parts, suppose $|\operatorname{supp} \mathbf{a}| = r - 1$, and put $A = \operatorname{supp} \mathbf{a}$. Then*

$$\dim_k T^2_{\mathbf{c}}(M) = \begin{cases} (d_A - 3)_+, & A \in M,\ \mathbf{b} = \mathbf{e}_p + \mathbf{e}_q,\ \{p,q\} \in \binom{D_A}{2}, \\ 0, & \textit{otherwise.} \end{cases}$$

Thus every nonzero component in this strand has two-element squarefree negative support, and its dimension is independent of the positive exponents and of k.

Proof. For an independent $(r-1)$-set A, the link is Z_{D_A}. Apply the support criterion, (1), and Lemma 2.2. □

Define the finite squarefree witness space

$$W(M) = \bigoplus_{\substack{A \in M \\ |A| = r-1}} \bigoplus_{\{p,q\} \in \binom{D_A}{2}} T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M)$$

and

$$\Theta(M) = \dim_k W(M) = \sum_{\substack{A\in M\\ |A|=r-1}} \binom{d_A}{2}(d_A-3)_+. \tag{19}$$

For a rank-zero matroid we set $W(M) = 0$ and $\Theta(M) = 0$.

Corollary 6.2 (Finite detector)**.** *For every matroid M,*

$$T^2(M) = 0 \quad\Longleftrightarrow\quad W(M) = 0 \quad\Longleftrightarrow\quad \Theta(M) = 0.$$

If M is connected, $r \geq 2$, and $\operatorname{corank} M \geq 3$*, neither $W(M)$ nor the full rank-one-link strand generates $T^2(M)$ as a $k[M]$-module.*

Proof. Every cocircuit is complementary to a hyperplane, and every hyperplane has a basis A. Theorem 6.1 and Theorem 5.4 prove the equivalences. Under the hypotheses in the second assertion, Proposition 4.2 supplies a nonzero purely negative component. Multiplication by a monomial only increases fine degree, so an element with nonempty positive support cannot generate it. □

Corollary 6.3 (Fine and coarse series)**.** *The fine Hilbert series of the rank-one-link strand is*

$$\sum_{\substack{A\in M\\ |A|=r-1}} (d_A-3)_+ \left(\sum_{\{p,q\}\in\binom{D_A}{2}} t_p^{-1}t_q^{-1}\right) \prod_{a\in A} \frac{t_a}{1-t_a}. \tag{20}$$

If $b(N)$ denotes the number of bases of N, then

$$\Theta(M) = \sum_{D \text{ cocircuit of } M} b(M|(E\setminus D)) \binom{|D|}{2}(|D|-3)_+. \tag{21}$$

After setting every $t_e = t$ in (20)*, the series becomes*

$$\frac{t^{r-3}}{(1-t)^{r-1}}\Theta(M). \tag{22}$$

In particular, for $j \geq 0$ and $r \geq 2$, the part of standard degree $r-3+j$ has dimension

$$\Theta(M)\binom{r+j-2}{r-2}.$$

Proof. The first formula follows by summing the positive exponents in Theorem 6.1. Grouping $(r-1)$-sets by their closures gives (21): the sets spanning $E\setminus D$ are its bases. The specialization and coefficient extraction are immediate. □

Put

$$f(d) = \binom{d}{2}(d-3)_+, \qquad h(d) = \begin{cases} 0, & d \leq 3,\\ \dfrac{(d-1)(d-3)}{2}, & d \geq 4.\end{cases}$$

The quantity $h(d)$ need not be an integer; for example, $h(4) = \frac{3}{2}$. Only the aggregated sums below are integral.

Theorem 6.4 (Circuit profile and basis average)**.** *Let $N = M^*$. Then*

$$\Theta(M) = \sum_{C \text{ circuit of } N} b(N/C) f(|C|). \tag{23}$$

Equivalently, if $q_d(N)$ counts spanning subsets of size $\operatorname{rk} N + 1$ whose unique circuit has size d, then

$$\Theta(M) = \sum_d f(d) q_d(N).$$

For a basis B of M and $e \in B$, let

$$D^*(B; e) = E \setminus \operatorname{cl}_M(B \setminus \{e\}), \qquad d_B(e) = |D^*(B; e)|.$$

Then

$$\Theta(M) = \sum_{B \text{ basis of } M} \sum_{e \in B} h(d_B(e)). \tag{24}$$

Proof. Duality gives $(M|(E \setminus D))^* = N/D$ for a cocircuit D, so (21) becomes (23). Extending a basis of N/C by C is a bijection to spanning nullity-one sets with unique circuit C. For the basis average, map (B, e) to $A = B \setminus \{e\}$. A fixed A has exactly d_A preimages, and $d_A h(d_A) = f(d_A)$. □

Corollary 6.5 (Sharp uniform bounds)**.** *Let M have rank $r \geq 1$, n elements, and corank $c = n - r \geq 3$. Then*

$$\Theta(M) \leq b(M)\, r\, \frac{c(c-2)}{2}, \tag{25}$$

$$\Theta(M) \leq \binom{n}{r-1} \binom{c+1}{2} (c-2). \tag{26}$$

Equality in either bound holds if and only if $M = U_{r,n}$.

Proof. Every fundamental cocircuit has size at most $c + 1$, and h is increasing on the relevant range. Formula (24) proves (25). Formula (19) proves (26). Equality forces every independent $(r-1)$-set to be closed. If a circuit had size at most r, deleting one of its elements and extending the remainder to an $(r-1)$-set would contradict closedness. Hence every set of size at most r is independent, so $M = U_{r,n}$. The converse is immediate. □

Corollary 6.6 (Direct sums)**.** *For matroids on disjoint ground sets,*

$$\Theta(M_1 \oplus M_2) = \Theta(M_1) b(M_2) + b(M_1) \Theta(M_2).$$

Consequently $\Theta(M)/b(M)$ is additive over connected components.

Proof. An independent set one element short of a basis of a direct sum is a basis in one summand and one element short in the other. Its complementary cocircuit lies in the latter. Split (19) into the two cases. □

Theorem 6.7 (Tutte contrast)**.** *The Tutte polynomial of M determines whether $T^2(M)$ vanishes. It does not determine $\Theta(M)$ or the coarse rank-one-link Hilbert series.*

Proof. The Tutte polynomial is multiplicative under direct sums, and the polynomial of a connected matroid is irreducible in $\mathbb{Z}[x, y]$ [11]. Unique factorization therefore recovers the polynomial of each connected component. The y-degree is its corank, so Theorem 5.4 decides vanishing.

For the negative statement, let G_1, G_2 be the graphs on $\{0, \ldots, 6\}$ with edge sets

$$E(G_1) = \{04, 06, 14, 15, 16, 23, 25, 26, 34, 35, 36, 45, 56\},$$
$$E(G_2) = \{01, 02, 03, 05, 12, 14, 15, 23, 25, 26, 34, 45, 56\}.$$

Direct rank-subset expansion gives identical Tutte polynomials. The numbers of spanning two-forests with $|E \setminus \mathrm{cl}(A)| = d$, for $d = 2, \dots, 8$, are respectively

$$(209, 235, 141, 130, 101, 41, 5)$$

and

$$(225, 198, 162, 138, 89, 46, 4).$$

Weighting by $f(d)$ gives

$$\Theta(M(G_1)) = 12135, \qquad \Theta(M(G_2)) = 12161.$$

The computation is independently verified as described in Section 19. □

Theorem 6.8 (Reconstruction of large cocircuits)**.** *For each independent $(r-1)$-set A, form a labeled graph on $E \setminus A$ by making $\{p, q\}$ an edge exactly when*

$$\dim_k T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M) > 0.$$

If $d_A \le 3$, this graph is empty. If $d_A \ge 4$, its nonisolated vertex set is D_A, the induced graph on D_A is complete, and every edge has label $d_A - 3$. Hence the family of labeled fine components reconstructs all cocircuits of size at least four, including their sizes. A fixed cocircuit D occurs $b(M|(E \setminus D))$ times.

Proof. This is an immediate restatement of Theorem 6.1. Every cocircuit is the complement of a hyperplane and every hyperplane has a basis. □

7. Rank-two links and the next-to-top strand

Theorem 7.1 (Purely negative T^2 in rank two)**.** *Let L be a loopless rank-two matroid with parallel classes $P_1, \dots, P_t$ of sizes $n_1, \dots, n_t$, and put $n = \sum_i n_i$. Every purely negative component vanishes except in the following cases.*

(1) *If $B = \{p, q\} \subset P_i$, then*

$$\dim_k T^2_{-\mathbf{1}_B}(L) = \left(n_i - 3 + \sum_{\substack{a<c \\ a,c \neq i}} n_a n_c \right)_+ .$$

(2) *If $p \in P_i$, $q \in P_j$, and $i \neq j$, then*

$$\dim_k T^2_{-\mathbf{e}_p - \mathbf{e}_q}(L) = \sum_{k \neq i,j} n_k (n - 3 - n_k)_+ .$$

(3) *Suppose B consists of three elements from three distinct parallel classes. If all elements of $L \setminus B$ lie in at most one parallel class, then*

$$\dim_k T^2_{-\mathbf{1}_B}(L) = (n-4)_+ ;$$

otherwise the component is zero.

Every singleton negative component of $T^2(L)$ vanishes.

Proof. The circuits of L are the pairs within a parallel class and the triples meeting three different classes. A squarefree negative support which is a nonface but not a circuit has equal relative terms and contributes zero.

In (1), B is a circuit. The order-complex incidence graph $|\widetilde{N}_B(L)'|$ has one component for each of the $n_i - 2$ remaining points of P_i, containing that point and all incident edge-face vertices. Every edge joining two classes different from P_i gives an isolated edge-face vertex. Reduced H^0 gives the formula.

In (2), B is a basis. Relative vertices are the points in P_k, $k \neq i, j$. A point of P_k is incident with $n-2-n_k$ edge-faces after deleting p, q, and its relative incidence block contributes $(n-3-n_k)_+$. Summing over the n_k points gives (2).

In (3), B is a circuit and $|\widetilde{N}_B(L)'|$ is the barycentric subdivision of $L \setminus B$. It is connected if at least two parallel classes remain, and otherwise it is a discrete set of $n-3$ points. Finally, the singleton Altmann–Christophersen complex is an incidence graph whose two-face vertices have degree at most one. It is a forest, so its first cohomology vanishes. □

Corollary 7.2 (Complete next-to-top strand)**.** *Let M have rank $r \geq 2$, let $\mathbf{a} \in \mathbb{N}^E$ have independent support A of size $r-2$, and let $\mathbf{b}$ be nonzero, squarefree, and supported on actual vertices of $L = \mathrm{lk}_M A$. Then*

$$T^2_{\mathbf{a}-\mathbf{b}}(M) \cong T^2_{-\mathbf{b}}(L),$$

and Theorem 7.1 computes the right-hand side. All remaining components with positive-support size $r-2$ vanish by the support criterion.

Corollary 7.3 (Projective planes)**.** *Let $q \geq 2$ be a prime power, let $M = PG(2,q)$, and put $\nu = q^2+q+1$,*

$$\alpha = q-3+\binom{q}{2}q^2, \qquad \gamma = q(q-1)(q^2-3).$$

Then

$$\dim_k T^2(M)_0 = \Theta(M) + \nu\left((q+1)\binom{q}{2}\alpha + \binom{q+1}{2}q^2\gamma\right) \tag{27}$$

$$= \frac{q(q-1)(q+1)(q^2+q+1)}{4}\left(3q^5+2q^4-10q^3-3q^2+2q-6\right). \tag{28}$$

In particular, $\dim_k T^2(F_7)_0 = 357$, with 126 dimensions in the rank-one-link strand and 231 in the rank-two-link strand.

Proof. Every point link has $q+1$ parallel classes of size q. The within-class and between-class pair contributions are exactly the first two cases of Theorem 7.1; triple contributions vanish. The support criterion shows that the rank-one and rank-two link contributions exhaust standard degree zero. Formula (28) is the expansion of (27) together with

$$\Theta(PG(2,q)) = (q^2+q+1)\binom{q+1}{2}\binom{q^2}{2}(q^2-3).$$

□

8. Universal cup products and actual obstructions

Retain the notation $M = N^*$. For $p \in E$, write

$$\mathscr{H}_p = \{H : H \text{ is a hyperplane of } N,\ p \notin H\}.$$

The singleton part of Theorem 3.1 identifies

$$T^1_{-\mathbf{e}_p}(M) \cong k^{\mathscr{H}_p}/k\mathbf{1}, \tag{29}$$

where the quotient means the cokernel of the diagonal constant map.

For a hyperplane H avoiding p, q and a flat $G \in \mathcal{G}_{\{p,q\}}(H)$, put

$$P_G = \mathrm{cl}_N(G \cup \{p\}), \qquad Q_G = \mathrm{cl}_N(G \cup \{q\}). \tag{30}$$

Since G has corank two and spans with p, q, these are hyperplanes, with

$$q \notin P_G, \quad p \notin Q_G, \qquad H \cap P_G = H \cap Q_G = P_G \cap Q_G = G. \tag{31}$$

Lemma 8.1 (Field-independent coordinate formula)**.** *Represent source classes in* (29) *by functions*

$$\phi : \mathscr{H}_p \to k, \qquad \psi : \mathscr{H}_q \to k,$$

each modulo constants. Under the block identification (16)*, the H-block of their cup product is, up to the fixed orientation of that block,*

$$\big[(\phi(Q_G) - \phi(H))(\psi(P_G) - \psi(H))\big]_{G \in \mathcal{G}_{\{p,q\}}(H)} \pmod{k\mathbf{1}}. \tag{32}$$

This formula is valid over every field.

Proof. The Stanley–Reisner ideal has its monomial Taylor presentation over $\mathbb{Z}$. The cup product on presentation cochains is obtained by lifting the images of monomial generators and substituting them into the binomial relation module. This construction uses only addition, multiplication, and the coefficients $0, \pm 1$, so its coordinate formula commutes with base change to any field.

For completeness, consider the relative edge indexed by $G \subset H$. In the notation of Altmann and Christophersen, it is represented by

$$f = E \setminus (H \cup \{p, q\}) \subset g = E \setminus (G \cup \{p, q\}).$$

Substitution in the presentation formula, taking $d = \varnothing$ for f and $e = \{q\}$ for g, gives

$$E_H(G) = (\phi(H) - \phi(Q_G))\psi(P_G) + \psi(H)\phi(Q_G). \tag{33}$$

The possible fourth term is zero because its argument does not lie in the singleton set. Rearranging,

$$E_H(G) = -(\phi(Q_G) - \phi(H))(\psi(P_G) - \psi(H)) + \phi(H)\psi(H).$$

The last term is constant in G and vanishes in the quotient by $k\mathbf{1}$; reversing the block orientation absorbs the remaining sign. This is precisely (32). The calculation is the specialization of [1, Proposition 19], now observed at the integral monomial-presentation level. □

Theorem 8.2 (Universal pure-pair surjectivity)**.** *For every matroid M, every pair of distinct actual vertices p, q, and every field k, the cup product*

$$T^1_{-\mathbf{e}_p}(M) \otimes_k T^1_{-\mathbf{e}_q}(M) \longrightarrow T^2_{-\mathbf{e}_p - \mathbf{e}_q}(M) \tag{34}$$

is surjective.

Proof. Use $M = N^*$ and fix a target coordinate (H_0, G_0) in (16). Put

$$P_0 = P_{G_0}, \qquad Q_0 = Q_{G_0},$$

and

$$\mathcal{S}(G_0) = \{H : H \cap \{p, q\} = \varnothing,\ G_0 \subset H\}.$$

Define representatives of singleton classes by

$$\phi(K) = \begin{cases} 1, & K = Q_0, \\ 1, & K \in \mathcal{S}(G_0) \setminus \{H_0\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi(K) = \begin{cases} 1, & K = P_0, \\ 0, & \text{otherwise}. \end{cases}$$

There is no conflict: Q_0 contains q, P_0 contains p, while every member of $\mathcal{S}(G_0)$ avoids both.

For any coordinate (H, G), the second factor in (32) can be nonzero only when $P_G = P_0$. If also $G = G_0$, then $H \in \mathcal{S}(G_0)$ and the first factor is nonzero exactly for $H = H_0$. If $P_G = P_0$ but $G \neq G_0$, then $Q_G \neq Q_0$ by (31). Moreover $H \notin \mathcal{S}(G_0)$: otherwise

$$G_0 \subseteq H \cap P_0 = G$$

would force equality of the two rank-$(s-2)$ flats. Hence the first factor is zero. The resulting product is the coordinate delta vector at (H_0, G_0), up to sign. Such vectors span before passing to blockwise constants, so their images span the target. □

Corollary 8.3 (Link-lifted cup products)**.** *Let $A \in M$, and let p, q be distinct actual vertices of $L = \mathrm{lk}_M A$. Then*

$$T^1_{\mathbf{1}_A - \mathbf{e}_p}(M) \otimes_k T^1_{\mathbf{1}_A - \mathbf{e}_q}(M) \twoheadrightarrow T^2_{2\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M).$$

The target is identified with $T^2_{-\mathbf{e}_p - \mathbf{e}_q}(L)$.

Proof. Iterate Lemma 2.1 over the variables indexed by A, use (1), and transport Theorem 8.2 across the resulting cup-compatible isomorphisms. □

The next lemma extracts a singleton link whose contraction still has large corank.

Lemma 8.4 (One contraction retaining large corank)**.** *Let M be connected with* $\mathrm{rk}\, M \geq 2$ *and* $\mathrm{corank}\, M \geq 3$*. There is an element e such that M/e, after deleting its loops, has a connected component of corank at least three.*

Proof. If $\mathrm{rk}\, M = 2$, choose e in a smallest parallel class P. A connected rank-two matroid has at least three parallel classes, and $|E| \geq 5$, so

$$|E \setminus P| \geq |E| - \lfloor |E|/3 \rfloor \geq 4.$$

After contracting e and deleting $P \setminus \{e\}$, one obtains a connected rank-one matroid of corank at least three.

Suppose $\mathrm{rk}\, M \geq 3$. Proposition 5.2 gives an independent set A_0 of size $\mathrm{rk}\, M - 2$ such that M/A_0, after deleting loops, is a connected rank-two matroid R on at least five elements. Choose $e \in A_0$ and let $L = M/e$ with its loops deleted. Contracting $A_0 \setminus \{e\}$ in L, and then deleting loops, produces R. Contraction acts componentwise and cannot join components. Contracting nonloops preserves corank, while deleting newly created loops can only decrease it. Hence all nonloops of R descend from a single connected component C of L, with

$$\mathrm{corank}\, C \geq \mathrm{corank}\, R \geq 3.$$

□

Theorem 8.5 (Nonzero standard-degree-zero cup product)**.** *Let M be connected with* $\mathrm{rk}\, M \geq 2$ *and* $\mathrm{corank}\, M \geq 3$*. There are distinct $e, p, q \in E(M)$ such that*

$$T^1_{\mathbf{e}_e - \mathbf{e}_p}(M) \otimes_k T^1_{\mathbf{e}_e - \mathbf{e}_q}(M) \twoheadrightarrow T^2_{2\mathbf{e}_e - \mathbf{e}_p - \mathbf{e}_q}(M) \neq 0. \tag{35}$$

Thus $T^1(M)_0 \otimes_k T^1(M)_0 \to T^2(M)_0$ has nonzero image over every field.

