

# NON-MAXIMAL PERSISTENCE IN VIETORIS–RIPS COMPLEXES OF TORUS GRIDS

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**ABSTRACT.** Adams, Adetowubo, Barriga-Acosta, Feng, and Sterling asked whether every inclusion-induced map in third homology along the Vietoris–Rips filtration of a square torus grid has the largest rank allowed by the Betti numbers of its source and target. We answer the question in the negative over  $\mathbb{F}_2$  and describe a systematic mechanism behind the failure. For  $3 \leq k \leq 7$ , the inclusion

$$\mathrm{VR}(T_{3k-1,3k-1}; k) \longrightarrow \mathrm{VR}(T_{3k-1,3k-1}; k+1)$$

induces the zero map on  $H_3$ , although the source and target have third Betti numbers  $6k-2$  and 1, respectively. The zero-map assertion is conceptual: the source is assembled from  $6k-2$  horizontal and vertical circle-grid Rips complexes, and every resulting third-homology generator dies after the scale increases by one.

In the complementary range  $k \geq 8$ , we prove that the same diagonal inclusion has rank exactly two over every field. There the cycle-section maps are homotopy equivalences; adjacent parallel sections become contiguous after the scale increase, while the two coordinate projections detect independent horizontal and vertical survivors. Published computations of Adams et al. give target third Betti number three over  $\mathbb{F}_2$  for  $k = 8, 9$ , so these two maps are also non-maximal; those two computations are cited as published and are not independently rerun here. Thus the family gives seven consecutive counterexamples over  $\mathbb{F}_2$ . A separate signed-chain construction gives a third independent rational target class for every  $k \geq 8$ , so the entire complementary diagonal family is non-maximal over  $\mathbb{Q}$ . For  $k \geq 11$ , a signed projection further proves that the target has integral third-homology rank at least three and that the map is non-maximal over every field of characteristic different from two and three. Reproducible finite-field Ripser computations determine the first five target dimensions and critical barcode portions. We also prove a general boundary-rank formula for inclusion maps, provide a dependency-free exact certificate for the smallest example, and verify that  $T_{8,8}$  at scales  $3 \rightarrow 4$  is the smallest counterexample among square torus grids through side length eight. For each of those five examples the degree-three barcode contains

$$(6k-2)[k, k+1] \sqcup [k+1, d_k], \quad d_k > k+1,$$

so the one-dimensional target class is newly born rather than inherited from the source.

## 1. INTRODUCTION

For a finite metric space  $X$  and a scale  $r$ , the Vietoris–Rips complex  $\mathrm{VR}(X; r)$  contains a simplex for each finite subset of diameter at most  $r$ . The inclusions obtained by increasing  $r$  form a filtration, and persistent homology records not only the dimensions of the homology groups at each scale but also the maps between them. This distinction is essential: dimensions at two scales do not determine the rank of the intervening map.

We study the square torus grid

$$T_{n,n} = C_n \square C_n = (\mathbb{Z}/n\mathbb{Z})^2,$$

where  $\square$  denotes the Cartesian product of graphs. Thus  $T_{n,n}$  is the square torus grid with vertex set  $(\mathbb{Z}/n\mathbb{Z})^2$ , with its graph metric, equivalently the periodic  $\ell^1$  metric. Adams et al. [1] determined

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several infinite families of homotopy types and tabulated homology through a wide range. Their Question 8.9 asks whether every map

$$H_3(\mathrm{VR}(T_{n,n}; s)) \longrightarrow H_3(\mathrm{VR}(T_{n,n}; t)), \quad s \leq t,$$

has rank equal to the minimum of the two third Betti numbers. Their computations use  $\mathbb{Z}/2$  coefficients, and that is the interpretation addressed here.

For  $k \geq 3$  write

$$K_k = \mathrm{VR}(T_{3k-1, 3k-1}; k), \quad L_k = \mathrm{VR}(T_{3k-1, 3k-1}; k+1).$$

Our result shows that the proposed maximal-rank property fails in seven consecutive members of a natural diagonal family over  $\mathbb{F}_2$  and in the entire post-wall diagonal family over  $\mathbb{Q}$ . More importantly, the proof identifies a sharp transition in the map rank. At the first critical scale  $k$  for the grid of side length  $3k-1$ , all source  $H_3$  is generated by horizontal and vertical copies of  $\mathrm{VR}(C_{3k-1}; k) \simeq S^3$ . The inclusion into scale  $k+1$  kills each of these copies for  $k \leq 7$ , by the classification of Rips complexes of cycles. Reproducible computation then shows that the target still has one third-homology class, so that class must be born at the same transition at which all old classes die. For every  $k \geq 8$ , however, the cycle sections remain in the same winding chamber. Their maps are homotopy equivalences, and a contiguity argument together with coordinate retractions shows that the ambient map has rank exactly two over every field. Published  $\mathbb{F}_2$  computations in Tables 1–2 of Adams et al. [1] give target third Betti number three for  $k = 8, 9$ , making those maps non-maximal as well. We do not independently rerun these two larger computations. A uniform signed-chain argument proves  $\beta_3(L_k; \mathbb{Q}) \geq 3$  for every  $k \geq 8$ , without determining the exact target dimension. Its signed integral projection also gives  $\mathrm{rank} H_3(L_k; \mathbb{Z}) \geq 3$  and non-maximality over fields of characteristic different from two and three when  $k \geq 11$ . For  $k \geq 10$ , Proposition 9.1 further reduces the target calculation to two explicit correction groups in the sum-metric comparison sequence.

**Theorem 1.1** (Pre-wall counterexamples). *Let  $3 \leq k \leq 7$  and use coefficients in  $\mathbb{F}_2$ . Then*

$$\beta_3(K_k) = 6k - 2, \quad \beta_3(L_k) = 1,$$

*while the inclusion-induced map  $H_3(K_k) \rightarrow H_3(L_k)$  is zero.*

The theorem produces the parameters

$$(n, k) = (8, 3), (11, 4), (14, 5), (17, 6), (20, 7).$$

The smallest case has source dimension 16, target dimension 1, and map rank 0. Consequently it violates the predicted rank 1.

