

A matching information bound for deterministic one-way teleportation

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Abstract

We establish a necessary mutual-information bound for deterministic finite-alphabet one-way teleportation of a Haar-distributed pure state in dimension $d \geq 2$. The protocol may use an arbitrary shared state, an arbitrary instrument on Alice's side, a fixed stochastic classical channel, and arbitrary trace-preserving recovery channels on Bob's side. If X and Y are the actual classical-channel input and output, with the input law induced by the Haar ensemble, then average fidelity greater than $2/(d+1)$ requires $I(X;Y) > \log_2 d - (1 - 1/d) \log_2(d+1)$. The proof reduces the quantum protocol to a bounded classical score whose expectation under the independent input-output law is exactly $1/d^2$, and applies relative-entropy data processing. Standard Bell/Weyl teleportation through symmetric noisy classical channels approaches the bound from above, so it is the exact unattained infimum in this class. The qubit value is the previously known 0.207518... bits; the converse here allows general instruments and recoveries. The result does not cover arbitrary interactive protocols or determine the optimum for each fixed resource state.

1 Introduction

Quantum teleportation transfers an unknown quantum state using shared entanglement and classical communication [1]. A natural question is how much classical information a noisy communication channel must transmit before teleportation can exceed the fidelity available to a measure-and-prepare strategy. For a Haar-distributed pure state in dimension d , that classical benchmark is $2/(d+1)$ [2, 4]. We use the Shannon mutual information between the actual classical-channel input and output as the communication cost, following the noisy-channel formulation of Banik and Gazi [3].

The numerical boundary is already known in restricted settings. Banik and Gazi obtained a qubit sufficient cost arbitrarily close to 0.208 bits from above with a singlet resource and a noisy four-symbol channel. Their necessity argument assumes an isotropic transformation of the teleported state [3]. Weinar, Laskowski, and Pawłowski obtained the expression

$$C_{\min}(d) = \log_2 d - \left(1 - \frac{1}{d}\right) \log_2(d+1) \quad (1)$$

in their analysis of standard Bell-basis teleportation and conjectured applicability beyond that setting [4]. For general two-qubit resources in Bell-measurement protocols with noisy classical communication, Bej, Halder, and Sengupta formulated resource-dependent cost minimizations and derived stationarity conditions rather than a general closed-form optimum [5].

The present result is a converse for the deterministic finite-alphabet one-way class defined below, with matching limiting achievability. Its proof follows the scientific source repository [6]. It assumes neither a Bell measurement nor isotropy, a Werner or maximally entangled resource, unitary recovery, uniform outcome probabilities, or a four-symbol alphabet. The essential quantum constraint is that discarding Alice's instrument outcome cannot signal to

Bob. Applied to normalized Choi branches, this constraint fixes a product-law score expectation and permits a purely classical relative-entropy argument. The result addresses a specified one-way class; it does not resolve the unrestricted interactive problem. No novelty or priority claim is made.

2 Protocol model and information convention

Let A be the unknown input and B Bob's final output, both of dimension $d \geq 2$. The parties initially share an arbitrary state $\omega_{A'B_0}$, independent of the input. Alice acts on AA' with a complete local instrument $\{\mathcal{A}_x\}_{x \in \mathcal{X}}$. Her local ancillary systems, private randomness, measurements, and encoding of the transmitted symbol x are included in this instrument. The alphabet $\mathcal{X}$ is finite, and $\sum_x \mathcal{A}_x$ is trace preserving. The resulting branches on Bob's pre-recovery system are completely positive trace-nonincreasing maps

$$\mathcal{E}_x(\rho) = \text{Tr}_{A_{\text{out}}} [(\mathcal{A}_x \otimes \text{id}_{B_0})(\rho \otimes \omega_{A'B_0})]. \quad (2)$$

No restriction to a particular shared-state family is imposed.