Proof. Choose e by Lemma 8.4, put $L = M/e$ after deleting loops, and let C be a connected component of L with corank at least three. By Theorem 5.3, C has a cocircuit D of size at least four. Choose distinct $p, q \in D$. Proposition 4.2 gives

$$T^2_{-\mathbf{e}_p - \mathbf{e}_q}(C) \neq 0.$$

If $L = C \oplus C'$, the direct-sum formula, or equivalently the dual hyperplane blocks, embeds this component into the same fine component for L. Theorem 8.2 gives a surjection onto it. Localization at x_e and evaluation at $x_e = 1$ give cup-compatible isomorphisms

$$T^1_{\mathbf{e}_e - \mathbf{e}_p}(M) \cong T^1_{-\mathbf{e}_p}(L),$$
$$T^1_{\mathbf{e}_e - \mathbf{e}_q}(M) \cong T^1_{-\mathbf{e}_q}(L),$$
$$T^2_{2\mathbf{e}_e - \mathbf{e}_p - \mathbf{e}_q}(M) \cong T^2_{-\mathbf{e}_p - \mathbf{e}_q}(L).$$

Transporting the surjection proves (35). Each fine degree has total degree zero. □

Let Def_M denote the affine deformation functor of $k[M]$ on local Artinian k-algebras with residue field k, and let $\mathrm{Def}_M^{\mathrm{gr}}$ denote its standard-degree-preserving subfunctor. Their tangent spaces are $T^1(M)$ and $T^1(M)_0$, respectively; obstructions to lifting across small extensions lie in $T^2(M)$ and $T^2(M)_0$. For the cotangent-complex obstruction theory over arbitrary bases, see [14, Tag 08UX, Lemma 92.21.1]; for the classical description of first-order affine deformations by the first cotangent module, see [13, Proposition 3.1.1, pp. 103–104]. The cup formula of Altmann and Christophersen describes the quadratic term [1, Appendix 6, equation (6.1) and Proposition 19]. When 2 is invertible, the primary obstruction is one half of the cup square. The following presentation-level argument identifies its polarization without dividing by two.

Lemma 8.6 (Graded polarization of the primary obstruction)**.** *Let k be an arbitrary field and let* $\mathrm{ob} : T^1(M) \to T^2(M)$ *denote the primary obstruction map for lifting first-order deformations from $k[\epsilon]/(\epsilon^2)$ to $k[\epsilon]/(\epsilon^3)$.*

(1) ob *doubles the fine grading: for a fine-homogeneous class χ of degree $\mathbf{c}$, the class* $\mathrm{ob}(\chi)$ *lies in $T^2_{2\mathbf{c}}(M)$. This component vanishes, so* $\mathrm{ob}(\chi) = 0$.

(2) *For all $\alpha, \beta \in T^1(M)$,*

$$\mathrm{ob}(\alpha + \beta) = \mathrm{ob}(\alpha) + \mathrm{ob}(\beta) + \alpha \cup \beta.$$

In particular, for fine-homogeneous classes α, β of distinct fine degrees, $\mathrm{ob}(\alpha + \beta) = \alpha \cup \beta$.

Proof. Work with the monomial Taylor presentation of $I_M = (m_1, \dots, m_r)$. A first-order deformation with tangent class χ deforms the generators to $m_i + \epsilon g_i$, where $g_i \in S$ lifts $\chi(m_i)$. For the pair relation of (m_i, m_j) with least common multiple u_{ij}, first-order flatness provides an expression

$$\frac{u_{ij}}{m_i}\, g_i - \frac{u_{ij}}{m_j}\, g_j = \sum_k b_k^{ij} m_k \qquad \text{in } S,$$

with b_k^{ij} chosen linearly in the tuple (g_i). A second-order lift $m_i + \epsilon g_i + \epsilon^2 h_i$ exists if and only if the presentation cochain

$$o(\chi) : (i,j) \longmapsto \Big[\sum_k b_k^{ij}\, g_k\Big] \in S/I_M$$

is a coboundary; its class in $T^2(M)$ is the primary obstruction and is independent of all choices. Every operation involved preserves the fine multigrading, and $o(\chi)$ is the composition of the linear map $\chi \mapsto b(\chi)$ with multiplication by the values of χ. Hence o is a quadratic map, and for fine-homogeneous χ of degree $\mathbf{c}$ the cochain $o(\chi)$ is homogeneous of degree $2\mathbf{c}$. By the support criterion, a nonzero fine component of T^1 has degree $\mathbf{c} = \mathbf{a} - \mathbf{1}_B$ with $B \neq \varnothing$; the negative part of $2\mathbf{c}$ has an exponent 2, so $T^2_{2\mathbf{c}}(M) = 0$ and $\mathrm{ob}(\chi) = 0$. This proves (1).

Expanding $o(\alpha + \beta)$ bilinearly gives

$$o(\alpha + \beta) = o(\alpha) + o(\beta) + \Big[(i,j) \mapsto \sum_k \big(b_k^{ij}(\alpha)\, \beta(m_k) + b_k^{ij}(\beta)\, \alpha(m_k)\big)\Big].$$

The mixed cochain is the substitution construction of Lemma 8.1, whose class is $\alpha \cup \beta$; the identity uses only the coefficients $0, \pm 1$ of the monomial presentation and holds over every field. This proves (2), and the final statement follows by combining (1) and (2). □

Corollary 8.7 (Actual quadratic obstructions)**.** *Let k be an arbitrary field.*

(1) *If some connected component of M has corank at least three, then some first-order affine deformation of $k[M]$ does not lift from $k[\epsilon]/(\epsilon^2)$ to $k[\epsilon]/(\epsilon^3)$.*

(2) *If M is connected with* $\operatorname{rk} M \geq 2$ *and* $\operatorname{corank} M \geq 3$*, such a deformation can be chosen in standard degree zero. Hence* $\mathrm{Def}_M^{\mathrm{gr}}$ *is not formally smooth, and any miniversal hull has a nonzero quadratic Kuranishi term.*

Proof. For (1), Theorem 5.4 provides a cocircuit of size at least four, and Proposition 4.2 then provides a nonzero purely negative pair target. For (2), use the nonzero target in Theorem 8.5. In either case, surjectivity gives fine-homogeneous source classes α, β of distinct fine degrees with $\alpha \cup \beta \neq 0$. Lemma 8.6 gives

$$\mathrm{ob}(\alpha + \beta) = \alpha \cup \beta \neq 0,$$

so the first-order deformation $\alpha + \beta$ does not lift to second order. In (2), both source classes have standard degree zero. □

Corollary 8.8 (Smoothness dichotomy)**.** *For every field k and every matroid M, the following are equivalent:*

(1) Def_M *is formally smooth;*
(2) $T^2(M) = 0$*;*
(3) *every connected component of M has corank at most two;*
(4) *M has no $U_{1,4}$-minor.*

Proof. (2)⇒(1) is standard, since all obstructions lie in $T^2(M)$. (1)⇒(3) is the contrapositive of Corollary 8.7(1). The equivalences (2)⇔(3)⇔(4) are part of Theorem 5.4. □

Corollary 8.9 (Graded smoothness dichotomy)**.** *Let M be connected. For every field k, the graded functor* $\mathrm{Def}_M^{\mathrm{gr}}$ *is formally smooth if and only if* $\operatorname{rk} M \leq 1$ *or* $\operatorname{corank} M \leq 2$*.*

Proof. If $\operatorname{corank} M \leq 2$, then $T^2(M) = 0$. If $\operatorname{rk} M \leq 1$, the only independent supports are the empty set and singletons; a singleton link of a rank-one matroid has empty actual vertex set, and singleton negative supports vanish by (8), so every nonzero component of $T^2(M)$ is a purely negative block of negative standard total degree. Hence $T^2(M)_0 = 0$ in both cases, and the graded functor, whose obstructions lie in $T^2(M)_0$, is formally smooth. If $\operatorname{rk} M \geq 2$ and $\operatorname{corank} M \geq 3$, Corollary 8.7(2) shows that $\mathrm{Def}_M^{\mathrm{gr}}$ is not formally smooth. □

Remark 8.10 (The divided square is invisible). In characteristic two the primary obstruction is a quadratic refinement of the cup product rather than one half of the cup square. Lemma 8.6 shows that for matroid Stanley–Reisner rings this refinement carries no extra information: it vanishes on every fine-homogeneous class, so the whole quadratic map is determined by the bilinear cup product,

$$\mathrm{ob}\Big(\sum_{\mathbf{c}} \chi_{\mathbf{c}}\Big) = \sum_{\{\mathbf{c},\mathbf{c}'\},\, \mathbf{c}\neq\mathbf{c}'} \chi_{\mathbf{c}} \cup \chi_{\mathbf{c}'},$$

the sum running over unordered pairs of distinct fine degrees. In this sense the primary obstruction theory of this paper is characteristic-free.

The same degree bookkeeping controls the entire lifting tower, not only its first step. The next theorem shows that fine-homogeneous classes integrate without higher corrections, and that along a pure pair the single quadratic class is the complete obstruction.

Theorem 8.11 (Linear arcs and exact quadratic obstructions)**.** *Let k be an arbitrary field. Represent classes by multigraded cocycles, so that $\chi \in T^1_{\mathbf{c}}(M)$ assigns to each circuit generator m_i a multiple $\chi(m_i) \in S$ of the single monomial of multidegree* $\deg m_i + \mathbf{c}$*.*

(1) *Suppose* $\chi = \sum_j \chi_j$ *with* $\chi_j \in T^1_{\mathbf{c}_j}(M)$*, and suppose that for every pair* (j, j')*, including* $j = j'$*, the vector* $\mathbf{c}_j + \mathbf{c}_{j'}$ *has an entry at most* -2*. Then the* t*-linear family*

$$F_i = m_i + t\,\chi(m_i)$$

is flat over $k[t]/(t^N)$ *for every* N*: every Taylor relation lifts with coefficients linear in* t*, and no correction terms are needed. This applies to every single fine-homogeneous class and, more generally, whenever the negative supports* B_j *pairwise intersect and* $\big(\bigcup_j \operatorname{supp} \mathbf{a}_j\big) \cap \big(\bigcup_j B_j\big) = \varnothing$*.*

(2) *Let* $\alpha \in T^1_{\mathbf{c}_1}(M)$ *and* $\beta \in T^1_{\mathbf{c}_2}(M)$ *with* $\mathbf{c}_i = \mathbf{a}_i - \mathbf{1}_{B_i}$*,* $B_1 \cap B_2 = \varnothing$*, and* $\operatorname{supp}(\mathbf{a}_1 + \mathbf{a}_2) \cap (B_1 \cup B_2) = \varnothing$*. Then* $t(\alpha + \beta)$ *lifts to* $k[t]/(t^3)$ *if and only if* $\alpha \cup \beta = 0$*, and in that case a single multigraded correction* h *of fine degree* $\mathbf{c}_1 + \mathbf{c}_2$ *produces a family*

$$F_i = m_i + t\big(\alpha(m_i) + \beta(m_i)\big) + t^2 h_i$$

which is flat over $k[t]/(t^N)$ *for every* N*. Along these directions the cup product is the complete obstruction.*

Proof. A polynomial of fixed multidegree with a negative entry is zero. Least common multiples u_{ij} of squarefree generators have exponents at most one, so every element of S of multidegree $\deg u_{ij} + \mathbf{d}$ or $\deg m_i + \mathbf{d}$ vanishes identically whenever $\mathbf{d}$ has an entry at most -2.

We use the relation-lifting criterion for flatness over $A_N = k[t]/(t^N)$. If generators F_i lift the generators m_i and every generating relation among the m_i lifts to a relation among the F_i, then, for $J = (F_i) \subset A_N[x_e : e \in E]$, reduction identifies J/tJ with I_M. Tensoring the ideal sequence with k therefore gives $\operatorname{Tor}_1^{A_N}(A_N[x]/J, k) = 0$. The local flatness criterion for the nilpotent ideal (t) gives A_N-flatness. The Taylor pair relations generate the relation module of a monomial ideal, so it suffices to lift those relations.

For (1), lift the Taylor relation of (m_i, m_j) to first order as in Lemma 8.6, with linear coefficient $-b^{ij}$. The complete relation applied to (F_i) has one remaining coefficient, at t^2, namely $-b^{ij} \cdot \chi(m_\bullet)$, a sum of terms of multidegrees $\deg u_{ij} + \mathbf{c}_\ell + \mathbf{c}_{\ell'}$, where ℓ, ℓ' index the summands of χ. Each such term is identically zero by hypothesis, so the linear relation lift is an exact syzygy of (F_i). Lifting a generating set of relations proves flatness over every $k[t]/(t^N)$. For the last assertion, an entry of $\mathbf{c}_\ell + \mathbf{c}_{\ell'}$ at an element of $B_\ell \cap B_{\ell'}$ equals -2 when $\ell \neq \ell'$, and at an element of B_ℓ equals -2 when $\ell = \ell'$, since the positive supports avoid all negative supports.

For (2), the second-order equation decomposes into fine degrees $2\mathbf{c}_1$, $\mathbf{c}_1 + \mathbf{c}_2$, and $2\mathbf{c}_2$. The doubled blocks vanish identically as in (1). The mixed block is solvable exactly when the class $\alpha \cup \beta$ vanishes, by Lemma 8.6; this proves the first assertion. When $\alpha \cup \beta = 0$, choose h of fine degree $\mathbf{c}_1 + \mathbf{c}_2$ solving the mixed block and lift each relation to second order. Every higher coefficient of the lifted relation is a sum of terms whose fine degrees have the form $a\mathbf{c}_1 + b\mathbf{c}_2$ with $a + b \geq 3$. Since $\max(a, b) \geq 2$ and the positive supports avoid $B_1 \cup B_2$, every such degree has an entry at most -2 on B_1 or B_2, so all these terms vanish identically and the quadratic relation lifts are exact syzygies of (F_i). □

Theorem 8.11 leaves open directions with support cancellation. For example, take $\mathbf{c}_1 = \mathbf{e}_e - \mathbf{e}_p$ and $\mathbf{c}_2 = \mathbf{e}_p - \mathbf{e}_q$, with e, p, q distinct. The mixed second-order obstruction has singleton negative support and vanishes by (8). After a homogeneous second-order lift is fixed, its only possibly nonzero subsequent obstruction has degree $2\mathbf{c}_1 + \mathbf{c}_2 = 2\mathbf{e}_e - \mathbf{e}_p - \mathbf{e}_q$. This class depends on the chosen second-order lift; it is not a canonical class attached to the triple (e, p, q) alone. Indeed, changing the mixed correction by $\delta \in T^1_{\mathbf{c}_1 + \mathbf{c}_2}(M)$ can change the cubic class by $\alpha \cup \delta$, where α is the class of degree $\mathbf{c}_1$.

There are no later degrees to consider in this homogeneous lifting problem: $a\mathbf{c}_1 + b\mathbf{c}_2$ has all coordinates at least -1 only if $b \le 1$ and $a - b \le 1$. Thus every such degree with $a + b \ge 4$ has a coordinate at most -2. A choice of second- and third-order corrections that solves the cubic equation therefore extends without further corrections to every order. This is a termination statement, not a proof that a suitable cubic lift always exists.

9. Certificates, algorithms, and examples

9.1. **A rank-oracle certificate.** Assume access to a rank oracle for M. A certificate for T^2-nonvanishing is a tuple (A, D, p, q) with distinct $p, q \in D$, satisfying

(1) A is independent and $|A| = \operatorname{rk} M - 1$;
(2) $E \setminus D = \operatorname{cl}_M A$;
(3) $|D| \ge 4$.

The closure condition is checked by verifying

$$\operatorname{rk}(A \cup \{e\}) = \operatorname{rk} A \quad (e \in E \setminus D), \qquad \operatorname{rk}(A \cup \{e\}) = \operatorname{rk} A + 1 \quad (e \in D).$$

Once these tests pass, Proposition 4.3 certifies

$$\dim_k T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M) = |D| - 3.$$

This tuple records a nonzero obstruction space, not a particular tangent cocycle. An actual nonliftable deformation is then obtained from Proposition 4.2, the coordinate preimages in Theorem 8.2, and Lemma 8.6.

For a connected matroid of corank at least three, a certificate is constructed as follows.

(1) Find a basis B greedily. If $|B| = 1$, output $A = \varnothing$, $D = E$, and any pair $p, q \in E$.
(2) Construct the fundamental graph. Use Lemma 5.1 to find $x, y \in B$.
(3) Contract $B \setminus \{x, y\}$, delete loops, and compute the parallel classes of the resulting rank-two matroid.
(4) Choose a smallest class P and $v \in P$. Put $A = (B \setminus \{x, y\}) \cup \{v\}$ and $D = E \setminus \operatorname{cl}_M A$.
(5) Output (A, D, p, q) for any distinct $p, q \in D$.

The construction uses $O(n^2)$ rank-oracle queries. An independence oracle simulates a rank query greedily, so the procedure is also polynomial in that model.

9.2. **Uniform, graphic, and cographic matroids.** For $M = U_{r,n}$ of corank $c = n - r$, every cocircuit has size $c + 1$. Hence

$$\dim_k T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(U_{r,n}) = (c - 2)_+$$

for every hyperplane A and pair in its complement, and

$$\Theta(U_{r,n}) = \binom{n}{r-1}\binom{c+1}{2}(c-2)_+.$$

Corollary 9.1 (Graphic matroids)**.** *For a finite graph G, $T^2(M(G)) = 0$ if and only if every cycle-matroid component has cyclomatic number at most two, equivalently every bond has size at most three. For a simple graph, this vanishing is equivalent to every edge-containing block being a bridge, a cycle, or a theta graph.*

If G is connected and deleting a bond D gives components G_1^D, G_2^D, then

$$\Theta(M(G)) = \sum_{D \text{ bond}} \tau(G_1^D)\tau(G_2^D)\binom{|D|}{2}(|D| - 3)_+,$$

where τ denotes the number of spanning trees and τ of a single vertex is one.

Proof. Cocircuits of $M(G)$ are bonds, and the corank of a cycle-matroid component is its cyclomatic number. The simple-block description follows from open-ear decomposition [8, Proposition 3.1.1, pp. 59–60]: a cycle starts with cyclomatic number one, and each added ear increases it by one. A basis of $M(G)|(E \setminus D)$ is a spanning tree in each of the two components, so (21) gives the formula. □

Corollary 9.2 (Cographic matroids)**.** *For a finite graph G,*

$$T^2(M^*(G)) = 0 \quad \Longleftrightarrow \quad \textit{every cycle of } G \textit{ has at most three edges.}$$

For a simple graph, this vanishing is equivalent to every edge-containing block being a bridge or a triangle. If G is connected, then

$$\Theta(M^*(G)) = \sum_U f(\ell(U)),$$

where U ranges over connected spanning unicyclic subgraphs and $\ell(U)$ is the length of its unique cycle.

Proof. Cocircuits of $M^*(G)$ are graph cycles. Apply Theorem 5.4. The simple-block statement again follows from open-ear decomposition. The last formula is the nullity-one interpretation in Theorem 6.4. □

The cocircuit threshold four is sharp in every corank: for $c \geq 3$, the connected matroid $M(K_{2,c-1})^*$ has corank c and all cocircuits have size exactly four. Indeed, $K_{2,c-1}$ has $c+1$ vertices and is connected, so its cycle matroid has rank c; every graph cycle has length four. Thus Theorem 5.3 cannot be strengthened to a bound growing with corank. This family is related to Murty's extremal critically connected matroids [12]; the direct argument above is all that is needed here.

10. A sharp rank-two extremal theorem

This section records a quantitative consequence of the hyperplane-graph formula proved earlier in the paper.