**Theorem 1.2** (Post-wall rank theorem). *For every field  $\mathbb{F}$  and every integer  $k \geq 8$ , the inclusion  $K_k \hookrightarrow L_k$  has rank exactly two on  $H_3(-; \mathbb{F})$ . Moreover,  $H_3(L_k; \mathbb{Z})$  contains a split direct summand isomorphic to  $\mathbb{Z}^2$ .*

**Theorem 1.3** (Secondary-product classes). *For every integer  $k \geq 8$ ,*

$$\beta_3(L_k; \mathbb{Q}) \geq 3.$$

*Consequently the rank-two map  $H_3(K_k; \mathbb{Q}) \rightarrow H_3(L_k; \mathbb{Q})$  is not of maximal rank. For every  $k \geq 11$  one also has*

$$\mathrm{rank} H_3(L_k; \mathbb{Z}) \geq 3.$$

*Moreover, if  $\mathbb{F}$  is a field of characteristic different from two and three, then  $\beta_3(L_k; \mathbb{F}) \geq 3$ , and the rank-two map  $H_3(K_k; \mathbb{F}) \rightarrow H_3(L_k; \mathbb{F})$  is not of maximal rank.*

At  $k = 8, 9$ , the published target dimensions cited above imply that the rank-two maps of Theorem 1.2 are also non-maximal over  $\mathbb{F}_2$ .

The paper is organized as follows. Section 2 fixes conventions. Section 3 recalls the structure theorem at the source scale and makes the generating property of the circle sections explicit. Section 4 proves the zero-map, post-wall rank, and secondary-product theorems. Section 5 gives a general computable formula for an inclusion-induced image. Section 6 verifies the target dimensions in Theorem 1.1 and records the exact smallest-case certificate. Section 7 explains the two independent computational routes. Section 8 gives the minimality audit. Section 9 separates the established map classification from the remaining target-homology problem and gives a correction-theoretic reduction of that problem.

## 2. TORUS GRIDS AND CONVENTIONS

Write vertices as pairs  $(a, b)$  with  $0 \leq a, b < n$ . Define

$$\delta_n(a, c) = \min\{|a - c|, n - |a - c|\}, \quad d_n((a, b), (c, d)) = \delta_n(a, c) + \delta_n(b, d).$$

The complex  $\text{VR}(T_{n,n}; r)$  is the clique complex of the graph joining distinct vertices at distance at most  $r$ . We use closed Rips complexes: equality with the scale is allowed. For  $k \geq 3$  the complexes  $K_k$  and  $L_k$  are as in the introduction.

For a complex  $M$  and field  $\mathbb{F}$ , let  $C_q(M)$ ,  $Z_q(M)$ , and  $B_q(M)$  be its simplicial chains, cycles, and boundaries. Let  $c_q(M) = \dim C_q(M)$  and  $\partial_q^M : C_q(M) \rightarrow C_{q-1}(M)$ . If  $K \subseteq L$ , the ordered simplex bases identify  $C_q(K)$  with a coordinate subspace of  $C_q(L)$ .

**Definition 2.1.** For  $a \in \mathbb{Z}/(3k-1)$ , the vertical section  $V_a(k)$  is the full subcomplex of  $K_k$  on vertices with first coordinate  $a$ . For  $b \in \mathbb{Z}/(3k-1)$ , the horizontal section  $H_b(k)$  is the full subcomplex on vertices with second coordinate  $b$ .

Each section is naturally isomorphic to  $\text{VR}(C_{3k-1}; k)$ . We use the same notation for the corresponding full subcomplexes at scale  $k+1$ .

## 3. THE CRITICAL SOURCE AND ITS GENERATORS

The source homotopy type is Theorem 5.12 of Adams et al. [1] and is the starting point of our map-level analysis.

**Theorem 3.1** (Adams et al.). *For every  $k \geq 3$ ,*

$$K_k \simeq \bigvee^{6k-3} S^2 \vee \bigvee^{6k-2} S^3.$$

*In particular  $\beta_3(K_k; \mathbb{F}) = 6k - 2$  over every field.*

Their proof decomposes  $K_k$  as a core homotopy equivalent to  $S^2$  together with the  $3k-1$  horizontal and  $3k-1$  vertical circle sections. The pairwise intersections of distinct sections contain at most one vertex. One distinguished horizontal section and one distinguished vertical section meet the core contractibly; each remaining section meets the preceding union in a circle. Repeated use of the standard null-homotopic union lemma

$$A \cup B \simeq A \vee B \vee \Sigma(A \cap B)$$

produces one  $S^3$  from every section and one additional  $S^2$  from every nondistinguished section.

The following map-level refinement is implicit in that construction and will be crucial.

**Proposition 3.2** (Section generation). *For every  $k \geq 3$  and every field  $\mathbb{F}$ , the homomorphism*

$$\bigoplus_a H_3(V_a(k); \mathbb{F}) \oplus \bigoplus_b H_3(H_b(k); \mathbb{F}) \longrightarrow H_3(K_k; \mathbb{F})$$

*induced by the section inclusions is an isomorphism.*

*Proof.* By the cycle classification recalled in Section 4, every section is homotopy equivalent to  $S^3$ . Order the sections as in the proof of Theorem 3.1, and write  $X_j = X_{j-1} \cup S_j$  for the partial union after the  $j$ th section is attached. The cited decomposition proves that  $X_{j-1} \cap S_j$  is either contractible or homotopy equivalent to  $S^1$ . Consequently its homology vanishes in degrees two and three. The Mayer–Vietoris sequence therefore contains a natural short exact segment

$$0 \longrightarrow H_3(X_{j-1}) \oplus H_3(S_j) \longrightarrow H_3(X_j) \longrightarrow 0.$$

Thus the actual inclusion of  $S_j$ , not merely a chosen wedge equivalence, contributes an independent generator. Starting from the core, whose third homology vanishes, induction shows that the fundamental classes of all sections form a basis. There are  $2(3k-1) = 6k-2$  such classes, agreeing with Theorem 3.1.  $\square$

*Remark 3.3.* The proposition asserts more than the numerical homotopy type: it identifies concrete geometric carriers for every degree-three generator. This is what allows the inclusion map, rather than just its source and target dimensions, to be controlled.

#### 4. THE DIAGONAL MAP ACROSS THE $2/5$ WALL

**4.1. Death of the cycle sections.** We now use the homotopy classification of clique complexes of cyclic graph powers [2, Corollary 6.7]. For integers  $m, r$  in the relevant range, the homotopy type changes when  $r/m$  crosses the fractions  $\ell/(2\ell+1)$ . In particular,

$$\text{VR}(C_m; r) \simeq S^3 \quad \text{if} \quad \frac{1}{3} < \frac{r}{m} < \frac{2}{5},$$

whereas at and beyond  $2/5$  the reduced homology is concentrated in dimensions at least four until the next odd-sphere chamber.

For  $m = 3k-1$ , one has  $k/(3k-1) > 1/3$  and  $k/(3k-1) < 2/5$  for every  $k \geq 3$ . Thus the sections at scale  $k$  are 3-spheres. At the next scale,

$$\frac{k+1}{3k-1} \geq \frac{2}{5} \quad \Longleftrightarrow \quad k \leq 7.$$

This elementary inequality explains exactly the range in Theorem 1.1. At the endpoints,  $k=3$  gives the complete graph  $C_8^4$ , while  $k=7$  lands exactly on the  $2/5$  wall; in both cases the target section has vanishing third homology.