Alice sends x through a fixed stochastic channel $q(y|x)$ with finite output alphabet $\mathcal{Y}$. On receiving y , Bob applies a completely positive trace-preserving (CPTP) map $\mathcal{R}_y : B_0 \rightarrow B$. The effective channel is

$$\mathcal{T} = \sum_{x,y} q(y|x) \mathcal{R}_y \circ \mathcal{E}_x. \quad (3)$$

The protocol is deterministic in the sense that every run contributes to this trace-preserving channel and to its fidelity; there is no conditioning on a successful subset of runs. Local randomness is allowed. Finite sequences of one-way channel uses without feedback are included by taking X and Y to be the complete input and output transcripts in this representation.

Writing $\psi = |\psi\rangle\langle\psi|$ and taking normalized Haar measure $d\psi$, define

$$F_{\text{av}}(\mathcal{T}) = \int d\psi \langle \psi | \mathcal{T}(\psi) | \psi \rangle. \quad (4)$$

The law of the actual transmitted symbol is specifically

$$p_x = \int d\psi \text{Tr} \mathcal{E}_x(\psi) = \text{Tr} \mathcal{E}_x(I_A/d), \quad p_{xy} = p_x q(y|x), \quad p_y = \sum_x p_{xy}. \quad (5)$$

All logarithms are base two, and the cost, in bits, is

$$I(X; Y) = \sum_{x,y} p_{xy} \log_2 \frac{p_{xy}}{p_x p_y} = D_2(p_{XY} \| p_X p_Y). \quad (6)$$

Zero terms have their usual limiting interpretation. This is mutual information for the induced law, rather than a maximization over input laws defining channel capacity. In particular, the alphabet size is not the cost. The distinction matters for a general instrument, whose outcome law need not be uniform.

For a reference $R \simeq A$, let

$$|\Phi\rangle_{RA} = \frac{1}{\sqrt{d}} \sum_{j=0}^{d-1} |j\rangle_R |j\rangle_A, \quad \Phi_{RA} = |\Phi\rangle\langle\Phi|, \quad (7)$$

and define entanglement fidelity with respect to the maximally mixed input by

$$F_e(\mathcal{T}) = \text{Tr} [\Phi_{RB}(\text{id}_R \otimes \mathcal{T})(\Phi_{RA})]. \quad (8)$$

Here the output basis identifies B with A . For a CPTP channel in dimension d , the average and entanglement fidelities obey [2]

$$F_{\text{av}} = \frac{dF_e + 1}{d + 1}. \quad (9)$$

Thus non-classical average fidelity, meaning $F_{\text{av}} > 2/(d + 1)$, is equivalent to $F_e > 1/d$.

3 Main theorem

For $p, q \in [0, 1]$ with $0 < q < 1$, write

$$D_2(\text{Bern}(p) \parallel \text{Bern}(q)) = p \log_2 \frac{p}{q} + (1 - p) \log_2 \frac{1 - p}{1 - q}. \quad (10)$$

Theorem 1 (One-way instrument converse and matching infimum). *For every deterministic finite-alphabet one-way protocol of Section 2,*

$$I(X; Y) \geq D_2(\text{Bern}(F_e(\mathcal{T})) \parallel \text{Bern}(1/d^2)). \quad (11)$$

Consequently,

$$F_{\text{av}}(\mathcal{T}) > \frac{2}{d + 1} \implies I(X; Y) > C_{\min}(d). \quad (12)$$

Over this class, allowing the shared state, instrument, classical channel, and recovery to vary,

$$\inf \left\{ I(X; Y) : F_{\text{av}}(\mathcal{T}) > \frac{2}{d + 1} \right\} = C_{\min}(d). \quad (13)$$

The infimum is not attained under the strict fidelity requirement. Shared classical randomness is subject to the fixed-channel qualification in Section 6.