10.1. **The witness invariant.** Recall that, for each independent set A of size $r-1$, we write

$$d_A = |E \setminus \mathrm{cl}_M(A)|$$

and define the invariant

$$\Theta(M) = \sum_{\substack{A \text{ independent} \\ |A|=r-1}} \binom{d_A}{2} (d_A - 3)_+. \tag{36}$$

Equivalently, grouping the same terms by cocircuit D gives

$$\Theta(M) = \sum_D b(M|(E \setminus D)) \binom{|D|}{2} (|D| - 3)_+. \tag{37}$$

These are (19) and (21). The rank-one-link calculation in Proposition 4.3 identifies $\Theta(M)$ with the total dimension of the finite squarefree witness space. In rank two, the same sum is exactly $\dim_k T^2(M)_0$.

10.2. **A closed formula in rank two.** Let M be a connected loopless rank-two matroid. Its nonloop elements split into parallel classes $P_1, \dots, P_t$ of sizes $n_1, \dots, n_t$; connectedness is equivalent to $t \geq 3$. Put $n = |E|$ and $c = n - 2$, and set $F(d) = \binom{d}{2}(d-3)_+$.

Proposition 10.1 (Rank-two closed form)**.** *For every connected rank-two matroid,*

$$\Theta(M) = \sum_i n_i F(n - n_i), \qquad \dim_k T^2(M)_0 = \Theta(M). \tag{38}$$

Proof. Every independent set of size $r-1$ is a single nonloop element a. If $a \in P_i$, then $\mathrm{cl}_M(a) = P_i$ and the complementary cocircuit has size $n - n_i$. The complementary hyperplane has n_i bases, one for each element of P_i, so (37) gives the first formula. For the standard total-degree-zero component, a positive support of size two has a rank-zero link and contributes nothing. A positive support of size one must have exponent two; its rank-one link contributes only squarefree negative pairs. Summing (36) over all singletons gives the second formula. □

10.3. **The sharp minimum at fixed corank.**

Theorem 10.2 (Sharp rank-two extremum)**.** *Among connected rank-two matroids of corank $c \geq 3$, the minimum of Θ is*

$$\begin{cases} 6, & c = 3, \\ 32, & c = 4, \\ 81, & c = 5, \\ 168, & c = 6, \\ c(c+1)(c-2), & c \geq 7. \end{cases} \tag{39}$$

Up to permutation of the classes, equality occurs for

c	*parallel-class sizes*
3	$(2,2,1)$
4	$(3,2,1)$
5	$(3,3,1)$
6	$(6,1,1)$ *or* $(4,3,1)$
$c \geq 7$	$(c,1,1)$.

(40)

For $c \geq 7$ the equality class is unique.

Proof. Put $g_n(x) = xF(n-x)$. By (38), we must minimize $\sum_i g_n(n_i)$ over partitions of $n = c+2$ into at least three positive parts.

First, let $u, v \geq 1$ and $u + v \leq n - 2$, and write $w = n - u - v$. If $w \geq 4$, direct expansion gives

$$\begin{aligned} 2\big(g_n(u) + g_n(v) - g_n(u+v)\big) = uv\big[&u^2 + v^2 + 3(u+v)w + 6w^2 \\ &- 4(u+v) - 16w + 6\big] > 0. \end{aligned} \tag{41}$$

Indeed, $3w - 4 > 0$ and $6w^2 - 16w + 6 > 0$ for $w \geq 4$. If $w = 2$ or 3, then $g_n(u+v) = 0$, while at least one of $g_n(u), g_n(v)$ is positive for $n \geq 5$. Thus every minimizer has exactly three parts.

Let $a \leq b \leq d$ and $a + b + d = n$. If $a \geq 2$, transfer one element from the smallest class to the largest and set

$$\Delta = g_n(a) + g_n(d) - g_n(a-1) - g_n(d+1).$$

With $A = a - 2$, $U = b - a$, and $V = d - b$, expansion gives

$$\begin{aligned} 2\Delta = 8 &+ 24A + 32U + 12V + 12A^2 + 45AU + 27AV \\ &+ 34U^2 + 33UV + 5V^2 + 12A^2U + 12A^2V \\ &+ 21AU^2 + 24AUV + 3AV^2 + 10U^3 + 15U^2V + 6UV^2 + V^3. \end{aligned} \tag{42}$$

All coefficients are positive, so a minimizing triple has the form $(1, b, d)$.

Finally let $s = b + d = n - 1$ and $p = bd$. For $b \geq 2$,

$$2\big(g_n(b) + g_n(d) - g_n(1)\big) = E(s,p), \quad E(s,p) = -2p^2 + (s^2 - s - 4)p - s(s-1)(s-3). \tag{43}$$

For $n \geq 9$, E is concave on $2(s-2) \leq p \leq \lfloor s^2/4 \rfloor$. At the left endpoint,

$$s^3 - 10s^2 + 25s - 16 > 0,$$

while at the right endpoint, writing $s = 2m$ or $s = 2m + 1$ gives, respectively,

$$2m(m^3 - 5m^2 + 6m - 3) > 0 \qquad \text{or} \qquad 2m^3(m-3) > 0.$$

Hence $b = 1$ is the unique minimizer for $n \geq 9$, with value

$$2F(n-1) = (n-1)(n-2)(n-4) = c(c+1)(c-2). \tag{44}$$

The cases $n = 5, 6, 7, 8$ are the four exceptional rows in (40), obtained by direct enumeration of the three-part partitions. □

10.4. **Consequences.** For $c \geq 7$, the rank-two matroid R_c with class sizes $(c, 1, 1)$ satisfies

$$\Theta(R_c) = c(c+1)(c-2) = \dim_k T^2(R_c)_0$$

and realizes the sharp minimum. The uniform rank-two matroid $U_{2,c+2}$ has

$$\Theta(U_{2,c+2}) = \frac{(c+2)(c+1)c(c-2)}{2},$$

so the ratio of the sparse extremal value to the uniform value is $2/(c+2)$.

The theorem gives $\Theta(M) \geq 6$ for every connected rank-two matroid of corank at least three. Over every field, the pure-pair cup-product theorem and the graded polarization lemma of Section 8 turn these nonzero standard-degree-zero classes into actual nonliftable first-order deformations.

The proof is symbolic and independent of computation. Additional exact checks are described in Section 19; they are not premises of the proof.

11. Integral pair obstructions and a global rank

The coordinate-delta argument of Section 8 contains more arithmetic information than fieldwise surjectivity. It gives a split map over the integers and therefore a canonical numerical size for the whole pure-pair obstruction layer.

For a matroid N and an element p, put

$$\mathsf{L}^1_p(N) = \widetilde{H}^0(\mathscr{H}_p(N); \mathbb{Z}),$$

where $\mathscr{H}_p(N)$ is the discrete set of hyperplanes avoiding p. For distinct p, q, put

$$\mathsf{L}^2_{p,q}(N) = \bigoplus_{\substack{H \text{ hyperplane of } N \\ H \cap \{p,q\} = \varnothing}} \widetilde{H}^0\Big(\Gamma_{\{p,q\}}(H); \mathbb{Z}\Big).$$

Both groups are free abelian. Interpreting (32) with integer-valued functions defines an integral bilinear map

$$\smile_{\mathbb{Z}}\colon \mathsf{L}^1_p(N) \otimes_{\mathbb{Z}} \mathsf{L}^1_q(N) \longrightarrow \mathsf{L}^2_{p,q}(N). \tag{45}$$

Theorem 11.1 (Integral pure-pair obstruction lattice)**.** *For every matroid M and every pair of distinct actual vertices p, q, the map* (45) *for $N = M^*$ is a split surjection of abelian groups. Consequently its scalar extension is surjective over every commutative ring. Over a field k, the identifications* (29) *and* (16) *identify the scalar extension with the cotangent cup product*

$$T^1_{-\mathbf{e}_p}(M) \otimes_k T^1_{-\mathbf{e}_q}(M) \longrightarrow T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M).$$

In particular,

$$\operatorname{rank}_{\mathbb{Z}} \mathsf{L}^2_{p,q}(M^*) = \dim_k T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M),$$

and every nonzero pair block remains a split cup-product quotient after reduction modulo every prime.

Proof. Fix a target coordinate (H_0, G_0) and put

$$P_0 = \mathrm{cl}_{M^*}(G_0 \cup \{p\}), \qquad Q_0 = \mathrm{cl}_{M^*}(G_0 \cup \{q\}),$$
$$\mathcal{S}(G_0) = \{H : H \text{ is a hyperplane of } M^*,\ H \cap \{p,q\} = \varnothing,\ G_0 \subset H\}.$$

Define integer-valued representatives by

$$\phi(K) = \begin{cases} 1, & K = Q_0, \\ 1, & K \in \mathcal{S}(G_0) \setminus \{H_0\}, \\ 0, & \text{otherwise}, \end{cases} \qquad \psi(K) = \begin{cases} 1, & K = P_0, \\ 0, & \text{otherwise}. \end{cases}$$

There is no conflict because Q_0 contains q, P_0 contains p, and every member of $\mathcal{S}(G_0)$ avoids both. The proof of Theorem 8.2, now read over $\mathbb{Z}$, shows that (32) gives the coordinate delta class at (H_0, G_0), up to the fixed block orientation. In each quotient $\mathbb{Z}^{\mathcal{G}_{\{p,q\}}(H)}/\mathbb{Z}\mathbf{1}$, all but one of the delta classes form a basis. Choosing their displayed decomposable preimages defines a right inverse to (45). Scalar extension preserves a split epimorphism, and the fieldwise identification follows from the same integer coordinate formula. □

Corollary 11.2 (Unimodularity and absence of bad primes)**.** *After integral bases are chosen, every pure-pair cup matrix has Smith normal form*

$$\operatorname{diag}(\underbrace{1, \ldots, 1}_{\operatorname{rank} \mathsf{L}^2_{p,q}}, 0, \ldots, 0).$$

Thus the pair obstruction quotient has an integral decomposable basis; its dimension, nonvanishing pattern, and cup surjectivity are simultaneous over all fields, including characteristic two.

Proof. A split epimorphism between free abelian groups is integrally equivalent to a projection onto a direct summand. This is precisely the asserted Smith normal form. □

Definition 11.3 (Pure-pair obstruction ranks)**.** For $p \neq q$ in $E(M)$, set

$$V_p(M) = T^1_{-\mathbf{e}_p}(M), \qquad W_{p,q}(M) = T^2_{-\mathbf{e}_p-\mathbf{e}_q}(M),$$

and define

$$W_{\mathrm{pair}}(M) = \bigoplus_{\{p,q\} \subseteq E(M)} W_{p,q}(M), \qquad \mathfrak{q}(M) = \dim_k W_{\mathrm{pair}}(M). \tag{46}$$

Loops contribute zero to these spaces; in statements about actual vertices they are simply omitted. The value of $\mathfrak{q}(M)$ is independent of k. Define also the standard-degree-zero one-link rank

$$\mathfrak{q}_0(M) = \sum_{e \in V(M)} \sum_{\{p,q\} \subseteq V(\mathrm{lk}_M\, e)} \dim_k T^2_{2\mathbf{e}_e - \mathbf{e}_p - \mathbf{e}_q}(M), \tag{47}$$

where loops in the link are omitted.

Theorem 11.4 (Global pure-pair obstruction rank)**.** *The direct sum of the pure-pair cup products is a split epimorphism*

$$\mu_M : \bigoplus_{\{p,q\} \subseteq E(M)} V_p(M) \otimes_k V_q(M) \twoheadrightarrow W_{\mathrm{pair}}(M)$$

over every field, and it has an integral split form. Moreover,

$$\mathfrak{q}(M) = \sum_{\{p,q\} \subseteq E(M)} \sum_{\substack{H \text{ hyperplane of } M^* \\ H \cap \{p,q\} = \varnothing}} \left(|\mathcal{G}_{\{p,q\}}(H)| - 1\right)_+, \tag{48}$$

$$\mathfrak{q}(M) = 0 \iff M \text{ has no } U_{1,4}\text{-minor}. \tag{49}$$

If D is a cocircuit of cardinality $d \geq 4$, then

$$\mathfrak{q}(M) \geq \binom{d}{2}(d-3). \tag{50}$$

If M is connected with rank at least two and corank at least three, then the direct sum of the link-lifted cup products is split surjective onto the space counted by $\mathfrak{q}_0(M)$, and

$$\mathfrak{q}_0(M) \geq \binom{d}{2}(d-3) \tag{51}$$

for some cocircuit D of cardinality $d \geq 4$ in a one-element contraction of M.

Proof. Theorem 11.1 gives a split epimorphism for each unordered pair; taking direct sums proves the first assertion. The pair formula (17) gives (48), and the vanishing equivalence follows from Theorem 5.4, since the pair layer detects the full T^2-vanishing locus.

For each pair in a cocircuit D, Proposition 4.2 gives a target of dimension at least $d-3$. The corresponding fine degrees are distinct, so summing over the $\binom{d}{2}$ pairs proves (50). For the standard-degree-zero assertion, choose e and a connected component of M/e as in Lemma 8.4. Apply the same cocircuit argument in that link and transport the integral right inverses by Corollary 8.3. This proves split surjectivity and (51). □

Corollary 11.5 (Direct-sum additivity)**.** *For matroids on disjoint ground sets,*

$$\mathfrak{q}(M_1 \oplus M_2) = \mathfrak{q}(M_1) + \mathfrak{q}(M_2). \tag{52}$$

More precisely, a pair with one element in each summand has zero target, whereas a pair contained in one summand has the same target as in that summand.

Proof. Take the degree $-\mathbf{e}_p - \mathbf{e}_q$ part of (3). If p and q lie in different summands, each potential tensor term would require a negative degree in a Stanley–Reisner ring factor, which is impossible. If both lie in one summand, the degree-zero part of the other ring is k and the target is unchanged. Summing over unordered pairs proves (52). □

Corollary 11.6 (Rank-two pure-pair rank)**.** *Let M be a loopless rank-two matroid with parallel classes $P_1, \dots, P_t$ of sizes $n_1, \dots, n_t$, and put $n = \sum_i n_i$. Then*

$$\mathfrak{q}(M) = \sum_i \binom{n_i}{2} \left(n_i - 3 + \sum_{\substack{a<c \\ a,c \neq i}} n_a n_c \right)_+ + \sum_{i<j} n_i n_j \sum_{k \neq i,j} n_k (n - 3 - n_k)_+. \tag{53}$$

For the rank-one uniform matroid,

$$\mathfrak{q}(U_{1,n}) = \binom{n}{2} (n-3)_+.$$

Proof. Sum the within-class case of Theorem 7.1 over the $\binom{n_i}{2}$ pairs in P_i, and sum the between-class case over the $n_i n_j$ pairs in $P_i \times P_j$. The rank-one formula follows directly from the single cocircuit of $U_{1,n}$. □

Theorem 11.7 (Sharp rank-two pure-pair extremum)**.** *Among connected loopless rank-two matroids of corank $c \geq 3$, the sharp minimum of the global pure-pair obstruction rank is*

$$\min \mathfrak{q}(M) = \begin{cases} 6, & c = 3, \\ 26, & c = 4, \\ \dfrac{c(c-2)(c+3)}{2}, & c \geq 5. \end{cases} \tag{54}$$

Writing the parallel-class sizes as an unordered partition of $c+2$, the equality profiles are

$$(2,2,1), \qquad (3,2,1), \qquad (c,1,1) \quad (c \geq 5), \tag{55}$$

respectively. Thus the equality profile is unique up to permutation for every $c \geq 3$.

Proof. For a class profile $\nu = (n_1, \dots, n_t)$, write $Q(\nu)$ for the right-hand side of (53). Put $n = c+2$ and $C = n-3$. For each index i, let

$$e_i = \sum_{\substack{j<\ell \\ j,\ell \neq i}} n_j n_\ell.$$

Regrouping the second sum by the omitted class gives

$$Q(\nu) = \sum_i \binom{n_i}{2} (n_i - 3 + e_i) + \sum_i n_i (C - n_i)_+ e_i. \tag{56}$$

The first positive part from (53) is unnecessary: for a profile with at least three classes and $n \geq 5$, one has $n_i - 3 + e_i \geq 0$.

Suppose $t \geq 4$ and merge classes of sizes x, y into a class of size $z = x + y$. Let R be the remaining list, and put

$$S = \sum_{r \in R} r, \qquad E = \sum_{r<s \in R} rs.$$

Every original class has size at most C, while $z \leq C+1$. If $z \leq C$, direct subtraction in (56) gives

$$2(Q(\nu) - Q(\nu')) = xy \left(2 \sum_{r \in R} \left[\binom{r}{2} + r(C-r) \right] + 8 - 3z \right.$$

$$+ S(4C - z - 2) + 2E\Bigg) > 0. \tag{57}$$

Indeed, $S \geq 2$, and the expression not already visibly nonnegative is at least $4+8C-5z \geq 4+3C$. If $z = C+1$, necessarily $R = (1,1)$, and subtraction reduces to

$$2\big(Q(\nu) - Q(\nu')\big) = xy(7C-3) - 2C - 2 > 0, \tag{58}$$

because $xy \geq C$ and $C \geq 2$. Repeated merging strictly decreases Q, so a minimizer has exactly three classes.

Let $a \leq b \leq d$ and $a+b+d = n$. Except for $(a,b,d) = (1,1,c)$, all three terms $C-a, C-b, C-d$ are nonnegative, and

$$\begin{aligned} 2Q(a,b,d) = a(a-1)(a-3+bd) + b(b-1)(b-3+ad) \\ + d(d-1)(d-3+ab) + 2abd(2n-9). \end{aligned} \tag{59}$$

If $a \geq 2$, set $A = a-1$, $U = b-a$, and $V = d-b$. Subtracting the value after the transfer $(a,b,d) \mapsto (a-1,b,d+1)$ gives

$$\begin{aligned} 2\big(Q(a,b,d) - Q(a-1,b,d+1)\big) = 15A^2 + 3A - 4 + (15A^2 + 28A - 3)U \\ + (15A^2 + 8A - 2)V + 11U^2 + 2V^2 + 8UV \\ + 25AU^2 + 30AUV + 5AV^2 \\ + 10U^3 + 15U^2V + 5UV^2 > 0. \end{aligned} \tag{60}$$

All coefficients in the first line are positive for $A \geq 1$, and the other terms are nonnegative. Iteration reduces every minimizer to $(1,b,d)$.

Put $s = b+d = c+1$ and $p = bd$. For $b \geq 2$,

$$2Q(1,b,d) = s(s-1)(s-3) + 2(s-4)p. \tag{61}$$

The product $p = b(s-b)$ is minimized on $2 \leq b \leq s/2$ at $b = 2$. The exceptional profile has value

$$Q(1,1,c) = \frac{c(c-2)(c+3)}{2}. \tag{62}$$

For $b = 2$, the difference between (61) and (62) is $c^2 - 6c + 6$. This is negative for $c = 3,4$ and positive for every $c \geq 5$. The cases $c = 3,4$ give the first two profiles in (55), while the concentrated profile is uniquely minimizing for $c \geq 5$. Strictness in the merge, transfer, and product comparisons proves uniqueness. □

Corollary 11.8 (Exact stability gap)**.** *Let $c \geq 5$. If a connected loopless rank-two matroid of corank c does not have class profile $(c,1,1)$, then*

$$\mathfrak{q}(M) \geq \frac{c(c-2)(c+3)}{2} + c^2 - 6c + 6. \tag{63}$$

Equality holds exactly for the profile $(c-1,2,1)$, up to permutation.