**Lemma 4.1** (Cycle-section death). *For  $3 \leq k \leq 7$ , the inclusion*

$$\text{VR}(C_{3k-1}; k) \hookrightarrow \text{VR}(C_{3k-1}; k+1)$$

*induces the zero map on  $H_3$  over every field.*

*Proof.* The source is homotopy equivalent to  $S^3$ . The target has vanishing  $H_3$  by the cyclic-graph Rips classification and the inequality above. The induced map therefore has zero target and is zero.  $\square$

**Theorem 4.2** (Conceptual zero-map theorem). *For every field  $\mathbb{F}$  and  $3 \leq k \leq 7$ , the inclusion  $K_k \hookrightarrow L_k$  induces the zero map on  $H_3$ .*

*Proof.* By Proposition 3.2,  $H_3(K_k)$  is generated by the fundamental classes carried by the horizontal and vertical sections. The image of each such generator factors through the corresponding inclusion of cycle Rips complexes at scales  $k \rightarrow k+1$ . Every one of these maps is zero by Lemma 4.1. Hence the map is zero on a generating set and therefore zero.  $\square$

**4.2. Post-wall rank two.** For  $k \geq 8$ , the section ratio remains strictly below  $2/5$ . We now use the map-level cyclic-graph theory of Adamaszek and Adams to determine the ambient map in this complementary range.

*Proof of Theorem 1.2.* Set  $n = 3k - 1$ . For  $k \geq 8$  one has

$$\frac{1}{3} < \frac{k}{n} \leq \frac{k+1}{n} < \frac{2}{5}.$$

The last strict inequality is equivalent to  $k > 7$ . The identity on the vertices is a cyclic homomorphism from the directed cycle power  $\vec{C}_n^k$  to  $\vec{C}_n^{k+1}$ , and their winding fractions are  $k/n$  and  $(k+1)/n$ . Proposition 4.9 of Adamaszek and Adams [3] therefore shows that the inclusion

$$\text{VR}(C_n; k) \hookrightarrow \text{VR}(C_n; k+1)$$

is a homotopy equivalence.

Choose one vertical and one horizontal section of  $L_k$ . Coordinate projection defines simplicial maps

$$p_1, p_2 : L_k \longrightarrow \text{VR}(C_n; k+1),$$

because each coordinate distance is at most the torus  $\ell^1$  distance. On the vertical section,  $p_2$  is the identity and  $p_1$  is constant; on the horizontal section the roles are reversed. Hence the sum of the two section inclusions is split injective on integral third homology. This proves the split- $\mathbb{Z}^2$  assertion and, after passing to  $\mathbb{F}$ , shows that the two corresponding images from  $K_k$  are independent. The ambient map therefore has rank at least two.

For the reverse inequality, identify every vertical source section with  $A = \text{VR}(C_n; k)$  using its second coordinate. Let  $f_a : A \rightarrow L_k$  be the embedding at first coordinate  $a$ . The maps  $f_a$  and  $f_{a+1}$  are contiguous: if  $u, v$  belong to one simplex of  $A$ , then

$$d_n((a, u), (a+1, v)) = 1 + \delta_n(u, v) \leq k+1,$$

and pairs within either copy have distance at most  $k$ . Thus the union of the two images of every simplex is a simplex of  $L_k$ . This includes the cyclic wraparound from  $a = n-1$  to  $a = 0$ . Contiguous maps induce the same map on homology, so all vertical section generators have the same ambient image. By symmetry, all horizontal section generators also have one common image. Proposition 3.2 says that these section classes span  $H_3(K_k; \mathbb{F})$ ; hence the ambient image is spanned by at most one vertical and one horizontal class. Its rank is at most two, completing the proof.  $\square$

**Corollary 4.3** (Two published-computation consequences). *Over  $\mathbb{F}_2$ , the diagonal inclusions at  $k = 8$  and  $k = 9$  are not of maximal rank. More precisely, their source dimensions are 46 and 52, their target dimensions are both 3, and both maps have rank 2.*

*Proof.* Theorem 1.2 gives rank two. Tables 1–2 of Adams et al. [1], computed over  $\mathbb{Z}/2$ , record

$$\beta_3(\text{VR}(T_{23,23}; 9); \mathbb{F}_2) = 3, \quad \beta_3(\text{VR}(T_{26,26}; 10); \mathbb{F}_2) = 3.$$

These are  $L_8$  and  $L_9$ , respectively, while Theorem 3.1 gives source dimensions  $6k - 2 = 46, 52$ . Thus the maximal possible rank is three in each case, whereas the actual rank is two. The two target computations are used as published evidence and were not independently rerun in this work.  $\square$

#### 4.3. A secondary product class.

*Proof of Theorem 1.3.* Put  $s = k$ ,  $n = 3s - 1$ , and  $r = s + 1$ . Let

$$z = \sum_{i=0}^{n-2} [i, i+1] - [0, n-1]$$

be the oriented fundamental one-cycle of  $C_n$ . The following integral two-chain lies in  $\text{VR}(C_n; s)$ :

$$\begin{aligned} b_s = & 2 \sum_{i=1}^{s-2} [0, i, i+1] + [0, s-1, s] - [0, 1, s] \\ & - \sum_{i=1}^{s-2} [i, i+1, s+i] + \sum_{i=1}^{s-2} [i, s+i-1, s+i] \\ & + [s-1, 2s-2, 2s-1] + [0, s-1, 2s-1] + \sum_{j=2s-1}^{3s-3} [0, j, j+1]. \end{aligned}$$

Every displayed triangle has cyclic diameter at most  $s$ . Expanding the oriented boundaries makes the two ladder sums telescope, cancels all radial edges, and leaves precisely the consecutive edges of  $z$ ; hence  $\partial b_s = z$ .

In the product-filtered ordered-tuple model of Carlsson and Filippenko [6], the normalized Eilenberg–Zilber chain

$$t_s = \text{EZ}(b_s \otimes z) + \text{EZ}(z \otimes b_s)$$

has product filtration at most  $s+1 = r$ . The signed cross-product formula gives  $\partial t_s = 0$ .

The cyclic-group maps

$$m(x, y) = x + y, \quad p_1(x, y) = x, \quad p_2(x, y) = y$$

are distance non-increasing from the sum-metric torus to  $C_n$ . Hence they induce maps from  $L_k$  to  $\text{VR}(C_n; r)$ . Set  $\lambda = m_* - p_{1,*} - p_{2,*}$ . This map vanishes on the horizontal and vertical section classes: on either section, addition agrees with the nonconstant projection, while the other projection is constant.

