

The incompatibility probability of random binary qubit measurements: exact value, fixed-bias phase diagram, and critical laws at the boundary

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How likely is it that two random quantum measurements cannot be measured jointly? For two binary qubit measurements drawn uniformly from the natural Euclidean parameter domain, the numerical literature conjectured the answer to be exactly one quarter. We prove the exact value

$$\frac{2}{5} - \frac{8\pi^2}{525} = 0.2496060281\dots,$$

refuting the conjecture and closing the gap with the closed-form difference $(32\pi^2 - 315)/2100$; by the Lindemann–Weierstrass theorem the probability is transcendental. The computation reduces the eight-dimensional volume to a single fourfold integral, evaluates it by an exact differential certificate, and brackets it, at every truncation order, by purely rational bounds that already exclude $1/4$ at second order. Beyond the constant, we determine the full conditional phase diagram at fixed outcome biases u and v —a second, structural answer that stands independently of the exact value: the probability decreases in each bias (an erasure postprocessing argument valid for binary measurements in any dimension), vanishes on and only on the region $u + v \geq 1$ (a universal explicit parent construction), and the boundary of this region has its own critical laws—fifth-power decay of the probability at ordinary boundary points, linear decay along axial approach to the corners. An exact two-dimensional integral representation of the conditional probability yields the diagonal slice as the survival function of the common-erasure robustness of two random measurements, with exact tails. The proximity to one quarter is explained by a positive-series cancellation and by a factorization into a support-cone mass and a mean angular window.

I. INTRODUCTION

Two quantum measurements are *jointly measurable* when they can be realized as coarse grainings of one common measurement. Joint measurability fails exactly when the measurements are *incompatible*, and incompatibility is the operational root of quantum advantage: incompatible measurements enable steering [1, 2], Bell nonlocality [3, 4], and advantages in state discrimination [5] and communication [6]. How common is this resource among random measurements?

For the simplest nontrivial family—two binary measurements on a qubit—the question has a canonical formulation. Each binary qubit measurement is specified by an *effect bias* $x_0 \in [-1, 1]$ and a *Bloch vector* x with $\|x\| \leq 1 - |x_0|$, and the natural measure is the Euclidean volume on this domain. Zhang, Xiang, Li-Jost, and Fei [7] computed the incompatibility probability restricted to the unbiased slice $x_0 = y_0 = 0$, obtaining exactly $3/5$, and on the basis of numerical integration proposed the full-bias-domain conjecture: the probability equals $1/4$, equivalently the eight-dimensional incompatibility volume equals $\pi^2/9$. The conjecture has circulated as a reference value for the typicality of incompatibility [3].

This paper settles the question and, in doing so, upgrades it from a single constant to a full structural picture.

The exact probability is

$$P_{\text{inc}} = \frac{2}{5} - \frac{8\pi^2}{525} = \frac{210 - 8\pi^2}{525} = 0.2496060281\dots, \quad (1)$$

so the conjecture is false, but by less than four parts in ten thousand: the gap to $1/4$ is the closed form $(32\pi^2 - 315)/2100$. The π^2 arises from a single Basel-type endpoint integral after a finite algebraic reduction; the transcendence of P_{inc} follows from the Lindemann–Weierstrass theorem. The proof reduces the eight-dimensional volume to one fourfold positive integral J , evaluates J through an explicit differential certificate (antiderivatives given as rational functions of a logarithmic variable), and—independent of the closed form—brackets J between rational numbers at every truncation order of a positive series, the second-order bracket already excluding $1/4$. An independent Dirichlet–Stiefel computation in a five-dimensional spherical geometry reproduces the same constant [8].