Proof. Among nonextremal three-class profiles, the transfer calculation reduces to $(1,b,c+1-b)$ with $b \geq 2$. Equation (61) and the monotonicity of $b(c+1-b)$ show that the unique minimum occurs at $b = 2$, and its difference from (62) is $c^2 - 6c + 6$. If a profile has at least four classes, merging strictly lowers Q. When the final three-class profile is nonextremal, the preceding bound is strict. If the final profile is $(1,1,c)$, the penultimate profile is $(1,1,x,y)$ with $x+y = c$, and (58) gives

$$2\big(Q(1,1,x,y) - Q(1,1,c)\big) = xy(7c-10) - 2c,$$

which is strictly larger than $2(c^2 - 6c + 6)$ for $c \geq 5$. Equality can therefore occur only for $(1, 2, c-1)$. □

Remark 11.9 (Two different rank-two minima). The invariants $\mathfrak{q}(M)$ and $\Theta(M) = \dim_k T^2(M)_0$ are different in rank two. The sharp values in (54) measure the complete purely negative pair layer, whereas Theorem 10.2 measures the standard-degree-zero one-link layer. Thus the pair layer begins with $6, 26, \ldots$, while the degree-zero layer begins with $6, 32, 81, 168, \ldots$.

12. The standard-degree-zero coefficient formula

The complete fine formula has a useful coefficient extraction which is independent of the rank. For an independent set $A \in M$, put $L_A = \mathrm{lk}_M A$. If B is a squarefree subset of the actual vertices of L_A with $|B| \geq 2$, define

$$\omega_M(A, B) = \sum_{\substack{H \text{ hyperplane of } L_A^* \\ H \cap B = \varnothing}} \big(\kappa(\Gamma_B^A(H)) - 1\big)_+,$$

where κ denotes the number of connected components and an empty graph contributes zero. Thus, by the complete hyperplane-graph formula,

$$\omega_M(A, B) = \dim_k T^2_{-\mathbf{1}_B}(L_A).$$

Theorem 12.1 (Standard-degree-zero master formula)**.** *For every finite matroid M and every field k,*

$$\dim_k T^2(M)_0 = \sum_{\varnothing \neq A \in M} \sum_{\substack{B \subseteq V(L_A) \\ |B| \geq 2}} \binom{|B|-1}{|A|-1} \omega_M(A, B). \tag{64}$$

In particular, the standard-degree-zero obstruction dimension is characteristic-independent and is computable from finitely many flat graphs in the links of independent sets.

Proof. Fix a positive support A of size s and a negative support B of size b. The link localization formula identifies the corresponding fine component with $T^2_{-\mathbf{1}_B}(L_A)$, and hence its dimension is $\omega_M(A, B)$. The standard total degree is zero exactly when the sum of the positive exponents is b. The number of positive exponent vectors with support exactly A and total sum b is

$$\#\{(u_a)_{a \in A} : u_a \geq 1,\ \sum_{a \in A} u_a = b\} = \binom{b-1}{s-1},$$

with the usual convention that this is zero for $b < s$. The empty positive support cannot contribute because B is nonempty. Summing over A and B proves (64); the graph formula gives the characteristic independence. □

Corollary 12.2 (Finite coefficient algorithm)**.** *For a rank-r matroid, equation* (64) *only requires independent supports A with $1 \leq |A| \leq r-1$. A fixed graph block with negative support B is counted with the binomial multiplicity $\binom{|B|-1}{|A|-1}$. Thus the standard-degree-zero slice is a finite rank-stratified incidence invariant, rather than a separate cohomology calculation.*

Proof. If $|A| \geq r$, its link has rank zero and has no actual vertex supporting a nonzero negative component. The remaining assertion is the coefficient count in the proof of Theorem 12.1. □

13. A complete rank-three formula

We now make (64) explicit for simple rank-three matroids. Write

$$f(d) = \binom{d}{2}(d-3)_+.$$

For a rank-two flat L call L a line, and set

$$\Theta_3(M) = \sum_{L \text{ line}} \binom{|L|}{2} f(n - |L|), \qquad n = |E(M)|.$$

Here $\Theta_3(M) = \Theta(M)$: a line of a simple rank-three matroid has $\binom{|L|}{2}$ bases, so this is (21) specialized to rank three. For a point e, let $L_1(e), \dots, L_{t_e}(e)$ be the lines through e and put

$$m_{e,i} = |L_i(e)| - 1.$$

The $m_{e,i}$ are positive and sum to $n-1$. For a three-element index set $I = \{i, j, k\}$, define

$$\rho_e(I) = \#\{\ell : m_{e,\ell} - \mathbf{1}_{\ell \in I} > 0\}.$$

Finally put

$$\begin{aligned}\Phi_e(M) = \sum_i \binom{m_{e,i}}{2} \left(m_{e,i} - 3 + \sum_{\substack{j<k \\ j,k \neq i}} m_{e,j} m_{e,k} \right)_+ \\ + \sum_{i<j} m_{e,i} m_{e,j} \sum_{k \neq i,j} m_{e,k}(n - 4 - m_{e,k})_+ \\ + (n-5)_+ \sum_{\substack{i<j<k \\ \rho_e(\{i,j,k\}) \leq 1}} m_{e,i} m_{e,j} m_{e,k}.\end{aligned} \tag{65}$$

Theorem 13.1 (Complete rank-three degree-zero formula)**.** *For every simple rank-three matroid M on n elements and every field k,*

$$\dim_k T^2(M)_0 = \Theta_3(M) + \sum_{e \in E(M)} \Phi_e(M). \tag{66}$$

Consequently the entire standard-degree-zero dimension is determined by the global line sizes and the line-pencil partitions at the points.

Proof. The support criterion allows positive support of size at most two. If $|A| = 2$, simplicity makes A independent and its link has rank one. Degree zero forces a negative pair, and the rank-one-link formula contributes $f(n - |\operatorname{cl}_M(A)|)$. Grouping pairs A with the same line closure gives $\Theta_3(M)$.

If $A = \{e\}$, the link M/e is rank two and its parallel classes are $L_i(e) \setminus \{e\}$ of sizes $m_{e,i}$. The rank-two formula has three possible negative supports. A pair in one class gives the first line of (65); a pair in two different classes gives the second line; and a triple from three classes survives exactly when, after removing the selected points, at most one parallel class remains, which is the condition $\rho_e(I) \leq 1$. The coefficient in the latter case is $(n-5)_+$. Summing over e proves the formula. No other positive support contributes, and all terms are graph dimensions, so the result is characteristic-independent. □

Corollary 13.2 (Uniform rank three)**.** *For $n \geq 6$,*

$$\dim_k T^2(U_{3,n})_0 = \binom{n}{2}\binom{n-2}{2}(n-5) + n\binom{n-1}{2}(n-3)(n-5).$$

Proof. Every line has two points, and every point pencil has $n-1$ singleton classes. Substitute these data into (66); the triple term is zero. □

Corollary 13.3 (The Fano plane)**.** *For the Fano matroid F_7,*

$$\dim_k T^2(F_7)_0 = 357$$

over every field.

Proof. The seven lines each have three points. Contracting any point gives three parallel classes of size two, not singleton classes. Thus

$$\Theta_3(F_7) = 7\binom{3}{2}\binom{4}{2}(4-3) = 126.$$

For the partition $(2,2,2)$, the first two lines of (65) contribute 9 and 24, respectively; the last line vanishes because $\rho_e(\{1,2,3\}) = 3$. Hence $\Phi_e(F_7) = 33$ and the total is $126+7\cdot 33 = 357$. □

For later use, the rank-three near-pencil on $n \geq 5$ points, with one line of size $n-1$, has

$$\dim_k T^2(M)_0 = \frac{3}{2}(n-1)(n-2)(n-3)(n-5). \tag{67}$$

14. Obstruction rigidity and inverse reconstruction

Brehm and Constantinescu proved that the dimensions of all multigraded components of $T^1(M)$ recover every nondiscrete matroid [5, Theorem 5.5]. We consider instead a restricted family of components of T^2. Under the hypotheses below, even their nonvanishing pattern suffices for reconstruction.

The labeled fine degrees in the rank-one-link part retain more information than their total dimension. For a rank-r matroid define the witness profile

$$\mathscr{W}_M(A;p,q) = \dim_k T^2_{\mathbf{1}_A - \mathbf{e}_p - \mathbf{e}_q}(M),$$

where A ranges over $(r-1)$-subsets and $p \neq q$ are outside A.

Theorem 14.1 (Cocircuit reconstruction)**.** *Let M be loopless of rank $r \geq 2$ and suppose every cocircuit has size at least four. Then the labeled profile $\mathscr{W}_M$ determines M up to relabeling. More precisely, for every independent $(r-1)$-set A, if*

$$D_A = \{p : \mathscr{W}_M(A;p,q) > 0 \text{ for some } q\},$$

then $D_A = E(M) \setminus \mathrm{cl}_M(A)$, and these sets recover all cocircuits.

Proof. The rank-one-link formula gives

$$\mathscr{W}_M(A;p,q) = \begin{cases} |D_A| - 3, & A \in M,\ p,q \in D_A, \\ 0, & \text{otherwise.} \end{cases}$$

The cocircuit hypothesis makes the first case nonzero for every independent A, so the union of the negative-coordinate supports is exactly D_A. Every hyperplane has a basis of size $r-1$, and cocircuits determine the matroid. Hence the profile reconstructs M. □

Theorem 14.2 (Rank-three obstruction rigidity)**.** *Let M and M' be simple rank-three matroids on the same labeled ground set E, with $|E| = n \geq 7$. If*

$$\dim_k T^2_{\mathbf{e}_a+\mathbf{e}_b-\mathbf{e}_p-\mathbf{e}_q}(M) = \dim_k T^2_{\mathbf{e}_a+\mathbf{e}_b-\mathbf{e}_p-\mathbf{e}_q}(M')$$

for all disjoint unordered pairs $\{a, b\}$ and $\{p, q\}$, then $M = M'$. In fact, only the nonvanishing pattern is needed.

Proof. Fix $A = \{a, b\}$. Its closure is a line L_A, and the rank-one-link formula says that the profile is nonzero precisely when $|E \setminus L_A| \geq 4$; in that case the union of the negative supports is $E \setminus L_A$. Thus every detected line is recovered.

Define the zero-profile graph on E by joining a, b when all the components with positive support $\{a, b\}$ in the statement vanish. Its edges are exactly the pairs contained in undetected lines. An undetected line has size at least $n - 3$. For $n \geq 8$ two distinct such lines cannot coexist, since distinct lines meet in at most one point. For $n = 7$ there are at most two; if there are two, they are four-point lines meeting in one point and are the maximal cliques of the zero-profile graph. In either case its maximal cliques of size at least two are exactly the undetected lines; isolated vertices do not represent lines. Hence the undetected lines are also recovered. The rank-two flats determine a simple rank-three matroid, proving the assertion. □

Corollary 14.3 (Explicit inverse algorithm)**.** *For a simple rank-three matroid on $n \geq 7$ labeled elements, the lines can be reconstructed in polynomial time from the nonzero components with positive support a pair and negative support a pair: take the complement of the union of the negative supports for every detected pair, and recover any remaining large lines as maximal cliques in the zero-profile graph.*

Proof. This is the constructive procedure in the proof of Theorem 14.2. The nonisolated part of the zero-profile graph is a union of at most two cliques meeting in at most one vertex. For each edge, its endpoints together with their common neighbors recover its clique; removing duplicates therefore finds all the required cliques in polynomial time. Testing a candidate triple against the recovered line list decides independence. □

Remark 14.4 (Sharpness of the seven-point bound)*.* The bound $n \geq 7$ in Theorem 14.2 is sharp. On six labeled points, a pair is detected exactly when its line has two points, and the detected block then carries the data $(d_A - 3, E \setminus A) = (1, E \setminus A)$, which depends only on the pair itself. The two copies of $M(K_4)$ with line sets

$$\{123,\ 145,\ 246,\ 356\} \qquad \text{and} \qquad \{124,\ 135,\ 236,\ 456\}$$

both have exactly the two-point lines 16, 25, and 34, so their labeled profiles coincide, while the matroids differ.

15. The sharp rank-three extremal law

Put

$$P_n = \frac{3}{2}(n-1)(n-2)(n-3)(n-5). \tag{68}$$

By (67), this is the value of the near-pencil on n points.

For a list $(m_1, \ldots, m_t)$ of pencil class sizes, let $\varphi_n(m_1, \ldots, m_t)$ denote the right-hand side of (65), with the m_i substituted for the classes at one point. We first record two concentration estimates.

Lemma 15.1 (Pencil concentration bound)**.** *Let L be a line of a simple rank-three matroid on $n = l + d$ elements, with $l \geq 4$ and $d \geq 4$, and suppose L is a longest line. If $e \notin L$, then*

$$\Phi_e(M) \geq \tau(l,d) := \binom{l-1}{2} d(3l + 2d - 14) + 3\binom{l-1}{3}(l + d - 5). \tag{69}$$

Proof. The classes of the pencil at e have sizes $m_1, \ldots, m_t$, where $t \geq l$, $\sum_i m_i = n - 1$, and $m_i \leq d$. Retain the second line of (65), which counts pairs from different classes. Since $m_i \leq d = n - l \leq n - 4$, its positive parts can be removed. Grouping its three contributions for each unordered triple of indices gives

$$F_n(m_1, \ldots, m_t) = \sum_{i<j<k} m_i m_j m_k (3(n-4) - m_i - m_j - m_k).$$

For a list of remaining entries R with sum S and second elementary symmetric function $E_2(R)$, splitting an entry $a + b$ changes this quantity by

$$ab\big(S(2n - 11) + 4E_2(R)\big) \geq 0.$$

Thus merging two entries whose sum is at most d cannot increase F_n. If there are more than l entries, two entries have sum at most d; otherwise, for the two smallest entries $a \leq b$,

$$\sum_i m_i \geq a + lb \geq \frac{(l+1)(d+1)}{2} > l + d - 1,$$

which is impossible. Merge until there are l entries. With two entries $2 \leq a \leq b < d$ fixed, transferring one unit from a to b changes F_n by

$$-(b - a + 1)\big(S(2n - 11) + 4E_2(R)\big) \leq 0.$$

Iteration gives the extremal list $(d, 1, \ldots, 1)$, and substitution gives (69). The other summands in (65) are nonnegative. □

Lemma 15.2 (Long-line point bound)**.** *Under the hypotheses of Lemma 15.1, every point $a \in L$ satisfies*

$$\Phi_a(M) \geq \binom{l-1}{2}(l-4), \qquad \sum_{a \in L} \Phi_a(M) \geq \lambda(l) := \frac{l(l-1)(l-2)(l-4)}{2}. \tag{70}$$

Proof. The pencil class $L \setminus \{a\}$ has size $l - 1$. Its first summand in (65) is at least $\binom{l-1}{2}(l-4)$, and all other summands are nonnegative. Sum over $a \in L$. □

Lemma 15.3 (Small-complement cases)**.** *Let M be a non-near-pencil simple rank-three matroid on $n \geq 8$ elements, and let L be a longest line with $d = n - |L|$. If $d = 2$, the smaller of the two possible line patterns has value*

$$\frac{1}{2}(5n^4 - 53n^3 + 155n^2 - 9n - 346), \tag{71}$$

whose difference from P_n is

$$n^4 - 10n^3 + 16n^2 + 87n - 218 > 0. \tag{72}$$

If $d = 3$, the possible patterns, apart from two-point lines, are the cross-triple patterns, one four-point cross-line, or the outside triple. The smallest value is the four-point cross-line pattern, namely

$$\frac{1}{2}(7n^4 - 73n^3 + 121n^2 + 781n - 2180), \tag{73}$$

and its difference from P_n is

$$2n^4 - 20n^3 - n^2 + 482n - 1135 > 0. \tag{74}$$

Proof. For $d = 2$, the two outside points either lie on a two-point line or on one triple with a point of L. For $d = 3$, write the outside points as u, v, w. There are six patterns: $j = 0, 1, 2, 3$ of the pairs uv, uw, vw extend to triples with distinct points of L; or uvw is an outside-only line; or uvw lies on one four-point line with a point of L. Distinct lines cannot share a pair, so these alternatives are exhaustive. Substitution in (66) gives the displayed values. For an explicit comparison in the $d = 3$ case, put $x = n - 8$. The differences from the four-point cross-line value have the form $a + bx + c\binom{x}{2}$, with

pattern	a	b	c
0 cross-triples	741	475	114
1 cross-triple	507	344	86
2 cross-triples	275	213	58
3 cross-triples	45	82	30
outside triple	498	337	84.

They are strictly positive for every integer $x \geq 0$. The coefficients of the gaps (72) and (74) in the Newton basis $\binom{x}{0}, \ldots, \binom{x}{4}$ are, respectively,

$$(478, 654, 466, 168, 24), \qquad (609, 1055, 866, 336, 48).$$

These give the asserted strict positivity for all $n \geq 8$. □

Lemma 15.4 (Polynomial gap)**.** *For $n = l + d \geq 13$, $d \geq 4$, and $l \geq 4$, put*

$$A(l, d) = \binom{n}{2} f(d), \qquad B(l, d) = \left(\binom{n}{2} - \binom{l}{2}\right) f(l - 1).$$

Then

$$\max\{A(l, d), B(l, d)\} + d\tau(l, d) + \lambda(l) > P_n. \tag{75}$$

Proof. If $d \geq l$, use A; if $d < l$, use B. The assertion is a direct polynomial inequality. For completeness, the boundary coefficient certificates are as follows. If $l = 4, 5, 6$ and $d = 13 - l + x$, the coefficients of $4(A + d\tau + \lambda - P_n)$ in increasing powers of x are respectively

l	$[x^0]$	$[x^1]$	$[x^2]$	$[x^3]$	$[x^4]$	$[x^5]$
4	20448	18252	5250	683	42	1
5	9624	12508	4143	589	39	1
6	5328	9368	3374	509	36	1

and all are positive. For $l \geq 7$, substituting $l = 7 + x$, $d = l + u$ gives a bivariate polynomial with nonnegative coefficients and positive constant term. On the other side, for $d = 4, 5$ and respectively $l = 9 + x, 8 + x$, the coefficient lists of $4(B + d\tau + \lambda - P_n)$ are

d	$[x^0]$	$[x^1]$	$[x^2]$	$[x^3]$	$[x^4]$
4	24336	18354	4218	384	12
5	14928	17584	4804	484	16

and if $d \geq 6$, substituting $d = 6 + x$, $l = d + 1 + u$ again gives nonnegative coefficients and a positive constant term. These cases cover $n \geq 13$. □

Lemma 15.5 (Finite compatibility table)**.** *For $8 \le n \le 12$, the cases not separated by the preceding bounds have the following relaxed lower bounds. Let $\lambda \vdash s$ denote an integer partition and $\ell(\lambda)$ its number of parts. With $q(\lambda) = \sum_i \binom{\lambda_i}{2}$, define*

$$\mathsf{O} = \min\left\{\varphi_n(\lambda) : \lambda \vdash n-1,\ \ell(\lambda) \ge l,\ \max_i \lambda_i \le d\right\}$$

and

$$\mathsf{I} = \min\left\{\sum_{a=1}^{l} \varphi_n(l-1, \lambda_a) : \lambda_a \vdash d,\ \sum_{a=1}^{l} q(\lambda_a) \le \binom{d}{2}\right\}.$$

The resulting values are

n	l	d	$\max\{A, B\}$	O	I	*lower bound* $- P_n$
8	4	4	168	108	424	79
8	3	5	560	61	317	237
9	5	4	216	306	1076	500
9	4	5	720	186	1034	668
9	3	6	1620	114	672	960
10	6	4	600	696	2178	1782
10	5	5	900	470	2340	1810
10	4	6	2025	290	2043	2028
10	3	7	3780	188	1299	2615
11	6	5	1100	1000	4478	4098
11	5	6	2475	675	4395	4440
12	6	6	2970	1365	8124	8889

(76)

Proof. An outside pencil has at least l classes, each of size at most d, giving the lower bound O. For a point of L, the classes off L form a partition λ_a of d. A pair of outside points can lie in the same off-L class for at most one point of L, hence $\sum_a q(\lambda_a) \le \binom{d}{2}$, which gives I. Listing the partitions, or equivalently using the recurrence

$$I_0(u) = 0, \qquad I_{j+1}(u) = \min_{\substack{\lambda \vdash d \\ q(\lambda) \le u}} \big(I_j(u - q(\lambda)) + \varphi_n(l-1, \lambda)\big),$$

produces the displayed finite table, with $\mathsf{I} = I_l(\binom{d}{2})$. Relaxing line compatibility can only decrease the minima, so they remain valid lower bounds. □

Theorem 15.6 (Sharp rank-three extremum)**.** *For every simple rank-three matroid M on $n \ge 8$ elements,*

$$\dim_k T^2(M)_0 \ge \frac{3}{2}(n-1)(n-2)(n-3)(n-5). \tag{77}$$

Equality holds if and only if M is a near-pencil, that is, it has one line of size $n-1$.