Let  $w_s = m_{\#} \text{EZ}(b_s \otimes z)$ . Factor transposition in bidegree  $(2, 1)$  has sign  $(-1)^{2 \cdot 1} = 1$ , and  $m$  is symmetric, so the two addition images of the Eilenberg–Zilber summands coincide. Each coordinate projection sends either summand to a degenerate chain: a step in the collapsed factor repeats a vertex after projection. Writing  $i$  for the natural map from the product filtration to the true sum-metric filtration, one therefore has

$$\lambda(i_*[t_s]) = 2[w_s].$$

For  $s = 8, 9, 10$ , the ancillary exact checker reduces  $w_s$  modulo two and eliminates it against every target four-simplex boundary. The residual is nonzero in all three cases, so  $[w_s]$  is nonzero integrally.

For every  $s \geq 8$ ,

$$\frac{1}{3} < \frac{r}{n} < \frac{2}{5},$$

so the cycle classification of Adamaszek and Adams [3] gives  $\text{VR}(C_n; r) \simeq S^3$ . Its integral third homology is torsion-free. Thus  $2[w_s] \neq 0$  over  $\mathbb{Q}$  in the three finite cases.

Suppose now that  $s \geq 11$ . Define

$$f_s: C_n \longrightarrow C_8, \quad f_s(x) = \left\lfloor \frac{8x}{n} \right\rfloor.$$

The only nonedges of  $C_8^3$  are the four antipodal pairs. Preimages of antipodal vertices under  $f_s$  have cyclic distance at least

$$\left\lfloor \frac{3n}{8} \right\rfloor + 1 = s + \left\lfloor \frac{s-3}{8} \right\rfloor + 1 > s + 1.$$

Thus  $f_s$  induces a simplicial map  $\text{VR}(C_n; s+1) \rightarrow \text{VR}(C_8; 3)$ . The latter clique complex is the boundary of the four-dimensional crosspolytope. Orient its fundamental cycle  $\omega$  so that the coefficient of the facet  $\{0, 1, 2, 3\}$  is  $+1$ .

The coefficient-one part of  $b_s$ , reduced modulo two, is the sum of the triangle families

$$\begin{aligned} A_i &= [i, i+1, i+s] & (0 \leq i \leq s-2), \\ B_i &= [i, i+s-1, i+s] & (0 \leq i \leq s-1), \\ C &= [0, s-1, 2s-1], & D_j = [0, j, j+1] \quad (2s-1 \leq j \leq 3s-3). \end{aligned}$$

Write  $c_a = \lceil an/8 \rceil$ , so the  $a$ th projection block is  $[c_a, c_{a+1} - 1]$ . Expanding the three  $(2, 1)$  shuffles shows that this part has exactly one raw contribution to the crosspolytope facet  $\{0, 1, 2, 3\}$ . It comes from  $D_j$ , the ordinary cycle edge  $[q, q+1]$ , and the shuffle that takes the edge step first, with

$$q = c_3 - 1, \quad j = n + c_1 - c_3 - 1.$$

Indeed the four projected blocks are then  $(2, 3, 0, 1)$ , and  $2s-1 \leq j \leq 3s-3$ . Here is a direct exclusion of the other cells. Every projection block has at least four vertices, so a shuffle containing three consecutive vertices cannot meet four distinct blocks. The two remaining  $A$ - and  $B$ -shuffles consist of adjacent pairs separated by  $s$ ; they would require either  $c_3 - c_1 = s$  or  $n + c_1 - c_3 = s$ , whereas  $c_3 - c_1 \leq \lceil n/4 \rceil \leq s-1$ . Each  $C$ -shuffle needs a cyclic covering arc of length  $2s-1$ , longer than the arc occupied by blocks  $0, 1, 2, 3$ . Finally, the standard ordered wrap edge  $[0, n-1]$  either makes a shuffle degenerate, produces adjacent pairs separated by  $s-1$ , or introduces a vertex in block 7. Thus it contributes zero. This proves that the displayed  $D_j$  term is unique. Writing  $s = 8u + v$ ,  $0 \leq v < 8$ , gives the same twelve-cell interval audit residue by residue; the ancillary checker replays those eight cases.

The surviving  $D_j$  shuffle has positive sign. Its ordered block list is  $(2, 3, 0, 1)$ ; both the source-order and block-order sorting permutations are even. It therefore contributes  $+1$  to the signed integral coefficient.

It remains to restore the terms invisible modulo two, namely

$$2E_i = 2[0, i, i+1], \quad 1 \leq i \leq s-2.$$

The first two shuffles contain three consecutive vertices and cannot meet four blocks. In the remaining shuffle, two adjacent pairs must cross the boundaries  $c_1$  and  $c_3$ . There is exactly one admissible choice,

$$q = c_1 - 1, \quad i = c_3 - c_1 - 1,$$

because

$$2 \leq c_3 - c_1 \leq \left\lceil \frac{n}{4} \right\rceil \leq s-1,$$

whereas the reverse order would require  $i+1 = n + c_1 - c_3 > s-1$ . Its ordered source and block lists are

$$(c_1 - 1, c_1, c_3 - 1, c_3), \quad (0, 1, 2, 3).$$

The shuffle and both orientation signs are positive, so the contribution is  $+2$ . The wrap edge produces either a degenerate image or a block-seven vertex and contributes nothing.

Hence the signed coefficient of  $f_{s,\#}(w_s)$  on the distinguished facet is  $2 + 1 = 3$ . This projected chain is a cycle. The integral three-cycles in the crosspolytope boundary form an infinite cyclic group and are determined by the coefficient on one facet. Therefore

$$f_{s,*}[w_s] = 3[\omega].$$

This identity concerns the projected cycle; it is not a statement about a topological degree of the block map itself.

For  $s = 8, 9, 10$ , the two coordinate-section classes lie in  $\ker \lambda$  while  $\lambda(i_*[t_s]) = 2[w_s] \neq 0$  over  $\mathbb{Q}$ . They and the product class are therefore rationally independent. For  $s \geq 11$ , compose  $\lambda$  with  $f_{s,*}$ . This composite kills both section classes and takes the product class to

$$2f_{s,*}[w_s] = 6[\omega].$$

Over the integers this element has infinite order, so the product class and the two integrally independent section classes are independent. This proves  $\text{rank } H_3(L_k; \mathbb{Z}) \geq 3$ . Over a field the

displayed detector value is nonzero whenever the characteristic is neither two nor three, giving the corresponding Betti-number lower bound. It gives no conclusion in characteristic two or three.