Beyond the constant, the conditioning structure of the problem carries as much information as the number itself, and determining it is the second contribution. Fixing the two effect biases at $|a_0| = u$, $|b_0| = v$ and conditioning on the remaining randomness defines a function $p(u, v)$, the *fixed-bias phase diagram* of the problem, which the original numerics sketched but did not determine. We prove that p is nonincreasing in each argument (an erasure postprocessing argument valid for binary measurements in arbitrary dimension), symmetric in its arguments, and supported exactly on the diamond $\{u + v < 1\}$: for $u + v \geq 1$ an explicit parent construction makes any

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pair of binary $u + v \geq 1$ an explicit parent construction makes any pair of binary measurements jointly measurable, in any Hilbert space dimension, while for $u + v < 1$ a positive-measure family of incompatible pairs exists. The unbiased corner $p(0, 0) = 3/5$ is the global maximum. The boundary of the diamond carries its own critical laws: near ordinary boundary points the probability vanishes as the fifth power of the distance to the boundary, while axial approach to the two corners vanishes linearly—two distinct critical classes meeting at the corners, with exponents that we derive from the exact two-dimensional slice representation. On the diagonal, $p(u, u)$ is the survival function of the *common-erasure robustness* R that quantifies how much erasure noise two random measurements tolerate before becoming jointly measurable; the exact diagonal tail is $p(u, u) \sim \frac{256}{5}(1 - 2u)^5$ as $u \uparrow 1/2$, and $\mathbb{E}R = \int_0^{1/2} p(u, u) du$ is thus an exactly defined functional of the diagram whose closed form remains open.

Why was $1/4$ so close? Two separable mechanisms. The positive series that computes P_{inc} starts at $256/945 \approx 0.2709$ and loses its first two terms $\delta_1 = 1024/51975$ and $\delta_2 = 30208/23648625$; after one term the partial sum exceeds $1/4$ and after two it falls below, so any low-order numerical estimate sits near $1/4$ for an unremarkable reason. Geometrically, the incompatibility event factorizes into entering the support cone $A^2 + C^2 < 1$ (a four-dimensional region of probability $P_{\text{cone}} = 101/150 - 135\pi^2/4096$) and then having a sufficiently large relative angular window (mean retention $\bar{\lambda} = 8192(105 - 4\pi^2)/(7(206848 - 10125\pi^2))$), and $P_{\text{inc}} = P_{\text{cone}}\bar{\lambda}$.

The paper is organized as follows. Section II fixes the ensemble and reduces the problem to the core integral. Section III evaluates it. Section IV proves the phase-diagram theorems and the critical laws. Section V analyzes the proximity to one quarter. Section VI states precisely what is qubit-specific and what is universal. Appendix A gives the independent cross-check; Appendix B records the window derivation and endpoint asymptotics. The exact-fraction certificate script and the derivation history are archived in the source repository [8].

II. RANDOM BINARY QUBIT MEASUREMENTS

A. The ensemble

A binary qubit measurement $A = (A_1, A_2)$ is a pair of positive semidefinite operators summing to the identity. Writing $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ for the Pauli vector, every such measurement is

$$A_i = \frac{1}{2} \left((1 + (-1)^i a_0) I + (-1)^i a \cdot \sigma \right), \quad i = 1, 2, \quad (2)$$

with real parameters (a_0, a) satisfying positivity of both effects, which is exactly

$$V = \{(x_0, x) \in [-1, 1] \times \mathbb{R}^3 : \|x\| \leq 1 - |x_0|\}. \quad (3)$$

The parameter x_0 is the *effect bias*: at $x_0 = 0$ the two outcomes are equiprobable on every state, and $|x_0| \rightarrow 1$ drives one effect to zero. At fixed x_0 the section of V is the ball of radius $1 - |x_0|$, so

$$\text{vol } V = \int_{-1}^1 \frac{4\pi}{3} (1 - |x_0|)^3 dx_0 = \frac{2\pi}{3}, \quad (4)$$

and the ensemble draws (a_0, a) and (b_0, b) independently and uniformly from V with respect to four-dimensional Lebesgue measure, normalized on $V \times V$. We keep this plain Euclidean measure throughout; it is the measure under which the $1/4$ conjecture was posed [7], and it differs from Haar-type samplings of effects or states [9].