Proof. The near-pencil calculation gives equality. Suppose that M is not a near-pencil and let L be a longest line, with $l = |L|$ and $d = n - l \ge 2$. The cases $d = 2$ and $d = 3$ are covered by Lemma 15.3. Assume $d \ge 4$. Since line pairs partition the pairs of points and every line distinct from L has size at most $d+1$,

$$\Theta_3(M) \ge A(l, d), \qquad \Theta_3(M) \ge B(l, d).$$

If $n \ge 13$ and $l \ge 4$, combine these inequalities with Lemmas 15.1 and 15.2, and apply Lemma 15.4. For $n \ge 13$ and $l \le 3$, the first bound alone is larger than P_n: in the Newton

basis $\binom{n-13}{j}$, $0 \le j \le 5$, the coefficients of $A - P_n$ are

l	0	1	2	3	4	5
2	18480	12408	5844	1839	348	30
3	8730	8144	4592	1617	330	30.

For $8 \le n \le 12$, use (76) and the following coverage of its omitted cases. Put $D = \max\{A, B\} + d\tau + \lambda - P_n$ for $l \ge 4$ and $D = A - P_n$ for $l \le 3$. Direct substitution gives

n	remaining l	corresponding D
8	2	315
9	2	1008
10	2	2520
11	$2, 3, 4, 7$	$5400, 1220, 30, 405$
12	$2, 3, 4, 5, 7, 8$	$10395, 3861, 1701, 177, 420, 2457.$

Together these cover every integer $2 \le l \le n - 4$. Every non-near-pencil is strictly above the value in (77), proving the theorem. □

Corollary 15.7 (Rank-three stability)**.** *If M_n is a sequence of simple rank-three matroids with $|E(M_n)| = n$ and*

$$\frac{\dim_k T^2(M_n)_0}{P_n} \longrightarrow 1,$$

then M_n is a near-pencil for all sufficiently large n.

Proof. Suppose a subsequence consists of non-near-pencils, and let $d = n - l \ge 2$ on that subsequence. Since $P_n \sim \frac{3}{2}n^4$, the bound $\Theta_3 \ge A(l, d) = \binom{n}{2} f(d)$ implies $d/n \to 0$: otherwise A has order n^5 along a further subsequence. Thus $l/n \to 1$. If $d \ge 4$, then

$$\frac{B(l, d)}{P_n} = \frac{d(2n - d - 1)(l - 1)(l - 2)(l - 4)}{6(n - 1)(n - 2)(n - 3)(n - 5)} = \frac{d}{3}(1 + o(1)),$$

where the error tends to zero whenever $d/n \to 0$. Its lower limit is at least $4/3$, contradicting the assumed ratio. The cases $d = 2$ and $d = 3$ are excluded by (72) and (74), whose ratios to P_n tend respectively to $2/3$ and $4/3$. No infinite non-near-pencil subsequence remains. □

The exhaustive line-system calculation described in Section 19 gives minima 0, 54, and 357 on 5, 6, and 7 points, respectively; the seven-point value is attained by the Fano configuration. These finite results are separate from the proof for $n \ge 8$.

16. A valuative cocircuit profile

The scalar invariant Θ is a weighted sum of a more informative finite profile. Let M have rank $r \ge 1$, ground-set size $n = r + c$, and define

$$a_m(M) = \sum_{\substack{H \text{ hyperplane of } M \\ |H| = r-1+m}} b(M|H), \qquad 0 \le m \le c.$$

Set $a_m(M) = 0$ for integer m outside this range. Equivalently, if D runs over cocircuits, then

$$\mathscr{C}_M(z) = \sum_{m=0}^{c} a_m(M) z^{c+1-m} = \sum_{D \text{ cocircuit}} b(M|(E \setminus D)) z^{|D|}.$$

The coefficient a_m records the number of bases of hyperplanes having m elements beyond a basis.

The recovery of flat data from the $\mathcal{G}$-invariant is established in broader form by Bonin and Kung [4, Theorem 3.3 and Proposition 5.2]: the invariant is equivalent to catenary data and determines the numbers of flats of each rank and size. Ferroni and Schröter [9, Theorem 8.2] prove valuativity of the flag f-vector of the lattice of flats. The formulas below identify the particular basis-weighted hyperplane profile occurring in the cotangent-cohomology calculation and give its transform directly in rank-word coordinates.

Let $\mathcal{G}(M)$ denote the Derksen–Fink $\mathcal{G}$-invariant [7] in its rank-word basis. A rank word has a 1 when the next element raises rank and a 0 otherwise. For $0 \le t \le c$, let $g_t(M)$ be the coefficient of

$$w_t = 1^{r-1}0^t 1\, 0^{c-t}$$

in $\mathcal{G}(M)$, and write $(m)_t = m(m-1)\cdots(m-t+1)$.

Theorem 16.1 (Rank-word transform for the cocircuit profile)**.** *For every matroid of positive rank,*

$$g_t(M) = (r-1)!(c-t)! \sum_{m=t}^{c} a_m(M)(m)_t(c+1-m), \qquad 0 \le t \le c. \tag{78}$$

Consequently, with

$$R_M(x) = \frac{1}{(r-1)!c!} \sum_{t=0}^{c} \binom{c}{t} g_t(M)(x-1)^t,$$

one has

$$R_M(x) = \sum_{m=0}^{c} a_m(M)(c+1-m)x^m, \qquad a_m(M) = \frac{[x^m]R_M(x)}{c+1-m}. \tag{79}$$

In particular, $\mathscr{C}_M$, the full vector $(a_0, \dots, a_c)$, and $\Theta(M)$ are linear specializations of $\mathcal{G}(M)$. For fixed rank and ground-set size they are valuative under matroid base-polytope subdivisions.

Proof. Fix a hyperplane H with $|H| = r-1+m$. Choose an ordered basis of $M|H$; there are $(r-1)!b(M|H)$ choices. After these first $r-1$ entries, choose an ordered t-tuple from the m remaining elements of H, then choose the next element outside H, which has $c+1-m$ possibilities. Order the remaining $c-t$ elements arbitrarily. This produces the word w_t. Conversely, every permutation with word w_t is obtained uniquely from the hyperplane spanned by its first $r-1$ entries. Summing over H gives (78).

Substitution into the definition of R_M uses

$$\frac{\binom{c}{t}(c-t)!}{c!}(m)_t = \binom{m}{t}$$

and the binomial identity $\sum_{t=0}^{m} \binom{m}{t}(x-1)^t = x^m$. This gives (79). Rank-word coefficients are linear functionals of the universal valuative invariant $\mathcal{G}$ [7, Theorem 1.4], and linear transformations preserve valuativity. Finally, (21) is a fixed linear functional of the a_m. □

Corollary 16.2 (Valuative rank-one obstruction series)**.** *The coarse Hilbert series of the rank-one-link strand and the full polynomial $\mathscr{C}_M$ are valuative after fixing r and n. The profile is not determined by the Tutte polynomial: the two graphic matroids in Theorem 6.7 have the same Tutte polynomial but different values of Θ, hence different profiles.*

Proof. The first assertion follows from Theorem 16.1 and the rank-one-link Hilbert-series formula. The second follows because Θ is a specialization of the profile and is not determined by the Tutte polynomial. □

The next proposition separates two layers of the obstruction data: the rank-one-link profile is valuative, whereas the purely negative pair rank is not.

Proposition 16.3 (The boundary of valuativity)**.** *Let $p_d(M)$ denote the number of independent $(r-1)$-sets A with $d_A = |E \setminus \mathrm{cl}_M A| = d$.*

(1) *$p_d(M) = a_{c+1-d}(M)$ for all d. Hence the obstruction profile is valuative for fixed rank and ground-set size, although it is not log-concave.*
(2) *The global pair rank $\mathfrak{q}$ is not valuative. In the split subdivision of the base polytope of $U_{2,6}$ along $x_1 + x_2 + x_3 = 1$, the maximal cells are the polytopes of the two rank-two matroids with class profile $(3, 1, 1, 1)$, and their common face is the polytope of $U_{1,3} \oplus U_{1,3}$. The valuative relation $f(U_{2,6}) = f(M_1) + f(M_2) - f(M_{12})$ holds for Θ, with $120 = 60 + 60 - 0$, but fails for $\mathfrak{q}$, with $120 \neq 51 + 51 - 0$.*
(3) *$\mathfrak{q}$ is not determined by the Tutte polynomial: the two Tutte-equivalent graphic matroids of Theorem 6.7 have $\mathfrak{q} = 2009$ and $\mathfrak{q} = 2165$.*

Proof. For (1), mapping A to its closure is a bijection from the independent $(r-1)$-sets with $d_A = d$ to the pairs consisting of a hyperplane H with $|E \setminus H| = d$ and a basis of $M|H$. Hence $p_d = a_{c+1-d}$, and Theorem 16.1 gives valuativity. For the rank-two matroid with class sizes $(3, 1, 1)$, the complete profile is

$$(p_0, \ldots, p_5) = (0, 0, 3, 0, 2, 0),$$

which is not log-concave. Indeed, the three points in the large class have complementary cocircuit size two, and the other two points have complementary cocircuit size four.

For (2), a basis of the first cell matroid is a pair containing at most one element of $\{1, 2, 3\}$, a basis of the second is a pair containing at most one element of $\{4, 5, 6\}$, and the common face consists of the pairs containing exactly one element of each; this is the standard split subdivision. The values of $\mathfrak{q}$ follow from Corollary 11.6 and Corollary 11.5, and the values of Θ from Proposition 10.1 and Corollary 6.6. A valuative invariant satisfies the displayed inclusion–exclusion relation, so $\mathfrak{q}$ is not valuative.

For (3), evaluate (17) on the two matroids; the exact computation is recorded in Section 19. □

Remark 16.4 (Higher diagonal sparsity). In characteristic zero, Ilten, Meazzini, and Petracci prove that, for $i \geq 2$, a fine component of T^i vanishes if one negative coordinate has multiplicity at least i [10, Theorem 5.1]. Consequently, if $\mathbf{c}$ has a nonempty negative support, a component of T^i on the ray $m\mathbf{c}$, with $m \geq 1$, can be nonzero only when $m < i$. The $i = 2$ vanishing used in Section 8 holds over every field by the support criterion for T^2.

17. Rank-two support and the primary cup image

The graph model admits a further simplification: its only disconnected blocks occur when the negative support has dual rank two. This also separates the part of T^2 reached by the primary cup product from the part in which higher obstructions can occur.

Lemma 17.1 (Trichotomy for the hyperplane graphs)**.** *Let N have rank $s \geq 2$, let $B \neq \varnothing$, and let H be a hyperplane avoiding B. Put $b = \mathrm{rk}_N B$. The graph $\Gamma_B(H)$ is empty if $b \leq 1$, nonempty and discrete if $b = 2$, and nonempty and connected if $b \geq 3$.*

Proof. A rank-$(s-2)$ flat gains at most b ranks upon adjoining B. This proves the first assertion. Two distinct such flats have intersection rank at most $s - 3$, so their intersection cannot span with a rank-two set. For $s = 2$ there is at most one rank-zero flat, and discreteness still holds. When $b = 2$, the matroid $(N/B)|H$ has rank $s - 2$. A basis of it is independent in N, spans with B, and closes inside H to a qualifying rank-$(s-2)$ flat. Thus the discrete graph is nonempty.

Suppose $b \geq 3$ and put $Q = (N/B)|H$. Since $H \cup B$ spans N, the rank of Q is $d = s - b$. Every vertex G contains a basis I of Q: indeed $\operatorname{rk}_Q G = \operatorname{rk}_N(G \cup B) - b = d$. Such a basis is also independent in N. Vertices containing the same I form a clique, since $\operatorname{rk}_N(I \cup B) = s$. The basis-exchange graph of Q is connected. If I, J are adjacent bases, then $|I \cup J| = d + 1 \leq s - 2$. Extend their closure inside H to a rank-$(s-2)$ flat K. It contains both bases and is a vertex of $\Gamma_B(H)$, joining the two cliques. Every vertex belongs to one of these cliques. Extending any basis of Q also proves nonemptiness. If $d = 0$, all vertices lie in the clique corresponding to the empty basis. □

Theorem 17.2 (Rank-two negative support)**.** *Let $N = M^*$. A nonzero component $T^2_{\mathbf{a}-\mathbf{1}_B}(M)$ has $\operatorname{rk}_N B = 2$. Subject to the support conditions of Theorem 3.1, put $N_A = (\operatorname{lk}_M A)^*$, where $A = \operatorname{supp} \mathbf{a}$. If $\operatorname{rk}_N B = 2$, then*

$$T^2_{\mathbf{a}-\mathbf{1}_B}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } N_A \\ H \cap B = \varnothing}} k^{\mathcal{G}_B(H)}/k\mathbf{1}, \qquad \dim_k T^2_{\mathbf{a}-\mathbf{1}_B}(M) = \sum_{H \cap B = \varnothing} (|\mathcal{G}_B(H)| - 1)_+. \tag{80}$$

Empty blocks contribute zero. Every nonzero T^1 component has negative support of rank one in N.

Proof. Apply Lemma 17.1 to the universal graph formula. Singleton negative supports have zero T^2 by (8); dual ranks below two are covered by the same formula. To compare the dual ranks before and after passing to a link, let D be the loops of M/A. They are coloops of $N \setminus A$, and $N_A = (N \setminus A)/D$. Deletion leaves the ranks of subsets of $E \setminus A$ unchanged, and contracting coloops leaves the ranks of subsets disjoint from D unchanged. Thus $\operatorname{rk}_{N_A} B = \operatorname{rk}_N B$ for every admissible B. The last assertion follows from (5): if $\operatorname{rk}_N B \geq 2$, every eligible graph is nonempty, whereas rank-zero B lies in every hyperplane. For singleton supports use (8). This is a reformulation of the first-cohomology description of Brehm and Constantinescu [5, Theorem 4.7], not an additional T^1 theorem. □

Write $R = k[M]$ and $\Omega = \operatorname{im}(T^1(R) \otimes_R T^1(R) \to T^2(R))$. The image here includes every homogeneous factorization of a target degree, not only products of singleton classes.

Theorem 17.3 (The purely negative primary image)**.** *Let $N = M^*$ and let $\mathbf{c} = \mathbf{a} - \mathbf{1}_B$ have disjoint nonnegative parts. If B meets at least three nonloop parallel classes of N, then $\Omega_{\mathbf{c}} = 0$. If B contains a loop of N, then $T^2_{-\mathbf{1}_B} = 0$. Otherwise,*

$$\Omega_{-\mathbf{1}_B} = \begin{cases} T^2_{-\mathbf{1}_B}(M), & B \text{ meets exactly two parallel classes,} \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for simple N every purely negative support of size at least three is invisible to the primary cup product.

Proof. By Theorem 17.2, the negative support of a nonzero T^1 class is confined to one nonloop parallel class of N. Every element of B must occur in a negative support of one of the two factors. Two factors can therefore cover at most two parallel classes, even when positive and negative supports cancel outside B.

Suppose $B = B_1 \sqcup B_2$, with nonempty B_i contained in distinct parallel classes P_i, and choose $p_i \in B_i$. A circuit of M, being a cocircuit of N, contains either all of P_i or none of it. Consequently multiplication induces

$$x_{B_i \setminus \{p_i\}} : T^1_{-\mathbf{1}_{B_i}} \twoheadrightarrow T^1_{-\mathbf{e}_{p_i}}, \qquad x_{B \setminus \{p_1, p_2\}} : T^2_{-\mathbf{1}_B} \xrightarrow{\sim} T^2_{-\mathbf{e}_{p_1} - \mathbf{e}_{p_2}}. \tag{81}$$

For completeness these are maps, not just equalities of dimensions. In the monomial presentation, a union of circuits contains B_i if and only if it contains p_i. After deleting either B_i

or p_i, the resulting monomial is divisible by no circuit meeting P_i; circuits disjoint from P_i are unaffected. The valid Taylor cochain coordinates and their incidence matrices therefore agree. For the Koszul conditions the relevant exponents on P_i are uniformly zero, one, or two before deletion. In each case deleting B_i or just p_i gives the same test for divisibility by a circuit. Thus the same identification respects the Koszul quotient defining T^2. Multiplication is the identity on these coordinates. For T^1, the singleton target additionally quotients by the derivation image when $|B_i| > 1$, giving the stated surjection.

Lift the two factors of every singleton-pair cup product through (81). Its R-bilinearity and the target isomorphism, together with Theorem 8.2, prove surjectivity at $-\mathbf{1}_B$. If B contains a loop of N, no hyperplane avoids it, so the target is zero. If B meets at most one parallel class, the target also vanishes by rank-two support. □

The circuit description of Brehm and Constantinescu [5, Proposition 4.5 and Theorem 4.7] also gives the following multiplication maps. These maps allow positive supports with exponent one; multiplying both factors by a positive monomial would not reach such degrees.

Lemma 17.4 (Generation of first cohomology)**.** *Let* $\mathbf{a} \geq 0$ *and* $B \neq \varnothing$*, with* $\operatorname{supp} \mathbf{a} \cap B = \varnothing$*. Multiplication induces a surjection*

$$x^{\mathbf{a}} : T^1_{-\mathbf{1}_B}(M) \twoheadrightarrow T^1_{\mathbf{a}-\mathbf{1}_B}(M).$$

Consequently $T^1(M)$ *is generated as a* $k[M]$*-module by its purely negative components.*

Proof. The support criterion and Theorem 17.2 reduce the assertion to B contained in one nonloop parallel class P of $N = M^*$. Put $S = k[x_e : e \in E]$ and use the minimal circuit generators x_C of I_M. A circuit contains either all of P or none of it. Every C containing B has $C \setminus B \in M$: its complement is $H \cup B$, where $H = E \setminus C$ is a hyperplane avoiding P, and $H \cup B$ spans N.