Finally  $\beta_3(K_k) = 6k - 2 > 2$ , while the diagonal map has rank two by Theorem 1.2; in every coefficient case asserted above, its rank is therefore strictly below the minimum of the endpoint dimensions.  $\square$

## 5. A BOUNDARY-RANK FORMULA FOR INCLUSIONS

For exact finite verification it is convenient to avoid choosing homology bases.

**Proposition 5.1** (Image-rank formula). *Let  $K \subseteq L$  be finite simplicial complexes,  $q \geq 0$ , and*

$$P : C_q(L) \longrightarrow C_q(L)/C_q(K)$$

*the quotient. Over any field,*

$$\begin{aligned} \text{rank}(H_q(K) \rightarrow H_q(L)) &= c_q(K) - \text{rank } \partial_q^K - \text{rank } \partial_{q+1}^L \\ &\quad + \text{rank}(P\partial_{q+1}^L). \end{aligned}$$

*Proof.* The image is the image of  $Z_q(K)$  in  $Z_q(L)/B_q(L)$ , so its dimension is

$$\dim Z_q(K) - \dim(Z_q(K) \cap B_q(L)).$$

Because a boundary in  $L$  supported on  $q$ -simplices of  $K$  is automatically a cycle in  $K$ , the intersection equals  $C_q(K) \cap B_q(L)$ . The restriction of  $P$  to  $B_q(L)$  has this intersection as kernel. Rank-nullity gives

$$\dim(C_q(K) \cap B_q(L)) = \text{rank } \partial_{q+1}^L - \text{rank}(P\partial_{q+1}^L).$$

Finally  $\dim Z_q(K) = c_q(K) - \text{rank } \partial_q^K$ . Substitution proves the formula.  $\square$

In simplex bases,  $P\partial_{q+1}^L$  is obtained simply by deleting rows indexed by  $q$ -simplices of  $K$ . Thus every term is an ordinary matrix rank.

We record an abstract form of the carrier argument that may be useful beyond torus grids.

**Proposition 5.2** (Carrier-annihilation criterion). *Let  $K \subseteq L$  and suppose subcomplexes  $S_1, \dots, S_m \subseteq K$  have the property that*

$$\bigoplus_{i=1}^m H_q(S_i; \mathbb{F}) \longrightarrow H_q(K; \mathbb{F})$$

*is surjective. If, for every  $i$ , the composite  $H_q(S_i; \mathbb{F}) \rightarrow H_q(L; \mathbb{F})$  is zero, then  $H_q(K; \mathbb{F}) \rightarrow H_q(L; \mathbb{F})$  is zero.*

*Proof.* Every class in  $H_q(K; \mathbb{F})$  is a sum of images of classes carried by the  $S_i$ . Naturality identifies the image of each summand in  $H_q(L; \mathbb{F})$  with the stated composite, which is zero.  $\square$

Theorem 4.2 is Proposition 5.2 with the horizontal and vertical circle sections as carriers. No choice of a global wedge equivalence is required.

**Proposition 5.3** (Interval-rank count). *For a finite persistence module over a field, the map from parameter  $s$  to parameter  $t$  has rank equal to the number of barcode intervals containing both  $s$  and  $t$ , with the half-open endpoint convention.*

*Proof.* The interval decomposition theorem for finite persistence modules [7] decomposes the module as a direct sum of interval modules. On an interval summand the structure map is the identity precisely when both parameters lie in the interval, and is zero otherwise. Ranks add under direct sums.  $\square$

Thus a simultaneous death of all source intervals and birth of a target interval forces image rank zero even when both endpoint Betti numbers are positive.

## 6. FINITE-FIELD VERIFICATION OF THE FIVE TARGETS

The zero-map theorem is independent of computation. To turn it into a counterexample to maximal rank, it remains to prove  $H_3(L_k) \neq 0$ . Two separately implemented finite-field Ripser computations over  $\mathbb{F}_2$  give the same data in Table 1. Ripser performs persistence reduction over the specified prime field [4]. The interval listed as  $[k + 1, > k + 1)$  has a death strictly greater than  $k + 1$ ; its exact later death is immaterial to the inclusion  $K_k \rightarrow L_k$ .

TABLE 1. Reproduced degree-three persistence at the critical transition over  $\mathbb{F}_2$ .

| $k$ | $n = 3k - 1$ | $\beta_3(K_k)$ | $\beta_3(L_k)$ | image rank | critical intervals         |
|-----|--------------|----------------|----------------|------------|----------------------------|
| 3   | 8            | 16             | 1              | 0          | $16[3, 4) \sqcup [4, > 4)$ |
| 4   | 11           | 22             | 1              | 0          | $22[4, 5) \sqcup [5, > 5)$ |
| 5   | 14           | 28             | 1              | 0          | $28[5, 6) \sqcup [6, > 6)$ |
| 6   | 17           | 34             | 1              | 0          | $34[6, 7) \sqcup [7, > 7)$ |
| 7   | 20           | 40             | 1              | 0          | $40[7, 8) \sqcup [8, > 8)$ |

The computation uses the complete distance matrix, whose entries are integers, and a standard persistence column reduction. Since all filtration values are integral and every reported birth and death is checked to be integral, the closed filtration at integer scales is represented without threshold ambiguity despite floating-point storage of the distances. The portable ancillary verifier records the raw degree-three intervals, coefficient field, metric and filtration conventions, truncation threshold, software versions, and distance-matrix hashes. As a separate dependency-free exact certificate for the smallest case, we enumerate all cliques through dimension four and reduce the boundary matrices by binary Gaussian elimination.

For  $K = \text{VR}(T_{8,8}; 3)$  and  $L = \text{VR}(T_{8,8}; 4)$ , the direct ranks are:

| quantity          | $K$  | $L$    |
|-------------------|------|--------|
| vertices          | 64   | 64     |
| edges             | 768  | 1216   |
| 2-simplices       | 3072 | 9984   |
| 3-simplices       | 5632 | 42848  |
| 4-simplices       | 5696 | 107712 |
| rank $\partial_3$ | 2352 | 8831   |
| rank $\partial_4$ | 3264 | 34016  |

Hence  $\beta_3(K) = 5632 - 2352 - 3264 = 16$  and  $\beta_3(L) = 42848 - 8831 - 34016 = 1$ . Moreover

$$\text{rank}(P\partial_4^L) = 30736,$$

so Proposition 5.1 returns

$$5632 - 2352 - 34016 + 30736 = 0.$$

This agrees with the conceptual proof and supplies an exact map-rank check independent of the Ripser computation.

## 7. REPRODUCIBILITY AND FINITE-FIELD ELIMINATION

The dependency-free smallest-case verifier uses only finite enumeration and exact arithmetic over  $\mathbb{F}_2$ . Vertices are ordered lexicographically. A recursive clique search maintains the common-neighbor set and adds labels in increasing order, so every clique is emitted exactly once. For a simplex, the boundary column is the set of codimension-one faces. Over  $\mathbb{F}_2$ , columns are sets of row indices and addition is symmetric difference.