4 Proof of the converse

4.1 No-signalling reduction

Set $\sigma_{B_0} = \text{Tr}_{A'} \omega_{A'B_0}$ and $\mathcal{A} = \sum_x \mathcal{A}_x$. Since $\mathcal{A}$ is trace preserving, its adjoint obeys $\mathcal{A}^*(I) = I$. For every observable M on B_0 ,

$$\begin{aligned} \text{Tr} \left[M \sum_x \mathcal{E}_x(\rho) \right] &= \text{Tr} [(I_{A_{\text{out}}} \otimes M)(\mathcal{A} \otimes \text{id}_{B_0})(\rho \otimes \omega)] \\ &= \text{Tr} [(\mathcal{A}^*(I_{A_{\text{out}}}) \otimes M)(\rho \otimes \omega)] \\ &= \text{Tr}(\rho) \text{Tr}(M \sigma_{B_0}). \end{aligned} \quad (14)$$

It follows that

$$\boxed{\sum_x \mathcal{E}_x(\rho) = \text{Tr}(\rho) \sigma_{B_0}.} \quad (15)$$

This also verifies that (3) is trace preserving. Equation (15) uses the completeness of Alice's instrument and no special measurement or resource structure.

4.2 Normalized Choi branches and a bounded score

Define the normalized Choi operators

$$J_x = (\text{id}_R \otimes \mathcal{E}_x)(\Phi_{RA}). \quad (16)$$

Their traces are

$$\text{Tr } J_x = \text{Tr } \mathcal{E}_x(I_A/d) = p_x. \quad (17)$$

If $p_x = 0$, positivity implies $J_x = 0$, so such branches may be omitted. For every remaining branch, $\tau_x = J_x/p_x$ is a state. Applying the normalized Choi map to (15) gives

$$\sum_x p_x \tau_x = \frac{I_R}{d} \otimes \sigma_{B_0}. \quad (18)$$

This identity is the resource-dependent quantum input to the rest of the converse.

For each pair of classical labels define the score

$$f_{xy} = \text{Tr} [\Phi_{RB}(\text{id}_R \otimes \mathcal{R}_y)(\tau_x)]. \quad (19)$$

The recovered operator is a state, hence $0 \leq f_{xy} \leq 1$. Expanding (8) using (3) yields

$$F_e(\mathcal{T}) = \sum_{x,y} p_x q(y|x) f_{xy} = \mathbb{E}_{p_{XY}} f_{XY}. \quad (20)$$

Under the independent law with the same two marginals, (18) instead implies

$$\begin{aligned} \mathbb{E}_{p_X p_Y} f_{XY} &= \sum_y p_y \text{Tr} \left[\Phi_{RB} \left(\frac{I_R}{d} \otimes \mathcal{R}_y(\sigma_{B_0}) \right) \right] \\ &= \sum_y p_y \frac{1}{d^2} = \frac{1}{d^2}. \end{aligned} \quad (21)$$

In the second line, $\text{Tr}[\Phi(I_R/d \otimes \eta_B)] = 1/d^2$ for any unit-trace η_B , and every $\mathcal{R}_y$ is trace preserving. This is why the baseline is universal, even for nonuniform outcome laws and nonunitary recoveries.

4.3 Binary relative entropy and the strict threshold

Apply to each pair (x, y) the same stochastic map which produces a mathematical bit Z with

$$\Pr(Z = 1 \mid x, y) = f_{xy}, \quad \Pr(Z = 0 \mid x, y) = 1 - f_{xy}. \quad (22)$$

This bit is an auxiliary test in the proof, not an additional message in the physical protocol. Equations (20) and (21) show that its laws under p_{XY} and $p_X p_Y$ are respectively $\text{Bern}(F_e)$ and $\text{Bern}(1/d^2)$. Classical relative-entropy data processing therefore gives

$$D_2(p_{XY} \parallel p_X p_Y) \geq D_2(\text{Bern}(F_e) \parallel \text{Bern}(1/d^2)), \quad (23)$$

which proves (11). In particular, the contraction applies to a stochastic score test; no binary-valued assumption on f_{xy} is needed.