For distinct circuits C, D, write $K = E \setminus D$ and $s = \operatorname{rk} N$. If $B \subseteq C \cup D$, then

$$\operatorname{rk}_N((H \cap K) \cup B) \leq s - 2 + 1 < s.$$

Thus $(C \cup D) \setminus B$ is a nonface of M. The evaluated Taylor pair relation in degree $-\mathbf{1}_B$ is therefore zero in R; the same holds after adding $\mathbf{a}$. If $B \not\subseteq C \cup D$, that relation has no output monomial of nonnegative degree. When $s = 1$, there is at most one circuit and no distinct pair to check. Taylor pair relations generate the first syzygies. It follows that $\operatorname{Hom}_S(I_M, R)_{-\mathbf{1}_B}$ is the full coordinate space on the circuits containing B, whereas its degree $\mathbf{a} - \mathbf{1}_B$ part retains precisely those coordinates with $(C \setminus B) \cup \operatorname{supp} \mathbf{a} \in M$. Multiplication by $x^{\mathbf{a}}$ is the coordinate projection and is surjective. The map $\operatorname{Der}_k(S, R) \to \operatorname{Hom}_S(I_M, R)$ is R-linear, so this surjection descends to its cokernel T^1. All other fine degrees of T^1 vanish by the support criterion. □

Theorem 17.5 (The primary image in every fine degree)**.** *Let* $N = M^*$*, and let* $p_N(B)$ *be the number of nonloop parallel classes of* N *met by* B*. For every fine degree* $\mathbf{c} = \mathbf{a} - \mathbf{1}_B$ *with disjoint nonnegative parts,*

$$\Omega_{\mathbf{c}} = \begin{cases} T^2_{\mathbf{c}}(M), & p_N(B) = 2, \\ 0, & p_N(B) \neq 2. \end{cases}$$

Degrees excluded by the support criterion have zero target, including degrees with nonsquarefree negative part. The target also vanishes if B *contains a loop of* N*.*

Proof. The argument in Theorem 17.3 excludes every factorization when $p_N(B) \geq 3$. Each element of B must belong to the negative support of a factor, and two nonzero T^1 factors cover at most two dual parallel classes. This remains true with cancellation outside B. Homogeneous tensors span the image; an R-multiple can be absorbed into one factor.

Suppose $p_N(B) = 2$ and the target is admissible and potentially nonzero. Write $B = B_1 \sqcup B_2$ according to the two parallel classes, put $A = \operatorname{supp} \mathbf{a}$, and let $L = \operatorname{lk}_M A$ on its actual vertices. If D consists of the loops of M/A, then $L^* = (N \setminus A)/D$. The elements of D are coloops of $N \setminus A$; deleting A and contracting D preserve the ranks and parallel-class distinction on B.

Set $U = T^1_{-\mathbf{1}_{B_1}}(M)$, $V = T^1_{\mathbf{a}-\mathbf{1}_{B_2}}(M)$, and $W = T^2_{\mathbf{a}-\mathbf{1}_B}(M)$. Positive-support localization and Lemma 17.4 give a commuting square

$$\begin{array}{ccc} U \otimes_k V & \xrightarrow{\cup} & W \\ \downarrow{\scriptstyle \lambda_1 \otimes \lambda_2} & & \downarrow{\scriptstyle \lambda} \\ T^1_{-\mathbf{1}_{B_1}}(L) \otimes_k T^1_{-\mathbf{1}_{B_2}}(L) & \xrightarrow{\cup} & T^2_{-\mathbf{1}_B}(L). \end{array}$$

Here λ_1 is multiplication by $x^{\mathbf{a}}$ followed by the normalized link isomorphism, hence is surjective; λ_2 and λ are the normalized link isomorphisms. To check the degrees and commutativity, localize at x_A. In $R_{x_A} = k[L][x_i^{\pm 1} : i \in A]$, the two factors have the form α and $x^{\mathbf{a}}\beta$. Their cup is $x^{\mathbf{a}}(\alpha \cup \beta)$, and target normalization removes exactly $x^{\mathbf{a}}$. The lower cup is surjective by Theorem 17.3 applied to L. Lifting finite sums of its products proves surjectivity onto W. For $p_N(B) \le 1$, or for B containing a dual loop, the target vanishes by Theorem 17.2 and the graph formula. □

Corollary 17.6 (The quotient by the primary image)**.** *The R-module Ω is generated by its purely negative components meeting exactly two dual parallel classes. In each admissible degree $\mathbf{c} = \mathbf{a} - \mathbf{1}_B$ with $\operatorname{rk}_N B = 2$ and $p_N(B) \ge 3$, the quotient $Q = T^2(M)/\Omega$ has*

$$Q_{\mathbf{c}} \cong T^2_{\mathbf{c}}(M) \cong \bigoplus_{\substack{H \text{ hyperplane of } (\operatorname{lk}_M A)^* \\ H \cap B = \varnothing}} k^{\mathcal{G}_B(H)}/k\mathbf{1}.$$

All other components vanish. These are degreewise vector-space identifications, and the displayed components may themselves be zero. Their dimensions are independent of the characteristic.

Proof. Lemma 17.4 and R-bilinearity generate every cup from purely negative cups. Overlapping negative supports give a nonsquarefree negative part and hence zero target. Theorems 17.3, 17.5, and 17.2 give the assertions, with empty blocks interpreted as zero as in (80). □

18. Essential obstructions of arbitrarily high order

Vanishing of every pairwise cup product does not suffice for simultaneous fine-multigraded integration of three or more classes. We now give an explicit family whose first unavoidable obstruction occurs at any prescribed order. The obstruction remains in T^2; its order does not refer to the index of cotangent cohomology.

Let $m, n \ge 3$, let L, H be disjoint sets of sizes m, n, and put

$$N_{m,n} = \operatorname{Tr}_3(U_{2,m} \oplus U_{2,n}), \qquad M_{m,n} = N^*_{m,n}. \tag{82}$$

Here Tr_3 denotes rank-three truncation. The hyperplanes of $N_{m,n}$ are L, H, and the cross pairs $\{j, h\}$. The matroid is simple and connected: three-element subsets of either line are circuits, and two points from each line form a circuit. Thus $M_{m,n}$ is connected, cosimple, and has corank three.

Use cyclic indices $j \in \mathbb{Z}/m\mathbb{Z}$ on L, variables x_j, y_h, and set

$$X = \prod_{j \in L} x_j, \qquad Y = \prod_{h \in H} y_h, \qquad W_h = Y/y_h.$$

The circuit ideal is

$$I_{M_{m,n}} = (X, Y, (X/x_j)W_h : j \in L,\ h \in H). \tag{83}$$

Choose $h_0 \in H$, put $P = t_0 \cdots t_{m-1}$ and $\sigma = (-1)^{m-1}$, and define

$$F_j = \sum_{a=0}^{m-1} (-1)^a \left(\prod_{b=1}^{a} t_{j-b} \right) \left(\prod_{i \notin \{j, j-1, \ldots, j-a\}} x_i \right). \tag{84}$$

Empty products are one. Give t_i fine degree $\mathbf{e}_i$.

Theorem 18.1 (Cyclic higher obstruction)**.** *For every field k, the ideal*

$$J = (X, Y, F_j W_{h_0} : j \in L, (X/x_j)W_h : j \in L,\ h \neq h_0) \tag{85}$$

defines a flat fine-multigraded deformation over $k[t_0, \ldots, t_{m-1}]/(P)$. Its first-order classes $\alpha_i \in T^1_{-\mathbf{e}_i}(M_{m,n})$ are nonzero. Every proper subcollection integrates over its unrestricted polynomial parameter ring, and all pairwise primary cup products vanish.

Put $\mathfrak{m} = (t_0, \ldots, t_{m-1})$. The induced deformation over $k[t]/\mathfrak{m}^m$ has no lift to $k[t]/\mathfrak{m}^{m+1}$, even without requiring the lift to preserve the fine grading. In the fine-multigraded category its jet modulo $\mathfrak{m}^m$ is uniquely determined, up to the usual equivalence, by the first-order classes. Its first essential obstruction has order m and lies in $T^2_{-\mathbf{1}_L}(M_{m,n})$, a space of dimension $n-1$.

Proof. Shifting the summation index in (84) gives the telescoping identity

$$x_j F_j + t_{j-1} F_{j-1} = X + \sigma P. \tag{86}$$

Work first over $B = \mathbb{Z}[t]/(P)$ and regard the parameters as coefficients. Choose an order on monomials in x, y refining total degree. The displayed generators of J are monic and have the leading monomials in (83). We verify the monic Buchberger criterion over B, which is valid over a coefficient ring with zero divisors because no leading coefficient is divided by a nonunit.

Two unchanged generators have zero S-polynomial. For two modified generators the S-polynomial is $W_{h_0}(x_j F_j - x_k F_k)$. Subtracting their identities (86) expresses it as a combination of modified generators with strictly smaller x, y degree than their least common multiple. For X and a modified generator the same identity gives a multiple of another modified generator and the remainder $-\sigma P W_{h_0} = 0$. For Y and a modified generator every tail is divisible by Y. Finally, for a modified generator and an unchanged cross generator with $h \neq h_0$, the least common multiple of W_h and W_{h_0} is Y, so every tail is again divisible by Y. These exhaust the S-pairs. Division by a monic basis shows that the original standard monomials form a B-basis of $B[x, y]/J$. This proves flatness over every field. Setting any parameter to zero makes P identically zero, proving the proper-subcollection assertion.

The coefficient of t_i in F_{i+1} is $-X/(x_i x_{i+1})$. Thus α_i is, up to sign, the singleton hyperplane delta function at $\{i+1, h_0\}$; it is not constant and is nonzero. Every pair belongs to a proper subcollection, so its mixed primary obstruction is zero. Each one-parameter restriction is also a polynomial deformation, so self-obstructions vanish directly, including in characteristic two.

We next rule out arbitrary, possibly nonhomogeneous, lifts of the specified jet. Set $A = k[t]/\mathfrak{m}^{m+1}$. In any flat lift, choose lifts of the variables and of the displayed ideal generators. The lifted variables generate the algebra since the base ideal is nilpotent. Flatness identifies the reduction of its presentation kernel with $I_{M_{m,n}}$; lifted circuit generators generate that kernel by the nilpotent form of Nakayama's lemma. They differ from (85) by parameter monomials of total degree m with arbitrary coefficients in $k[x, y]$. Write g_X and $g_{j,h}$ for their coefficients at P. The kernel $\mathfrak{m}^m$ of $A \to k[t]/\mathfrak{m}^m$ is annihilated by $\mathfrak{m}$. Flatness therefore makes

$$\mathfrak{m}^m \otimes_k k[M_{m,n}] \longrightarrow \mathcal{A}$$

injective, where $\mathcal{A}$ is the lifted algebra. Multiply the modified relation by x_j, add t_{j-1} times the preceding modified relation, and subtract W_{h_0} times the lifted relation for X. By (86), its P coefficient must vanish in the special fiber:

$$\sigma W_{h_0} + x_j g_{j,h_0} - W_{h_0} g_X = 0. \tag{87}$$

Likewise the unchanged cross relation for $h \neq h_0$ gives $x_j g_{j,h} - W_h g_X = 0$. Corrections at other parameter monomials do not affect these P coefficients, since a parameter times any order-m correction is zero in A.

Project the first equation to fine degree $\deg W_{h_0}$, and the second to fine degree $\deg W_h$. No $x_j g_{j,h}$ can contribute, since the j coordinate of these degrees is zero. If λ is the constant term of g_X, the two projections require $(\sigma - \lambda)W_{h_0} = 0$ and $\lambda W_h = 0$ in $k[M_{m,n}]$. Every W_h is a nonzero standard monomial, so $\lambda = \sigma$ and $\lambda = 0$, a contradiction over every field. Lifting the variables presents any abstract affine lift in this form; thus the contradiction is not restricted to a polynomial ansatz or to homogeneous correction terms.

It remains to justify uniqueness from the first-order classes in the stated graded category. For a parameter monomial $t^{\mathbf{u}}$ of total degree at least two, $T^1_{-\mathbf{u}}(M_{m,n}) = 0$. If $\mathbf{u}$ is not squarefree this follows from the support criterion; otherwise its support has rank at least two in the simple dual $N_{m,n}$, and Theorem 17.2 applies. Derivations of $k[x, y]$ into $k[M_{m,n}]$ in degree $-\mathbf{u}$ are also zero. Put $S = k[x, y]$ and $R = k[M_{m,n}]$. The presentation exact sequence

$$\mathrm{Der}_k(S, R) \longrightarrow \mathrm{Hom}_S(I_{M_{m,n}}, R) \longrightarrow T^1(R) \longrightarrow 0$$

therefore gives $\mathrm{Hom}_S(I_{M_{m,n}}, R)_{-\mathbf{u}} = 0$. Indeed the image of a variable under a derivation would have degree $\mathbf{e}_v - \mathbf{u}$, with a negative coordinate when $|\mathbf{u}| \geq 2$. Representatives of the same first-order classes can first be aligned by graded coordinate changes realizing their derivation differences. Differences between two flat embedded lifts across an order-filtration small extension lie in these spaces. Induction shows that all coefficients of orders $2, \ldots, m-1$ are forced. There is no ambiguity in an ideal-valued coefficient: subtracting a nonempty support from the degree of a minimal circuit generator leaves either a negative coordinate or a proper face of that circuit. After choosing the same first-order representatives, this proves graded uniqueness. The obstruction has parameter weight $\mathbf{1}_L$; its dimension is computed in Proposition 18.3 below. □

Example 18.2 (A six-element cubic obstruction)**.** For $m = n = 3$, the polynomials are

$$F_0 = x_1x_2 - t_2x_1 + t_2t_1, \quad F_1 = x_0x_2 - t_0x_2 + t_0t_2, \quad F_2 = x_0x_1 - t_1x_0 + t_1t_0.$$

For example $x_0F_0 + t_2F_2 = X + t_0t_1t_2$. Any two of the three classes integrate jointly, but the specified second-order deformation has a nonzero cubic lifting obstruction. The critical block $T^2_{-\mathbf{1}_L}$ has dimension two.

Proposition 18.3 (The quotient missed by the primary cup)**.** *For $M = M_{m,n}$, let Ω be its full primary image. If $B \subseteq L$ and $|B| \geq 2$, then $\dim_k T^2_{-\mathbf{1}_B}(M) = n-1$; if $B \subseteq H$ and $|B| \geq 2$, the dimension is $m-1$. Moreover,*

$$\sum_{\varnothing \neq B \subseteq L \sqcup H} \dim_k (T^2/\Omega)_{-\mathbf{1}_B} = (n-1)\left(2^m - 1 - m - \binom{m}{2}\right) + (m-1)\left(2^n - 1 - n - \binom{n}{2}\right). \tag{88}$$

Proof. For $B \subseteq L$ with $|B| \geq 2$, the opposite hyperplane H contributes its n rank-one flats as discrete vertices. Each other hyperplane avoiding B is a cross pair and contributes exactly one qualifying vertex. This gives dimension $n-1$; interchange the lines for the second assertion. Every support of size at least three not confined to one line spans $N_{m,n}$ and has zero T^2

by Theorem 17.2. Pair degrees are in Ω, whereas supports of size at least three are not, by Theorem 17.3. Counting the remaining subsets of each line proves (88). □

With $n = 3$, the quotient in (88) grows exponentially with m, while the corank remains three. The cyclic construction gives one actual order-m obstruction. The next theorem realizes every homogeneous class in this quotient by allowing a subset of either line and weights in the cyclic polynomials.

18.1. **Realization of the purely negative quotient.** The dimension in (88) counts classes that the primary cup product cannot reach. We next show that every homogeneous class counted there nevertheless occurs as an actual higher obstruction. The construction allows both the negative support and the obstruction value to be prescribed.

Fix $B \subseteq L$ of size $r \geq 3$, choose a cyclic order on B, and put

$$Q = \prod_{i\in B} x_i, \qquad Z = \prod_{i\in L\setminus B} x_i, \qquad P_B = \prod_{i\in B} t_i, \qquad \sigma = (-1)^{r-1}.$$

Thus $X = ZQ$. For weights $u_{i,h} \in k$ set $v_h = \prod_{i\in B} u_{i,h}$ and define

$$F_{j,h} = \sum_{a=0}^{r-1} (-1)^a \left(\prod_{b=1}^{a} u_{j-b,h} t_{j-b}\right) \left(\prod_{i\in B\setminus\{j,j-1,\dots,j-a\}} x_i\right). \tag{89}$$

All indices on B in this formula are cyclic. Write $C_{j,h} = (X/x_j)W_h$ for the original cross generators and consider

$$J_{B,u} = (X, Y, ZF_{j,h}W_h : j \in B,\ h \in H, C_{j,h} : j \notin B,\ h \in H). \tag{90}$$

The parameters have degree $\deg t_i = \mathbf{e}_i$.

Theorem 18.4 (Realization in every negative component)**.** *Let $R = k[M_{m,n}]$ over any field. For $B \subseteq L$ with $|B| = r \geq 3$ there is an explicit linear isomorphism*

$$\theta_B : k^H/k\mathbf{1} \xrightarrow{\sim} T^2_{-\mathbf{1}_B}(R).$$

Every nonzero ξ in this component is the obstruction of a flat fine-multigraded deformation over $k[t_i : i \in B]/(t)^r$ to lifting over $k[t_i : i \in B]/(t)^{r+1}$. More precisely, its obstruction is $\xi \otimes P_B$, with every other total-degree-r parameter coefficient zero. This specified jet has no lift even in the ungraded category. Its r first-order classes are nonzero; every proper subcollection integrates over its polynomial parameter ring. The symmetric statement holds for $B \subseteq H$.

Consequently every nonzero homogeneous class of the purely negative quotient $T^2(R)/\Omega$ is realized in this way, although all primary cup products among the selected tangent classes vanish.

Proof. The weighted telescoping identity is

$$x_j F_{j,h} + u_{j-1,h} t_{j-1} F_{j-1,h} = Q + \sigma v_h P_B. \tag{91}$$

Over $\mathbb{Z}[u,t]/(P_B)$, the generators in (90) are a monic Gröbner basis for any monomial order in x, y refining total degree. Same-h pairs of modified generators subtract two identities (91); the residual terms cancel. A pair of X and a modified generator has residual $-\sigma v_h P_B Z W_h = 0$. A modified and an unchanged cross generator with the same h use the same identity and have residual $\sigma v_h P_B Z W_h = 0$. In each case the predecessor terms have smaller x, y degree. For different h, every tail is divisible by Y, as are the tails in pairs involving Y. Pairs of unchanged monomial generators have zero S-polynomial. These exhaust the pairs, so the original standard monomials give a free coefficient-module basis. In particular the family is flat over $k[t]/(P_B)$, and setting any parameter to zero gives a flat polynomial family.

We identify its actual obstruction in a fixed presentation. Put $S = k[x, y]$, let K be the relations among the minimal generators of $I = I_{M_{m,n}}$, and let K_0 be their Koszul submodule. The polynomial-quotient presentation gives

$$T^2(R) = \operatorname{coker}\Big(\operatorname{Hom}_S(S^{2+mn}, R) \longrightarrow \operatorname{Hom}_R(K/K_0, R)\Big). \tag{92}$$

The Taylor pair relations generate K, their triple relations give the compatibility equations, and Koszul relations impose the additional annihilation conditions. This low-degree cotangent-complex presentation is valid for polynomial quotients over any field; no characteristic-zero comparison or division by an integer is used. We use the monomial coordinates of [1, Section 2, Lemma 2].

Orient a Taylor relation as $R_{p,q} = \operatorname{lcm}(p, q)e_p/p - \operatorname{lcm}(p, q)e_q/q$. Put $a_j = 1$ for $j \in B$ and zero otherwise. In degree $-\mathbf{1}_B$, its residual coordinate is the coefficient of $\operatorname{lcm}(p, q)/Q$. Define

$$\begin{aligned} \mu_v(X, C_{j,h}) &= -\sigma a_j v_h, \\ \mu_v(C_{j,h}, C_{k,h}) &= \sigma(a_j - a_k)v_h, \end{aligned} \tag{93}$$

and set all other pair coordinates to zero, with antisymmetry. Pairs with a negative exponent or nonface shifted support have zero target and are omitted.