Incremental Gaussian elimination uses the largest nonzero row as pivot. If the pivot is already stored, the current column is added to the stored pivot column; otherwise it becomes a new pivot. Termination is immediate because every reduction lowers the largest nonzero row. Distinct terminal pivots are linearly independent and span the original columns, hence their number is the exact matrix rank.

For the family scan, the distance-matrix persistence calculation and the direct smallest-case certificate are deliberately different computational paths. The first computes a truncated barcode; the second computes the image rank through Proposition 5.1. Agreement therefore checks both the filtration reduction and the quotient-boundary construction. All smallest-case quantities can also be verified by substituting the printed ranks into the displayed formulas.

**Proposition 7.1** (Correctness of the direct certificate). *The direct algorithm returns exact simplex counts and exact boundary ranks over  $\mathbb{F}_2$ .*

*Proof.* The increasing-label recursion emits every clique exactly once, because every clique has a unique increasing ordering and the common-neighbor invariant retains precisely the vertices that can extend it. Each boundary column contains exactly the codimension-one faces of its simplex. Symmetric difference is vector addition over  $\mathbb{F}_2$ . Adding a stored pivot column preserves the column span and strictly lowers the largest pivot until a column vanishes or supplies a new pivot. Terminal columns have distinct pivots and are independent; reversing the reductions expresses every input column in their span. Their number is the exact rank. Deleting the source rows before applying the same routine is precisely the quotient projection in Proposition 5.1.  $\square$

The direct executable writes the simplex counts, five boundary ranks, two Betti numbers, and image rank to a machine-readable record. The portable Ripser verifier separately writes the raw interval lists and their audit metadata. These finite calculations do not support an unbounded theorem; Table 1 is intentionally stated only for the five reproduced runs.

## 8. MINIMALITY OF THE SMALLEST COUNTEREXAMPLE

We now justify the phrase “smallest example” within the square-grid family. Two separately implemented scans compute the degree-three barcode of every  $T_{n,n}$  for  $3 \leq n \leq 8$  and inspect every pair of integer scales from zero through the graph diameter. For  $n = 3, 4, 5, 6, 7$ , no pair of scales with nonzero homology at both ends violates maximal rank. The barcodes before  $n = 8$  are:

| $n$ | degree-three intervals |
|-----|------------------------|
| 3   | $\emptyset$            |
| 4   | $9[2, 3)$              |
| 5   | $\emptyset$            |
| 6   | $[3, 4)$               |
| 7   | $[3, 5)$               |

At  $n = 8$  the barcode contains  $16[3, 4) \sqcup [4, 6)$ , yielding the first violation at  $3 \rightarrow 4$ . Values  $n \leq 3$  are immediate from the complete small grids and have no competing nonzero  $H_3$  endpoints. Thus  $n = 8$  is minimal by side length and vertex count among square torus grids.

**Corollary 8.1.** *Over  $\mathbb{F}_2$ ,  $\text{VR}(T_{8,8}; 3) \hookrightarrow \text{VR}(T_{8,8}; 4)$  is the smallest square-torus-grid counterexample to Question 8.9.*

## 9. CONSEQUENCES AND CORRECTED QUESTIONS

Theorem 1.1 demonstrates a clean replacement phenomenon: all old degree-three classes die at the transition, while a new class is born. The target Betti number alone cannot distinguish this from the continuation of one source class. This illustrates why tables of Betti numbers, however extensive, cannot determine persistent maps.

The proof separates the scale transition of the horizontal and vertical section classes from the birth of genuinely two-dimensional global classes. For the diagonal  $n = 3k - 1$ , the section behavior is now completely determined: all section classes die for  $3 \leq k \leq 7$ , whereas for  $k \geq 8$  the parallel classes coalesce into one surviving vertical and one surviving horizontal class. The 2/5 wall in the Rips theory of cyclic graphs is exactly the transition between these regimes.

Theorem 1.2 determines the map rank for all  $k \geq 8$ . Corollary 4.3 shows from published finite computations that one additional mod-two target class is present at  $k = 8, 9$ . Theorem 1.3 supplies a third independent rational target class for every  $k \geq 8$ , and hence proves rational nonmaximality throughout the post-wall diagonal family. For  $k \geq 11$  it also proves integral third-homology rank at least three and nonmaximality over fields of characteristic different from two and three. For every field and every  $k \geq 8$ , coordinate projection still gives  $\beta_3(L_k; \mathbb{F}) \geq 2$ , so the maximal-rank property on this diagonal holds exactly when  $\beta_3(L_k; \mathbb{F}) = 2$ . Determining the remaining target classes, and possible integral torsion outside the split  $\mathbb{Z}^2$ , is a genuinely two-dimensional problem.

We next make this residual problem more precise. Let  $\mathcal{C}_n$  be the ordered-tuple Rips simplicial set of  $C_n$ , filtered by maximum edge length. For  $r \geq 11$ , put  $n = 3r - 4$  and

$$R_r = \text{VR}(T_{n,n}; r).$$

Let  $P_r$  be the scale- $r$  part of the product filtered simplicial set  $\mathcal{C}_n \times \mathcal{C}_n$ , where the product filtration value is the sum of the two factor filtration values. The maximum length in the true sum metric is at most this sum, giving a natural simplicial map  $P_r \rightarrow R_r$ . The ordered-tuple models have the homology of the corresponding ordinary Rips complexes.

**Proposition 9.1** (Sum-metric correction criterion). *For every field  $\mathbb{F}$  and every  $r \geq 11$ ,*

$$\beta_2(P_r; \mathbb{F}) = 0, \quad \beta_3(P_r; \mathbb{F}) = 3, \quad \beta_4(P_r; \mathbb{F}) = 2.$$

*Let  $\overline{H}_j(r; \mathbb{F})$  denote the degree- $r$  correction group in the natural long exact sequence from the product filtration to the true sum-metric filtration of Carlsson and Filippenko [6], and set*

$$\epsilon_r = \text{rank}(\overline{H}_4(r; \mathbb{F}) \rightarrow H_3(P_r; \mathbb{F})).$$

*Then*

$$\beta_3(R_r; \mathbb{F}) = 3 - \epsilon_r + \dim_{\mathbb{F}} \overline{H}_3(r; \mathbb{F}), \quad \epsilon_r \in \{0, 1\}.$$

*Consequently, for  $k = r - 1$ , the diagonal map  $K_k \rightarrow L_k = R_r$  has maximal rank if and only if*

$$\epsilon_r = 1 \quad \text{and} \quad \overline{H}_3(r; \mathbb{F}) = 0.$$

*Proof.* Writing  $(a, b)$  for a persistence interval supported at integer scales  $a \leq t < b$ , the cycle classification and same-chamber maps give, among the factor bars relevant at scale  $r$ ,

$$\begin{aligned} PH_0(C_n) &= (0, \infty) + (n - 1)(0, 1), \\ PH_1(C_n) &= (1, r - 1), \quad PH_2(C_n) = 0, \\ PH_3(C_n) &= (r - 1, d_r), \quad d_r = \left\lceil \frac{2n}{5} \right\rceil > r. \end{aligned}$$

Indeed,

$$\frac{r - 2}{n} < \frac{1}{3} < \frac{r - 1}{n} < \frac{r}{n} < \frac{2}{5},$$

and  $n$  is not divisible by three. The persistent Künneth theorem and the interval formulas of Carlsson and Filippenko [6, Proposition 2.6, Theorem 1.6] now apply.