For fixed $q_0 \in (0, 1)$ and $0 < p < 1$,

$$\frac{\partial}{\partial p} D_2(\text{Bern}(p) \parallel \text{Bern}(q_0)) = \log_2 \frac{p(1 - q_0)}{q_0(1 - p)}. \quad (24)$$

The derivative is positive for $p > q_0$; monotonicity extends to $p = 1$ by continuity. Since $d \geq 2$ gives $1/d > 1/d^2$, the condition $F_e > 1/d$ implies

$$\begin{aligned} I(X; Y) &> D_2(\text{Bern}(1/d) \| \text{Bern}(1/d^2)) \\ &= \frac{1}{d} \log_2 d + \left(1 - \frac{1}{d}\right) \log_2 \frac{d}{d+1} \\ &= \log_2 d - \left(1 - \frac{1}{d}\right) \log_2(d+1). \end{aligned} \quad (25)$$

Together with (9), this proves (12), including its strictness.

5 Matching limiting achievability

Use a maximally entangled resource and standard d^2 -outcome Bell/Weyl teleportation [1, 4]. For labels $x = (u, v) \in \{0, \dots, d-1\}^2$, take the Weyl operators

$$W_{u,v}|j\rangle = e^{2\pi i v j/d} |j+u \pmod{d}\rangle.$$

They form a unitary error basis with

$$\text{Tr}(W_y^\dagger W_x) = d \delta_{xy}. \quad (26)$$

Choose the Bell-label convention so that outcome x produces the pre-correction error W_x , and Bob's correction for label y is $W_y^\dagger$. Alice's outcome is uniform, $p_x = 1/d^2$. The normalized Choi branch is the maximally entangled state with error W_x on Bob's system. Since $\langle \Phi | (I \otimes U) | \Phi \rangle = \text{Tr}(U)/d$, (26) gives

$$f_{xy} = \frac{|\text{Tr}(W_y^\dagger W_x)|^2}{d^2} = \delta_{xy}. \quad (27)$$

Transmit x through the symmetric classical channel

$$q_a(y|x) = \begin{cases} a, & y = x, \\ (1-a)/(d^2-1), & y \neq x, \end{cases} \quad 0 \leq a \leq 1. \quad (28)$$

Its output is also uniform. Equations (20), (9), and (27) yield

$$F_e = a, \quad F_{\text{av}} = \frac{da+1}{d+1}. \quad (29)$$

Let $h_2(a) = -a \log_2 a - (1-a) \log_2(1-a)$. The conditional entropy is $H(Y|X) = h_2(a) + (1-a) \log_2(d^2-1)$, so

$$\begin{aligned} I_a &= 2 \log_2 d - h_2(a) - (1-a) \log_2(d^2-1) \\ &= D_2(\text{Bern}(a) \| \text{Bern}(1/d^2)). \end{aligned} \quad (30)$$

For every $a > 1/d$, the fidelity is strictly above the classical benchmark and $I_a > C_{\min}(d)$. At $a = 1/d$, direct substitution gives $I_a = C_{\min}(d)$, but the fidelity is exactly $2/(d+1)$. By continuity,

$$\lim_{a \downarrow 1/d} I_a = C_{\min}(d). \quad (31)$$

This proves (13). The converse excludes attainment of that value at strict non-classical fidelity, completing the proof of Theorem 1. The construction is an optimization over allowed resources and channels; it does not claim achievability for every fixed resource or fixed classical channel.