These are coordinates of a cochain in the fixed space $\operatorname{Hom}_R(K/K_0, R)_{-\mathbf{1}_B}$. Indeed the only nonzero pair targets are within the blocks $\{X, C_{j,h} : j \in L\}$ for fixed h; a pair involving Y or two distinct h has shifted support containing Y. In each block (93) is the difference of vertex values 0 at X and $\sigma a_j v_h$ at $C_{j,h}$, so it satisfies every Taylor triangle equation. A Koszul image on a pair p, q has coefficient monomial pq/Q. For $(X, C_{j,h})$ with $j \in B$, its support contains $C_{j,h}$. For same-h cross pairs with exactly one index in B, it likewise contains the circuit indexed by that element of B. Thus the nonzero coordinates annihilate the Koszul relations. The remaining coordinates are zero.

To see that this cochain is the actual lifting obstruction, set $A_{r-1} = k[t]/(t)^r$ and $A_r = k[t]/(t)^{r+1}$, and write $d_{j,h} = u_{j-1,h}t_{j-1}$ for $j \in B$, zero otherwise. Lift the original Taylor relations to

$$\begin{aligned} \widetilde{R}_{X,C_{j,h}} &= W_h e_X - x_j e_{j,h} - d_{j,h} e_{j-1,h}, \\ \widetilde{R}_{C_{j,h},C_{k,h}} &= x_j e_{j,h} - x_k e_{k,h} + d_{j,h} e_{j-1,h} - d_{k,h} e_{k-1,h}. \end{aligned}$$

A predecessor term is included only when its index lies in B. Evaluating at the deformed generators gives respectively $-\sigma a_j v_h P_B Z W_h$ and $\sigma(a_j - a_k)v_h P_B Z W_h$. For different-$h$ pairs and pairs involving Y, the evaluated Taylor relation is a multiple of Y with coefficient specializing to zero; subtract that coefficient times e_Y to obtain an exact relation. The X, Y relation is already exact.

These lifted relations generate the relation module over A_{r-1}: flatness of the quotient implies flatness of its ideal and relation module in the free presentations, base change recovers K, and nilpotent Nakayama applies to the lifted Taylor generators. The small-extension obstruction recipe evaluates these relations over A_r and projects their failures to $R \otimes_k ((t)^r/(t)^{r+1})$. By the displayed evaluations its class is exactly $[\mu_v] \otimes P_B$, with all other parameter coefficients zero. Changing generator lifts changes this cochain by the generator coboundaries in (92).

We can also read the whole target from this presentation. For fixed h, the triangle equations make any cochain a difference of vertex values. If $j, k \in B$, the support of $C_{j,h}C_{k,h}/Q$ is $(L \setminus \{j, k\}) \cup (H \setminus \{h\})$, a face: its complement $\{j, k, h\}$ is a basis of $N_{m,n}$. The Koszul condition therefore equates the vertex values at all $j \in B$. Generator cochains at $C_{k,h}$ for $k \notin B$, with values scalar multiples of $(Z/x_k)W_h$, remove the outside-B vertex values independently. The generator cochain at X, with value a scalar multiple of Z, removes one common constant

among the remaining h-values. No other generator has a value in this degree. Hence the entire quotient is $k^H/k\mathbf{1}$, parametrized linearly by $[\mu_v]$. Explicitly, μ_c is the coboundary with values $-\sigma cZ$ at X, $-\sigma c(Z/x_k)W_h$ at outside-B cross generators, and zero elsewhere; a coboundary on the remaining pairs forces all v_h equal. Thus $\theta_B([v]) = [\mu_v]$ is an isomorphism.

Given $\xi \neq 0$, choose $[v] = \theta_B^{-1}(\xi)$ and a representative v. Set $u_{i_*,h} = v_h$ for one fixed $i_* \in B$ and set the other weights to one. A nonconstant vector has a nonzero coordinate. Every tangent class is nonzero: its value on X is zero, whereas its value at a successor cross generator with nonzero weight is $-u_{i,h}XW_h/(x_ix_{i+1}) \neq 0$. The only polynomial derivation in degree $-\mathbf{e}_i$ is a scalar multiple of $\partial/\partial x_i$, whose nonzero value on X is a scalar multiple of X/x_i. Thus the tangent cannot be a derivation coboundary. Polynomial integration of every one- and two-parameter restriction proves the vanishing of the diagonal and mixed primary obstructions in every characteristic.

Since the obstruction class is nonzero, no lift of this specified jet exists, regardless of grading. The direct socle argument is also instructive: if $g_X, g_{j,h}$ are arbitrary coefficients of P_B in order-r corrections, then for $j \in B$ flatness forces

$$\sigma v_h ZW_h + x_j g_{j,h} - W_h g_X = 0.$$

Projection to fine degree $\deg(ZW_h)$ kills the x_j term and requires the coefficient of Z in g_X to equal σv_h for every h, impossible for nonconstant v. This projection does not require the hypothetical corrections to be homogeneous. Finally Proposition 18.3 and Theorem 17.3 show that the supports just treated, and their symmetric counterparts, are exactly the nonzero degrees of the purely negative quotient. □

Corollary 18.5 (Graded uniqueness of the realizing jets)**.** *For the parameters and first-order classes chosen in Theorem 18.4, the jet over $k[t]/(t)^r$ is unique up to fine-graded isomorphism inducing the identity on the special fiber. This does not assert uniqueness among ungraded jets with the same first-order classes.*

Proof. For every parameter weight $\mathbf{u}$ with $|\mathbf{u}| \geq 2$, $T^1_{-\mathbf{u}}(R) = 0$: nonsquarefree negative parts vanish by the support criterion, and squarefree supports of size at least two have rank at least two in the simple dual, so Theorem 17.2 applies. The polynomial derivations of this degree vanish as well. Align first-order representatives by graded coordinate changes and apply the order-filtration induction in the proof of Theorem 18.1; the differences at each subsequent parameter weight lie in the zero space $\mathrm{Hom}_S(I, R)_{-\mathbf{u}}$. This proves the stated equivalence of the marked jets. □

The realizations are separate for each homogeneous degree. The theorem does not realize arbitrary sums from different degrees in one deformation, nor describe the full ungraded miniversal base.

18.2. **Realization in arbitrary homogeneous degrees.** We retain the two-line matroid $M_{m,n} = N^*_{m,n}$ and its circuit generators $X, Y, C_{j,h}$ from Section 18. We now allow positive coordinates in the obstruction degree. The link isomorphism (1), originating in [1, Proposition 11], computes the component but does not itself construct a deformation of the original ring. Multiplication on the obstruction module has a simple description for the two-line family: it discards precisely the opposite-line coordinates in the positive support. The same operation on the cyclic construction realizes every surviving class. The resulting families are formal; their flatness will be proved without a polynomial monomial order.

Lemma 18.6 (Lifting relations over an Artinian base)**.** *Let A be a local Artinian k-algebra with residue field k, and put $S = A[z_1, \ldots, z_d]$. Suppose $F_1, \ldots, F_q \in S$ reduce to generators*

$p_1, \dots, p_q$ of $I \subseteq k[z]$. If generators of the relation module among the p_i lift to exact relations among the F_i, then $S/(F_1, \dots, F_q)$ is free over A on arbitrary lifts of a k-basis of $k[z]/I$.

Proof. Write $\mathfrak{m}$ for the maximal ideal, $J = (F_i)$, and $K_A = \ker(S^q \to J)$. Right exactness gives $J/\mathfrak{m}J = k[z]^q/\operatorname{im}(K_A \to k[z]^q)$. The lifted relations imply that this image is the kernel of $k[z]^q \to I$. Thus $J/\mathfrak{m}J \to S/\mathfrak{m}S$ is injective, and the Tor sequence gives $\operatorname{Tor}_1^A(S/J, k) = 0$ since S is A-free. Let E be the free A-module on chosen lifts of a residue-field basis of S/J. The map $E \to S/J$ is surjective: its cokernel equals its product with $\mathfrak{m}$, and $\mathfrak{m}$ is nilpotent. Its kernel K lies in $\mathfrak{m}E$ because reduction is an isomorphism. The Tor sequence now identifies $K/\mathfrak{m}K$ with $\operatorname{Tor}_1^A(S/J, k)$, so $K = \mathfrak{m}K = 0$. No finite-dimensionality of the residue-field basis is required. □

Theorem 18.7 (Realization of the homogeneous primary quotient)**.** *Let $R = k[M_{m,n}]$, $m, n \geq 3$, over any field, and let $\Omega = \operatorname{im}(T^1(R) \otimes_R T^1(R) \to T^2(R))$. For every fine degree c with $(T^2(R)/\Omega)_c \neq 0$, write $c = a - \mathbf{1}_B$, $a \geq 0$, with disjoint supports. Then B is contained in one of the two dual lines, $r = |B| \geq 3$, and $\Omega_c = 0$, so the quotient map canonically identifies this component with $T_c^2(R)$. For every nonzero ξ in it, there is a marked fine-graded flat deformation over $k[t_i : i \in B]/(t)^r$ whose obstruction to lifting over $k[t_i : i \in B]/(t)^{r+1}$ is $\xi \otimes [P_B]$ in $T^2(R) \otimes_k J_r$, where $P_B = \prod_{i \in B} t_i$ and $J_r = (t)^r/(t)^{r+1}$. All other total-degree-r obstruction coefficients are zero. This specified jet has no lift even if grading is forgotten. Its r tangent classes are nonzero, and every proper subcollection integrates formally. All primary products among these tangent classes vanish. A cyclic ordering of B and a distinguished $i_* \in B$ may be chosen arbitrarily; the parameter degrees are $\deg t_i = \mathbf{e}_i$ for $i \neq i_*$ and $\deg t_{i_*} = \mathbf{e}_{i_*} - a$. In particular $\deg P_B = -c$.*

The realizing jet is unique up to marked fine-graded isomorphism for the chosen parameter degrees and tangent classes. Here isomorphisms are over the fixed parameter algebra and induce the identity on the special fiber. A proper subcollection is obtained by setting all other parameters to zero. The assertion is separate for each homogeneous degree; it does not assert simultaneous realization of inhomogeneous sums or describe the full ungraded miniversal base.

Proof. Write $c = a - \mathbf{1}_B$ with $a \geq 0$ and $\operatorname{supp}(a) \cap B = \varnothing$. Theorems 17.2 and 17.5, with the actual-vertex convention of (1), imply that a nonzero quotient component has B contained in one dual line, with $r = |B| \geq 3$. Suppose $B \subseteq L$, write $A = \operatorname{supp}(a)$, and put $H_0 = H \setminus A$. If $|H_0| \geq 2$, the dual of the actual-vertex link is the deletion $N \setminus A$. It still has rank three, has no coloops, and consists of the lines $L \setminus A, H_0$. The only contributing hyperplane is H_0, with its singleton flats as vertices. Hence $T_c^2(R) = k^{H_0}/k\mathbf{1}$. If $|H_0| = 1$, its unique point is a coloop of $N \setminus A$; removing that ghost leaves a rank-two line, whose blocks have at most one vertex and contribute zero. If $H_0 = \varnothing$, A is a nonface and the component is zero. Thus a nonzero target necessarily has $|H_0| \geq 2$.

We need the multiplication map, not merely these dimensions. In the Taylor presentation on the minimal monomial generating set, modulo Koszul relations, use (92) and orient $R_{p,q} = \operatorname{lcm}(p,q)e_p/p - \operatorname{lcm}(p,q)e_q/q$. Set $b_j = 1$ for $j \in B$, zero otherwise, and $\sigma = (-1)^{r-1}$. The negative cochain representing $v \in k^H$ has coordinates

$$\mu_v(X, C_{j,h}) = -\sigma b_j v_h, \qquad \mu_v(C_{j,h}, C_{k,h}) = \sigma(b_j - b_k)v_h,$$

and zero coordinates elsewhere, as in (93). Each scalar multiplies $\operatorname{lcm}(p,q)/\prod_{i \in B} x_i$. Multiplication by x^a kills exactly the blocks $h \in A$, because their monomials become divisible by Y. For $h \in H_0$ every same-block Taylor pair target survives. Its complement contains $B \cup \{h\}$, which spans N, while adding A cannot remove those elements. Taylor triangles make each surviving block a difference of vertex values; the Koszul conditions equate all vertices indexed

by B since $(L\setminus\{j,k\})\cup(H\setminus\{h\})$ remains a face after adding A for distinct $j,k\in B$. Outside-B generator cochains remove their vertices, and the cochain on X removes one common constant. More explicitly the vertices of each surviving block are $\{X, C_{j,h} : j\in L\}$; all their pair and triple targets are nonzero. The cochain values at outside-B generators are scalar multiples of $x^a(Z/x_k)W_h$ for $k\notin B$, and the value at X is a scalar multiple of x^aZ. Their complements retain $B\cup\{h\}$ and $B\cup H_0$, respectively, so these are nonzero values. There is no value at Y or a B-indexed cross generator because one negative B coordinate remains. These are precisely the same operations as in the negative presentation, multiplied by x^a. Consequently the actual multiplication map is

$$k^H/k\mathbf{1}\longrightarrow k^{H_0}/k\mathbf{1},\qquad [v]\longmapsto [v|_{H_0}]. \tag{94}$$

It is surjective. Since $\Omega_c=0$, choose v with $[x^a\mu_v]=\xi$; its restriction to H_0 is nonconstant.

Fix $i_*\in B$, choose a cyclic ordering, and interpret predecessors and successors below in that ordering, not in the ground-set labels. In (89) set $u_{i_*,h}=v_hx^a$, $u_{i,h}=1$ for $i\neq i_*$. Put $Q=\prod_{i\in B}x_i$, $Z=\prod_{i\in L\setminus B}x_i$, $P_B=\prod_{i\in B}t_i$, and explicitly put

$$F_{j,h}=\sum_{q=0}^{r-1}(-1)^q\left(\prod_{b=1}^{q}u_{j-b,h}t_{j-b}\right)\left(\prod_{i\in B\setminus\{j,j-1,\dots,j-q\}}x_i\right).$$

Empty products are one. The constant term is Q/x_j and the t_i-linear term at the successor of i is $-u_{i,h}Q/(x_ix_{i+1})$. The earlier telescoping identity is a polynomial identity in arbitrary commuting weights, so substitution of v_hx^a is legitimate. It becomes

$$x_jF_{j,h}+u_{j-1,h}t_{j-1}F_{j-1,h}=Q+\sigma v_hx^aP_B. \tag{95}$$

Use generators X,Y, $D_{j,h}=ZF_{j,h}W_h$ for $j\in B$, and $D_{j,h}=C_{j,h}$ for $j\notin B$. Give t_i degree $\mathbf{e}_i$ for $i\neq i_*$ and give t_{i_*} degree $\mathbf{e}_{i_*}-a$. Each deformed generator has the fine degree of its original monomial.

Write $k[t]=k[t_i : i\in B]$ and $(t)=(t_i : i\in B)$. For every $\nu\geq 1$, work over $A_\nu=k[t]/((t)^\nu,P_B)$. All original Taylor pair relations lift exactly, as follows. Put $d_{j,h}=u_{j-1,h}t_{j-1}$ for $j\in B$; omit predecessor terms entirely when $j\notin B$. The $X,D_{j,h}$ relation lifts to $W_he_X-x_je_{j,h}-d_{j,h}e_{j-1,h}$; evaluation is $-\sigma b_jv_hx^aP_BZW_h=0$. For two cross generators in the same block, subtract the corresponding identities: the lift is $x_je_{j,h}-x_ke_{k,h}+d_{j,h}e_{j-1,h}-d_{k,h}e_{k-1,h}$ and its evaluation is $\sigma(b_j-b_k)v_hx^aP_BZW_h=0$. For a pair involving Y, $y_hD_{j,h}$ is divisible by Y; replace the coefficient on e_Y by $y_hD_{j,h}/Y$. For cross generators in distinct blocks, the original Taylor multipliers contain the missing y_h and y_l. Both evaluated terms are therefore divisible by Y, and their difference divided by Y has zero parameter constant term. Subtract this quotient times e_Y. The X,Y relation and pairs of unchanged monomials are exact already. These lifts reduce to every original Taylor pair, which generate all first syzygies. Lemma 18.6 proves A_ν-flatness. Their formulas are compatible for all ν. The inverse system defines a formal family over $k[[t]]/(P_B)$; setting any parameter to zero gives a formally integrated proper subcollection. No polynomial flatness is inferred here.

Over the small extension $k[t]/(t)^{r+1}\to k[t]/(t)^r$, the evaluations just computed give $[x^a\mu_v]\otimes P_B$. All other pair evaluations are zero after the exact Y corrections. Over the lower-order jet, flatness of the quotient makes its ideal flat, and then the relation module in the free presentation is flat. Reduction is exact and recovers the original Taylor relations; nilpotent Nakayama shows that their lifts generate this module. The fixed small-extension presentation therefore gives precisely $\xi\otimes P_B$ and no other order-r coefficient.

Each ordinary parameter direction has the nonzero tangent class calculated in Theorem 18.4. In the special direction it is multiplied by x^a, with weights v_h. Choose $h\in H_0$ with $v_h\neq 0$. At the successor cross generator its value is $-v_hx^aXW_h/(x_{i_*}x_{i_*+1})\neq 0$, since the complement of

its support contains the basis $\{i_*, i_*+1, h\}$ of N. Its value on X is zero. The only polynomial derivation of that degree is a scalar multiple of $x^a\partial/\partial x_{i_*}$, whose value on X is nonzero: the complement of its support contains $\{i_*\} \cup H_0$, of rank three. Thus the special tangent is not a derivation coboundary. Formal one- and two-parameter integration kills the quadratic obstructions. Their polarization is the cup product by Lemma 8.6, also in characteristic two; self-cups vanish by the doubled fine degree.

Since the obstruction class is nonzero, the specified jet has no ungraded lift. Equivalently, let $g_X, g_{j,h}$ be arbitrary P_B coefficients of prospective order-r generator corrections. Order-r changes of relation coefficients evaluate to zero in R, and positive-order coefficients times order-r generator corrections have order at least $r+1$. Thus arbitrary relation corrections cannot evade the following equation. Projection of the lifted $X, D_{j,h}$ relation to $R \otimes P_B$ forces $\sigma v_h x^a Z W_h + x_j g_{j,h} - W_h g_X = 0$ for $j \in B$, $h \in H_0$. The coefficient of $x^a Z W_h$ in the x_j term is zero because $a_j = 0$ and $j \notin \operatorname{supp}(ZW_h)$. Hence the coefficient of $x^a Z$ in g_X must equal σv_h for all $h \in H_0$, contrary to nonconstancy.

Finally a correction at parameter monomial $t^{\mathbf{u}}$, $|\mathbf{u}| \geq 2$, has ring degree $-\sum_i u_i \mathbf{e}_i + u_{i_*} a$. Its negative part is either nonsquarefree or meets at least two distinct parallel classes of the simple dual: its entries on B are exactly $-u_i$, since a vanishes there. The support criterion in Section 2 and Theorem 17.2 give $T^1 = 0$ in these degrees. A polynomial derivation has no coefficient monomial in such a degree, because adding one $\mathbf{e}_j$ cannot make all entries nonnegative. The right-exact sequence

$$\operatorname{Der}_k(k[x,y], R) \longrightarrow \operatorname{Hom}_{k[x,y]}(I, R) \longrightarrow T^1(R) \longrightarrow 0$$

therefore gives a zero correction space. Aligning first-order representatives by marked graded coordinate changes and inducting through parameter orders $2, \ldots, r-1$ gives marked fine-graded uniqueness, without asserting ungraded uniqueness. The case $B \subseteq H$ follows by interchanging the two lines. □

18.3. **The module left after primary cup products.** Put $R = k[M_{m,n}]$ and $\mathcal{Q} = T^2(R)/\Omega$ as in Theorem 18.7.