In total degree three, the two factor orders of

$$(r - 1, d_r) \otimes (0, \infty) = (r - 1, d_r)$$

contribute two classes, while

$$\text{Tor}_1((1, r - 1), (1, r - 1)) = (r, 2r - 2)$$

contributes one. Terms obtained by tensoring the degree-three bar with a finite degree-zero bar end at  $r$  and are absent at scale  $r$ . Thus  $\beta_3(P_r) = 3$ . Similarly, the only potentially relevant degree-two bars are

$$(1, r-1) \otimes (1, r-1) = (2, r), \quad \text{Tor}_1((1, r-1), (0, 1)) = (r-1, r),$$

so  $\beta_2(P_r) = 0$ . In degree four, the two factor orders of

$$(r-1, d_r) \otimes (1, r-1) = (r, \min\{d_r+1, 2r-2\})$$

are the only bars present at scale  $r$ , giving  $\beta_4(P_r) = 2$ .

The two tensor classes in  $H_3(P_r)$  are the horizontal and vertical coordinate-section classes. Coordinate projections from  $R_r$  distinguish them, so the map  $H_3(P_r) \rightarrow H_3(R_r)$  is injective on their two-dimensional span. The degree- $r$  part of the natural correction sequence [6, Proposition 4.4] is

$$\overline{H}_4(r) \longrightarrow H_3(P_r) \longrightarrow H_3(R_r) \longrightarrow \overline{H}_3(r) \longrightarrow H_2(P_r) = 0.$$

Exactness gives the dimension formula. Since the kernel of  $H_3(P_r) \rightarrow H_3(R_r)$  is disjoint from the two-dimensional section subspace inside a three-dimensional space, its dimension  $\epsilon_r$  is zero or one. Finally, Theorem 1.2 gives rank two for the diagonal map and the section retractions give  $\beta_3(R_r) \geq 2$ . Hence that map is maximal exactly when  $\beta_3(R_r) = 2$ , which is equivalent to the two displayed correction conditions.  $\square$

*Remark 9.2.* Proposition 9.1 alone does not compute either correction group. Theorem 1.3 proves survival of an explicit secondary product class over  $\mathbb{Q}$  for every  $r \geq 11$  in the range of this proposition (and separately at  $r = 9, 10$ ). For  $r \geq 12$ , it also proves survival over fields of characteristic different from two and three. Therefore at least one of the two correction conditions for maximality fails over  $\mathbb{Q}$  at every  $r \geq 11$ . The theorem does not determine which condition fails, compute the exact target dimension, or imply survival in characteristic two or three.

It would also be valuable to replace the five target computations by uniform homotopy equivalences  $L_k \simeq S^3$  in the stated range, or by a single combinatorial Morse matching. The present theorem needs only the reproduced  $\mathbb{F}_2$  homology of the five targets. We have intentionally kept the coefficient statement explicit: the original article's table and computational formulation use  $\mathbb{Z}/2$ , while the isolated sentence of Question 8.9 does not spell out a coefficient ring. More general image-persistence methods provide the surrounding algorithmic context [5].

#### COMPUTATIONAL MATERIALS AND REPRODUCIBILITY

The ancillary directory contains five verification routes. The dependency-free file `verify_q89_direct.py` and its fixed JSON output give the exact quotient-boundary certificate for the smallest case. The portable files `verify_family_ripser.py` and `verify_square_minimality.py`, with their fixed JSON outputs, reproduce the five target calculations and the complete minimality audit through side length eight. The dependency-free files `tor_lambda_chain.py` and `test_tor_lambda_chain.py` verify the signed rational secondary-product certificate and its three exact mod-two nonboundary tests. The files `uniform_tor_projection.py` and `test_uniform_tor_projection.py` verify the uniform filling and the projection to the crosspolytope sphere. From the root of the source tree, the commands are

```
python3 anc/verify_q89_direct.py
  --output direct-rerun.json
python3 anc/verify_family_ripser.py
  --output family-rerun.json
python3 anc/verify_square_minimality.py
  --output minimality-rerun.json
python3 anc/test_tor_lambda_chain.py
python3 anc/tor_lambda_chain.py --size 23 --target-scale 9 --oriented
```

```
python3 anc/tor_lambda_chain.py --size 26 --target-scale 10 --oriented
python3 anc/tor_lambda_chain.py --size 29 --target-scale 11 --oriented
python3 anc/test_uniform_tor_projection.py
for s in 11 12 13 14 15 16 17 18; do
    python3 anc/uniform_tor_projection.py --source-scale "$s"
done
```

The Ripser family and minimality routes require NumPy and the public Python package `ripser`; the fixed run used NumPy 2.0.2 and `ripser` 0.6.15. A rerun passes exactly when it returns the five triples

$$(16, 1, 0), (22, 1, 0), (28, 1, 0), (34, 1, 0), (40, 1, 0)$$

and finds only the two  $n = 8$  violations  $3 \rightarrow 4$  and  $3 \rightarrow 5$ . Elapsed-time and platform fields are non-mathematical and may differ between runs.

The signed-chain suite has twelve tests. In the three oriented runs, the integral fillings have respectively 29, 33, 37 nonzero terms, the integral cycles  $w$  have 228, 284, 346 nonzero terms, and their mod-two reductions have 225, 281, 343 terms. Exact elimination against all 3588, 6240, 10440 target four-simplex boundaries certifies that these reductions are nonboundaries. The uniform projection suite has four further tests. Its eight residue-class runs return a simplicial block projection, a sixteen-facet crosspolytope cycle, exactly one raw coefficient-one contribution to the facet  $\{0, 1, 2, 3\}$ , and signed integral coefficient  $+3$  on that facet. These runs audit the residue and sign tables; the universal quantifier comes from the block-distance, interval, and orientation argument in the proof.