6 Shared classical randomness

Use S for pre-shared classical randomness, independent of the unknown input, and keep it distinct from the quantum reference R . The parties may choose their local maps using S . Conditional on $S = s$, the previous argument gives

$$I(X; Y | S = s) \geq D_2(\text{Bern}(F_e^{(s)}) \| \text{Bern}(1/d^2)). \quad (32)$$

Averaging over s and using convexity in the first Bernoulli parameter yields

$$\begin{aligned} I(X; Y | S) &\geq \sum_s p_s D_2(\text{Bern}(F_e^{(s)}) \| \text{Bern}(1/d^2)) \\ &\geq D_2\left(\text{Bern}\left(\sum_s p_s F_e^{(s)}\right) \parallel \text{Bern}(1/d^2)\right). \end{aligned} \quad (33)$$

The overall entanglement fidelity is $F_e = \sum_s p_s F_e^{(s)}$. Individual conditional protocols need not each beat the classical benchmark.

For the fixed physical classical channel, $p(y|x, s) = q(y|x)$, so $S - X - Y$ is a Markov chain and $I(S; Y|X) = 0$. The mutual-information chain rule then gives

$$I(X; Y) = I(X; Y | S) + I(S; Y) \geq I(X; Y | S). \quad (34)$$

Equations (33) and (34) establish the same converse for the ordinary cost $I(X; Y)$. If side information changes the model so that this Markov condition fails, the final comparison is not established. When the accounting convention is instead $I(X; Y|S)$, equation (33) is the relevant bound.

7 Special cases and limitations

For qubits the exact boundary is

$$C_{\min}(2) = 1 - \frac{1}{2} \log_2 3 = 0.207518749639 \dots \text{ bits}. \quad (35)$$

At the boundary, the symmetric four-symbol channel has correct-label probability $1/2$ and three wrong-label probabilities $1/6$. This is the previously known noisy-channel special case [3]; the boundary channel itself reaches only $F_{\text{av}} = 2/3$. The distinction in the converse is its applicability to arbitrary Alice instruments and arbitrary CPTP Bob recoveries under the stated one-way hypotheses. In dimension four,

$$C_{\min}(4) = 2 - \frac{3}{4} \log_2 5 = 0.258553928834 \dots \text{ bits}, \quad (36)$$

consistent with the rounded 0.259-bit value reported in the standard-protocol analysis [4].

The result has substantive limits. Bob-to-Alice feedback and arbitrary interactive protocols are outside the instrument-fixed-channel-recovery representation used here. Postselected conditional fidelity is also outside the theorem: the no-signalling sum and the CPTP fidelity relation apply to the complete deterministic process. The information law is the Haar-induced law in (5); a non-Haar input ensemble or a different information accounting convention requires separate analysis. Uncharged side information that invalidates (34) is not covered by the unconditional shared-randomness comparison. Both classical alphabets are finite; no continuous-alphabet measure-theoretic extension is proved.

Finally, a universal necessary bound does not determine the exact resource-dependent infimum. In particular, this proof does not solve the minimization for a fixed general two-qubit state specified by correlation parameters t_{ii} , as considered in [5]. Matching achievability

here uses a maximally entangled resource that is free to be chosen. These limitations prevent interpreting the theorem as a complete resolution of the broader unrestricted teleportation communication problem.

Source and verification

The scientific source is the repository proof at commit `51200b595d910d073999971b1c254b5a249877e2`, attributed to Xiansheng Cai [6]. The supplied `SOLUTION.md` contains the derivation, while `VALIDATION.md` and `artifacts/INDEPENDENT_AUDIT.md` record checks of the no-signalling reduction, Choi normalization, product-law baseline, relative-entropy contraction, strict threshold, symmetric-channel achievability, and shared-randomness condition. The audit validates the theorem within its stated class and explicitly declines to treat it as a solution of the unrestricted problem. The result and the numerical specializations above are analytic; no additional experiments or computational claims are introduced.

References

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- [6] Xiansheng Cai, *A one-way no-signalling-instrument converse for teleportation*, scientific source repository, proof commit `51200b595d910d073999971b1c254b5a249877e2`. Supplied repository snapshot: `SOLUTION.md`, `VALIDATION.md`, `artifacts/INDEPENDENT_AUDIT.md`, and `docs/problem-context.md`.