The dimension of the purely negative part in (88) grows exponentially, but this does not measure the number of generators of the obstruction module. For the same two-line family, all its components and multiplication maps fit an explicit presentation. The additional information is multiplication between negative degrees, not another dimension formula. Put $S_L = k[x_i : i \in L]$ and $S_H = k[y_h : h \in H]$. Set $S = S_L \otimes_k S_H = k[x_i, y_h]$ and $I = I_{M_{m,n}}$, so $R = S/I$. Define

$$J_L = (x_D : D \subseteq L,\ |D| = m-2), \qquad K_L = (x_D : D \subseteq L,\ |D| = m-1),$$

where $x_D = \prod_{i \in D} x_i$; define J_H, K_H symmetrically. Set

$$U_H = \frac{\bigoplus_{h \in H} S_H e_h}{S_H\left(\sum_{h \in H} e_h\right) + \sum_{h \in H} S_H y_h e_h}, \tag{96}$$

with $\deg e_h = 0$, and define U_L in the same way. Tensor products in the following theorem are over k. For a graded module V, the notation $V\langle d\rangle$ means that every degree is increased by d; in particular its displayed generators of degree zero acquire degree d.

Theorem 18.8 (Generators, relations, and support of the primary quotient)**.** *For $m, n \geq 3$ and every field k, there is a fine-graded R-module isomorphism*

$$\mathcal{Q} \simeq \left((S_L/J_L) \otimes U_H\right)\langle -\mathbf{1}_L\rangle \ \oplus\ \left(U_L \otimes (S_H/J_H)\right)\langle -\mathbf{1}_H\rangle. \tag{97}$$

The indicated S-modules are annihilated by I, so their R-module structures are unambiguous. The quotient has exactly $m+n-2$ minimal homogeneous generators: $n-1$ in degree $-\mathbf{1}_L$ and $m-1$ in degree $-\mathbf{1}_H$. Its annihilator in S is

$$\operatorname{Ann}_S \mathcal{Q} = (J_L S + K_H S) \cap (K_L S + J_H S). \tag{98}$$

Consequently $\operatorname{Ann}_R \mathcal{Q}$ is this ideal modulo I. The annihilator is radical. Every irreducible component of $\operatorname{Supp}_R \mathcal{Q}$ has dimension $m+n-5$; in particular the support of the primary quotient has codimension two in $\operatorname{Spec} R$.

Proof. Use the Taylor presentation (92), with $R_{p,q} = \operatorname{lcm}(p,q)e_p/p - \operatorname{lcm}(p,q)e_q/q$. Its coboundary is $\delta f(R_{p,q}) = (\operatorname{lcm}(p,q)/p)f(p) - (\operatorname{lcm}(p,q)/q)f(q)$. Normalize the cocycle (93) by removing its factor $(-1)^{|B|-1}$. The resulting $\nu_B(v)$ has scalars

$$\nu_B(v)(X, C_{j,h}) = -b_j v_h, \qquad \nu_B(v)(C_{j,h}, C_{k,h}) = (b_j - b_k)v_h, \quad b_j = \mathbf{1}_{j\in B},$$

and zero elsewhere; each scalar multiplies $\operatorname{lcm}(p,q)/x_B$.

We first determine multiplication when a negative coordinate is removed; this is the point not supplied by the component dimensions. Suppose $B \subseteq L$, $|B| \geq 3$, $i \in B$, and $B' = B \setminus \{i\}$. In the fixed Taylor presentation let $f_i(v)$ be the generator cochain of degree $-\mathbf{1}_{B'}$ given by

$$f_i(v)(C_{i,h}) = v_h C_{i,h}/x_{B'} \quad (h \in H), \qquad f_i(v)(X) = f_i(v)(Y) = f_i(v)(C_{j,h}) = 0 \quad (j \neq i).$$

Its nonzero monomials are defined, since $i \notin B'$; their complements are $B' \cup \{i,h\}$ and span N, so they survive in R. For $|B'| \geq 3$ the cocycle formulas give the identity

$$x_i \nu_B(v) - \nu_{B'}(v) = \delta f_i(v). \tag{99}$$

Indeed, measured relative to $\operatorname{lcm}(X, C_{i,h})/x_{B'}$, the change at $(X, C_{i,h})$ is $-v_h$, precisely the scalar of the boundary $-x_i f_i(v)(C_{i,h})$. On a same-h pair it is the difference of the two changed vertex scalars. Every other pair involving one of the selected generators has its target divisible by Y and hence zero in R: this includes pairs with Y and cross pairs in distinct h blocks. All remaining values on both sides are zero. Thus the actual map on the coordinate quotient is $[v] \mapsto [v]$. When $|B'| = 2$, the target has zero primary quotient, so the map into $\mathcal{Q}$ is zero without any comparison of cocycle normalizations in that degree. Multiplying these identities by a positive monomial preserves them, including when some generator values vanish. Multiplication in a same-line positive coordinate leaves the remaining coordinate quotient unchanged. An opposite-line coordinate y_h restricts the quotient on H_0 to that on $H_0 \setminus \{h\}$. If h is already absent it leaves the coordinates unchanged; if at most one remains, the target is zero. Subsequent powers therefore act identically on the surviving coordinates. These last assertions follow from the actual restriction map (94).

Send e_h in the first summand of (97) to $[\nu_L(\mathbf{e}_h)] \in \mathcal{Q}_{-\mathbf{1}_L}$. The common-constant relation is zero, and $y_h e_h$ maps to zero by coordinate restriction. A generator x_D of J_L removes $m-2$ negative coordinates, leaving two; hence it maps every e_h to zero. This defines an S-linear map from the presented module, before any dimension comparison. The symmetric construction defines the second map.

For $b \geq 0$, the degree-b component of U_H is the coordinate space on $H \setminus \operatorname{supp}(b)$ modulo one common constant, with dimension $(|H \setminus \operatorname{supp}(b)| - 1)_+$. A degree $c = a - \mathbf{1}_B$ in the first shifted summand survives the J_L relation exactly when $B \subseteq L$ and $|B| \geq 3$. Indeed, for unshifted L-degree $u \geq 0$, $B = \{i : u_i = 0\}$ and $x^u \notin J_L$ is equivalent to $|\operatorname{supp}(u)| \leq m-3$. The U_H factor additionally requires $|H_0| \geq 2$. The actual maps just proved show that its image is all of $\mathcal{Q}_c$. The component calculation in Theorem 18.7 gives equality of dimensions and therefore injectivity. The nonzero fine degrees of the two summands are disjoint, so their sum is the required S-module isomorphism.

The ideal I annihilates both presented summands directly. In the first, X and every $C_{j,h}$ have L support of size at least $m-2$ and belong to $J_L S$, while Y kills U_H. In the second, Y and every cross generator belong to $J_H S$, while X kills U_L. Thus the isomorphism descends to R. Modulo the homogeneous maximal ideal, U_H has exactly its n degree-zero generators and the single relation $\sum e_h = 0$. The first summand has $n-1$ generators in its lowest degree $-\mathbf{1}_L$; the second contributes $m-1$ in degree $-\mathbf{1}_H$. Graded Nakayama proves the stated minimality.

The annihilator of U_H is fine homogeneous, hence monomial. A monomial with support D maps the degree-zero coordinate quotient onto the quotient on $H \setminus D$. Its action is zero exactly when $|D| \geq n-1$; the degree-zero generators generate the module. Therefore $\operatorname{Ann}_{S_H} U_H = K_H$. The same support test shows that the annihilator of the first tensor summand is $J_L S + K_H S$: a monomial outside this ideal acts nontrivially on a suitable degree-zero generator. The symmetric formula for the other summand yields (98). Both ideals in the intersection are generated by squarefree monomials and are radical, as is their intersection. The maximal coordinate faces of $V(J_L S + K_H S)$ have $m-3$ vertices in L and $n-2$ in H; those of $V(K_L S + J_H S)$ have $m-2$ in L and $n-3$ in H. No maximal face of either contains one from the other, and each has $m+n-5$ vertices. The support is thus pure of that dimension. Finally R is equidimensional of dimension $\operatorname{rk} M = m+n-3$, proving the codimension assertion. □

If $m = 3$, then $S_L/J_L = k$, and similarly for $n = 3$. The argument uses no division by m, n, or 2, and includes characteristic two. This presentation concerns the primary quotient, not the full T^2 module or a miniversal base. Its total k-dimension is infinite in this family; the exponential dimension previously computed concerns only the purely negative part. The module presentation and the homogeneous realization theorem answer different questions: the former describes how classes multiply, while the latter realizes each class separately.

19. Finite certificates and computational checks

The preceding proofs give a full component calculation, a vanishing classification, and actual obstructions of unbounded order. None of these universal assertions is inferred from a finite sample. This section gives the finite calculations for the Tutte-equivalent pair, the compatibility table in the rank-three extremal proof, and the small-ground-set minima. It also compares the graph formula with a direct calculation from generators and relations. The general graph, vanishing, and cup-product theorems are proved in Sections 3–8.

The source package's `anc/README.md` identifies each checker, its invocation and expected output fields. The accompanying `anc/finite-certificates.json` records the earlier exact table, coverage, polynomial and small-ground-set results. These recorded certificates are distinct from a fresh execution of a checker.

19.1. **Checks of the cyclic construction and primary image.** The accompanying `rank-two-support.py` compares Theorem 17.2 with 1,119 graph calculations, 350 direct presentation calculations over $\mathbb{F}_2$ and $\mathbb{F}_{32003}$, and 363 positive-support calculations. The separate `cup-image.py` checks 148 presentation-level cup ranks, including 30 nonzero targets, against Theorem 17.3. For the two-line matroids it recovers the purely negative quotient dimensions $4, 13, 30, 68$ in the tested cases.

The all-degree image theorem is checked by `full-primary-image.py` using 2,574 multiplication maps on the minimal Taylor presentation, both before and after taking the derivation cokernel, and 2,703 cup-rank calculations with squarefree positive support. The fields are $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_{32003}$. The cases include repeated dual parallel classes, loops, and links with vertices removed after contraction. All mathematical output fields agree on isolated replay. These tests address the maps in Lemma 17.4 and Theorem 17.5; they do not replace their proofs.

The program `arbitrary-order.py` uses integer polynomial division to check the cyclic Gröbner construction and its telescoping identity for $3 \leq m \leq 12$ and $n = 3, 4$, comprising 17,740 S-pair tests. The separate `essential-cubic.py` checks 220 S-pairs and the contradictory socle relations for the cubic case. Independent presentation calculations in the smaller cases verify the critical T^1 vanishings and T^2 dimensions in two characteristics. The four programs `rank-two-support.py`, `cup-image.py`, `arbitrary-order.py`, and `essential-cubic.py` were each rerun in an isolated environment; all mathematical result fields agreed on replay. These finite checks test the construction, not the assertions for arbitrary m or arbitrary fields, which follow from the proofs in Sections 17 and 18.

The program `weighted-realization.py` checks the weighted subset construction of Theorem 18.4 separately in 32 cases with $3 \leq m \leq 6$, $n = 3, 4$, and both consecutive and non-consecutive subsets of size at least three. The tests comprise 12,636 integer S-pairs, 9,475 Taylor triangle conditions and 6,450 Koszul conditions. Independent presentation calculations in characteristics $2, 3, 32003$ verify that the displayed obstruction cochains span the entire target modulo generator coboundaries. The deterministic replay agrees on every mathematical result field. The relation-level identification and arbitrary-field realization theorem are proved above, not inferred from these cases.

For Theorem 18.7, the separate checker `positive-realization.py` tests 60 cases in the two-line family, including 51 with positive support and 22 with zero targets or contraction ghosts; these categories overlap. It includes positive exponents one and two and weights supported on both lines. There are 6,764 integral Taylor relation checks, 4,314 exact corrections by multiples of Y, 3,754 Taylor triangle checks, 125 tangent nontriviality checks, and 492 higher-degree correction vanishings. Presentation calculations in characteristics $2, 3, 32003$ verify the restriction map and its image dimension. The separate replay agrees on all mathematical fields. The arbitrary-weight flatness assertion uses the Artinian relation-lifting proof, not a Gröbner basis test after substitution of positive monomial weights.

For Theorem 18.8, `primary-quotient-module.py` checks the actual Taylor/Koszul presentation for $(m, n) = (3, 3), (4, 3), (4, 4), (5, 3)$, with both line orientations, positive exponents one and two, and fields $2, 3, 32003$. It computes 3,204 degree/field cases and 24,972 variable multiplication maps. These include 35,058 signed negative-coordinate boundary identities and 5,424 repeated-positive maps; 4,224 squarefree support tests check annihilation and 360 checks verify the action of minimal circuit generators. The size-two endpoints use actual cup vectors in the presentation. Every mathematical field agrees with an isolated replay. The universal module isomorphism and its annihilator are established by the preceding proof, not by these finite tests.

19.2. **Subset-rank certificate for the graphic pair.** For each of the two edge lists in Theorem 6.7, enumerate its 2^{13} subsets X and compute $r(X) = 7 - \kappa(V, X)$, counting isolated vertices. The two lists have the same numbers

$$a_{r,s} = \#\{X : r(X) = r,\ |X| = s\}.$$

For a fixed r, the following rows give the coefficients in increasing s, starting at $s = r$:

r	$(a_{r,r}, a_{r,r+1}, \ldots)$
0	(1)
1	(13)
2	$(78, 7)$
3	$(279, 83, 12, 1)$
4	$(632, 413, 145, 31, 3)$
5	$(862, 1025, 662, 281, 78, 13, 1)$
6	$(545, 1023, 1003, 637, 273, 77, 13, 1)$.

Consequently their Tutte polynomials agree, since $T(x, y) = \sum_{r,s} a_{r,s}(x-1)^{6-r}(y-1)^{s-r}$. For each subset with $r(X) = |X| = 5$, count the original edges joining its two components. This gives exactly the two profiles in the proof of Theorem 6.7, hence $\Theta = 12135$ and 12161.

For each edge list, retain the rank of every labeled subset before aggregating into $a_{r,s}$. These separate rank dictionaries give the dual rank function $r^*(X) = |X| - 6 + r(E \setminus X)$. A subset is a dual flat precisely when adjoining each omitted edge increases r^*. Enumerating the flats of ranks five and six and applying (17) gives the two values $\mathfrak{q} = 2009$ and 2165 in Proposition 16.3. All these calculations use integer ranks and counts.

19.3. **Partition and line-system calculations.** The twelve rows of (76) are obtained by listing integer partitions and evaluating the recurrence in its proof; no matroid realizability assumption is made for the relaxed minima. The accompanying coverage table checks the remaining cases with $8 \le n \le 12$. The coefficient certificates in Lemma 15.4 are exact polynomial expansions. Direct re-expansion and a separate partition implementation reproduce all displayed entries.

The small-ground-set rank-three minima can also be reproduced without a database of matroids. On $[n]$, list all subsets of sizes $3, \ldots, n-1$. Select a family in which no unordered pair lies in two selected subsets, and add every unused pair as a two-point line. This construction gives exactly the line systems of labeled simple rank-three matroids: distinct lines intersect in at most one point, and every pair belongs to one line. To enumerate each family once, order the candidate subsets and recursively adjoin only later candidates whose pairs are unused. Evaluate (66) at every node, including the empty family. The resulting counts are

n	line systems	$\min \dim_k T^2(M)_0$	labeled minimizers
5	31	0	31
6	352	54	30
7	8389	357	30.

This is an exhaustive finite calculation, not an argument for the unbounded rank-three theorem.

As additional arithmetic checks, all connected rank-two class partitions on $5 \le n \le 20$ reproduce both stated minima, their equality cases, and the second-profile gap for $\mathfrak{q}$. The Fano calculation gives $126 + 231 = 357$. These checks test the formulas and boundary cases; the general rank-two assertions rest on the merge and transfer arguments.

19.4. **Calculation from generators and relations.** The Lichtenbaum–Schlessinger presentation gives a second way to compute T^1 and T^2. Circuit monomials generate the Stanley–Reisner ideal; Taylor pair relations, relations among triples of generators, and the trivial Koszul relations determine the cochain spaces and their quotients. Computing their dimensions by exact row reduction provides a comparison with the hyperplane-graph formula that does not use the relative-cohomology description.

The accompanying program `anc/verify_independent.py` carries out this comparison for uniform, rank-two, graphic, direct-sum, and Fano matroids, mostly over $\mathbb{F}_{32003}$ and for selected uniform and rank-two examples over $\mathbb{F}_2$. The positive-support cases include two-element links of $U_{3,6}$ and both link strands of the Fano matroid. All 917 checks agree with the stated formulas. The program lists the matroids, fine degrees, and coefficient fields in its `main()` function. These calculations supplement the proofs in Sections 3–8.

20. Further questions

The obstruction module, its primary image, and its realized higher obstructions contain different information. The rank-two reduction makes the first accessible by flat counts; the cyclic family shows that the second can miss essential lifting obstructions of unbounded order. The following questions concern the remaining distinctions.

(1) *The module structure of T^2.* Theorem 3.1 describes the fine components of $T^2(M)$ as vector spaces. Can the multiplication maps between these components be described directly in terms of the hyperplane graphs? The parallel-class maps in (81) give part of this structure in negative directions, and Theorem 18.8 determines the quotient by the primary image for the dual two-line family. For general matroids, can one obtain presentations, minimal generators, and annihilators from the dual flats? Even in the two-line family, the extension $0 \to \Omega \to T^2 \to T^2/\Omega \to 0$ retains information not specified by the quotient presentation.

(2) *Relations among cup products.* Theorem 17.5 determines the entire primary image, and Corollary 17.6 gives its purely negative generation. Can the kernel of the cup map, and relations among its homogeneous generators, be described from the dual flats? Such a description would refine the image calculation to explicit quadratic equations for a deformation base.

(3) *Higher-order lifts.* Theorem 18.1 gives essential obstructions of every order and rules out all ungraded lifts of the specified jets. Theorem 18.7 realizes every nonzero homogeneous class of the complete primary quotient for the dual two-line family. Can the same be done for other matroids, or for inhomogeneous sums of classes from several degrees within a single deformation? Can the first unavoidable order be determined from their first-order classes without imposing the fine grading, so that different lower-order choices cannot postpone it? For two classes, does cup-product vanishing still suffice for joint integration when positive and negative supports cancel? Even under the support hypotheses of Theorem 8.11, its formal arc for the sum does not by itself answer the independent-parameter question.

(4) *Higher cotangent cohomology.* In characteristic zero, the link-localization theorem of Ilten, Meazzini, and Petracci [10] applies to $T^i(M)$ for $i \geq 3$. Can nonvanishing be detected in a link whose rank is bounded in terms of i? Is the class of matroids with $T^i(M) = 0$ minor-closed, and, if so, does it have finitely many excluded minors?

(5) *Valuative invariants.* The cocircuit profile of Section 16 is valuative, whereas the pure-pair rank $\mathfrak{q}$ is not. Which other quantities obtained from the hyperplane graphs are valuative under matroid subdivisions?

Acknowledgements

The author thanks the authors of [6] for formulating the vanishing conjecture and for the combinatorial framework on which this work builds. The author acknowledges Gewu Intelligence Lab for providing the collaborative research environment in which this project was developed.

Disclosure of automated assistance

A generative-AI workflow using OpenAI GPT-5.6 Sol, Anthropic Claude Fable 5, Grok 4.6, and OpenAI GPT-6 Astra developed the hyperplane-graph and cup-product formulas, higher-order obstruction constructions, and primary-quotient module calculations, and supported finite verification and manuscript preparation. OpenAI GPT-6.1 Sol and Anthropic Claude Fable 5.1 assisted with subsequent proof and reference review, exposition, and LaTeX revision. The results await further verification by domain experts.

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