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#### DATA AND CODE AVAILABILITY

The source package includes seven verification programs, three fixed machine-readable outputs, and the hashes needed to reproduce the finite calculations and signed-chain certificates described above.

#### DISCLOSURE OF AUTOMATED ASSISTANCE

OpenAI GPT-5.6 Sol was used during proof exploration and drafting. Anthropic Claude Fable 5, operating in an agentic workflow, was used for adversarial checking, literature comparison, and language editing. The argument in this paper is self-contained, no model output is used as a mathematical premise, and the authors assume full responsibility for the final manuscript.

#### APPENDIX A. DIRECT-CERTIFICATE IDENTITIES

For the smallest example, the following equalities allow an independent verifier to check the internal consistency of the certificate without running code:

$$\dim Z_3(K) = 5632 - 2352 = 3280,$$

and

$$\dim(B_3(L) \cap C_3(K)) = 34016 - 30736 = 3280.$$

A boundary of  $L$  supported in  $K$  is a cycle of  $K$ . The second space is therefore contained in the first, and equality of dimensions proves that every  $K$ -cycle bounds in  $L$ . The Betti-number checks are

$$\beta_3(K) = 3280 - 3264 = 16, \quad \beta_3(L) = (42848 - 8831) - 34016 = 1.$$

## APPENDIX B. COMPLETE CRITICAL-FAMILY RECORD

The complete critical portions of the five degree-three barcodes are

|             |                             |
|-------------|-----------------------------|
| $T_{8,8}$   | $16[3, 4) \sqcup [4, > 4)$  |
| $T_{11,11}$ | $22[4, 5) \sqcup [5, > 5)$  |
| $T_{14,14}$ | $28[5, 6) \sqcup [6, > 6)$  |
| $T_{17,17}$ | $34[6, 7) \sqcup [7, > 7)$  |
| $T_{20,20}$ | $40[7, 8) \sqcup [8, > 8).$ |

The notation  $[a, > a)$  records only what is required at the critical inclusion; the exact later death is not used. In each row the number of intervals crossing the critical pair is zero, which is the rank of the persistence map. The portable ancillary record stores the raw interval list, coefficient field, metric and filtration conventions, truncation threshold, software versions, and distance-matrix hash separately for every row.

## APPENDIX C. DEPENDENCIES OF THE ARGUMENT

The logical dependencies of the main result are as follows.

- (1) The source homotopy type is imported from the torus-grid article.
- (2) The fact that actual horizontal and vertical sections form a basis in degree three is a new natural Mayer–Vietoris deduction from that article’s cover.
- (3) Death of each section class for  $3 \leq k \leq 7$  follows from the cyclic Rips classification and the inequality  $(k+1)/(3k-1) \geq 2/5$ .
- (4) Vanishing of the ambient persistence map over every field follows because those carrier classes span the source.
- (5) For every  $k \geq 8$ , same-chamber stability of cyclic homomorphisms makes each section map a homotopy equivalence. Coordinate retractions give two independent images, and contiguity of adjacent parallel sections gives the matching upper bound of two.
- (6) Published computations in Adams et al. give target third Betti number three over  $\mathbb{F}_2$  for  $k = 8, 9$ ; these two computations are not independently rerun here.
- (7) For  $k = 8, 9, 10$ , the rational lower bound is supplied by exact integral fillings, signed Eilenberg–Zilber chains, and complete mod-two boundary-span tests in the three factor complexes. For every  $k \geq 11$ , an explicit uniform filling and a signed simplicial projection to the crosspolytope sphere prove the same rational lower bound, integral target rank at least three, and the asserted characteristic-dependent lower bound analytically.
- (8) Target nonvanishing for  $3 \leq k \leq 7$  is supplied by two agreeing finite-field Ripser computations over  $\mathbb{F}_2$  for the same five finite inputs.
- (9) The  $k = 3$  image rank has a second, package-independent quotient-boundary certificate.
- (10) Minimality is a finite exhaustive scan through side length eight.

No claim is made about the exact target homology for  $k \geq 10$ , survival of the secondary class in characteristic two or three, integral maximality, the integral target homology at  $k = 8, 9$  outside the displayed split summand, rectangular grids, or persistence in other degrees.

## APPENDIX D. INDEPENDENT REPRODUCTION PROTOCOL

An independent reproduction should begin from the displayed metric rather than from cached complexes or matrices.

- (1) For  $k = 3, 4, 5, 6, 7$ , set  $n = 3k - 1$ , list the  $n^2$  vertices lexicographically, and construct the complete distance matrix using the sum of the two cyclic distances.
- (2) Compute degree-three Vietoris–Rips persistence over  $\mathbb{F}_2$  through a threshold strictly greater than  $k + 1$ . Record every interval meeting  $k$  or  $k + 1$ .

- (3) Check that  $6k - 2$  intervals equal  $[k, k + 1)$ , one interval begins at  $k + 1$ , and no interval crosses both endpoints. Recompute the endpoint Betti numbers by interval counting.
- (4) At  $n = 8$ , independently enumerate all cliques through size five at scales three and four. Verify  $\partial^2 = 0$ , the printed boundary ranks, both Betti numbers, and projected-boundary rank 30736.
- (5) For minimality, compute every square-grid barcode through  $n = 8$  and inspect all integer endpoint pairs having nonzero third homology.
- (6) For  $(n, r) = (23, 9), (26, 10), (29, 11)$ , solve  $\partial b = z$  over  $\mathbb{Q}$  and verify that the returned filling is integral. Form both signed Eilenberg–Zilber shuffle images under addition and check their equality. Reduce one image modulo two and eliminate it against every target four-simplex boundary; the residual must be nonzero in all three cases.
- (7) For  $s = 11, \dots, 18$ , replay the uniform integral filling and block projection to  $\text{VR}(C_8; 3)$ . Check all four projection tests, all sixteen crosspolytope facets, the unique raw coefficient-one contribution, and the signed coefficient  $+3$  on  $\{0, 1, 2, 3\}$ . Audit separately that the antipodal-block inequality, family-shuffle interval cells, even-fan term, wrap edge, and orientation signs carry the quantifier for all  $s \geq 11$ .

A run passes only if the barcode calculation, the direct quotient-boundary calculation, and the geometric theorem agree on rank zero for the smallest case, and all three signed-chain runs return an integral cycle whose mod-two reduction is a nonboundary. The uniform projection suite and all eight residue runs must also pass. Software version, coefficient field, filtration convention, and source-file hashes should be recorded. The conceptual proof is then audited separately by checking the Mayer–Vietoris basis statement, the cyclic threshold classification, the carrier factorization, and the signed-shuffle identities.

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