Two measurements A, B are *jointly measurable* if there exist positive semidefinite operators G_{ij} with $\sum_j G_{ij} = A_i$ and $\sum_i G_{ij} = B_j$; the G_{ij} form a parent measurement whose coarse grainings recover A and B . Define P_{inc} as the probability, under the above ensemble, that A and B are *not* jointly measurable.

B. The compatibility criterion

For binary qubit measurements joint measurability has an exact characterization [7, 10]. Introduce, for $(x_0, x) \in V$ with $x \neq 0$,

$$h(x_0, x) = \frac{\sqrt{(1 + x_0)^2 - \|x\|^2} + \sqrt{(1 - x_0)^2 - \|x\|^2}}{2}, \quad (5)$$

and abbreviate $h_a = h(a_0, a)$, $h_b = h(b_0, b)$. Then A and B are jointly measurable if and only if

$$(1 - h_a^2 - h_b^2) \left(1 - \frac{a_0^2}{h_a^2} - \frac{b_0^2}{h_b^2} \right) \leq (\langle a, b \rangle - a_0 b_0)^2. \quad (6)$$

The boundary where a square root in (5) vanishes or a denominator in (6) degenerates has measure zero, and there the parent definition is applied directly; these points do not affect any volume below.

C. Reduction to a fourfold integral

The rotational symmetry of V around the x_0 -axis is the first reduction. Write each Bloch vector in polar form $a = r \hat{a}$, $b = t \hat{b}$ with unit directions $\hat{a}, \hat{b}$ and radii r, t . The criterion (6) depends on the directions only through their inner product $s = \langle \hat{a}, \hat{b} \rangle$, and for independent uniform directions on the sphere, s is uniform on $[-1, 1]$. The sign pattern $(\text{sgn } a_0, \text{sgn } b_0)$ contributes four equal-measure sectors.

The second reduction reparametrizes each sector's radial variables. There is a bijective change of variables between $\{(u, r) : 0 < u < 1, 0 < r < 1 - u\}$ with $u = |a_0|$ and pairs of cosines $0 \leq B \leq A \leq 1$ through

$$u = AB, \quad r = \sqrt{(1 - A^2)(1 - B^2)}, \quad (7)$$

where $A = \cos \alpha$, $B = \cos \beta$ with $0 \leq \alpha \leq \beta \leq \pi/2$; surjectivity and injectivity both follow from the identity $(1 \pm AB)^2 - (1 - A^2)(1 - B^2) = (A \pm B)^2$. The same parametrization $(v, t) = (CD, \sqrt{(1 - C^2)(1 - D^2)})$ with $0 \leq D \leq C \leq 1$ applies to the second measurement. Carrying the spherical Jacobians, a fixed-sign sector acquires the radial weight

$$w(A, B) = (A^2 - B^2)\sqrt{(1 - A^2)(1 - B^2)} \quad (8)$$

with normalizing constant $I = \int_{0 \leq B \leq A \leq 1} w(A, B) dA dB = \frac{1}{12}$.

Under (7) the criterion simplifies remarkably. Set

$$\tau = \tan \alpha \tan \gamma, \quad v = \tan \beta \tan \delta, \quad (9)$$

and let $\kappa = \text{sgn}(a_0) \text{sgn}(b_0)$. After extracting the common factor $ABCD$ from all three terms, the criterion (6) becomes

$$(\tau^2 - 1)(v^2 - 1) \leq (\tau v s - \kappa)^2. \quad (10)$$

If the two factors on the left have opposite signs the inequality is automatic; if both are at most one in modulus, the inequality follows from $(\tau - v)^2 \geq 0$ (Appendix B).

Incompatibility can occur only for

$$\tau > 1 \quad \text{and} \quad v \geq \tau > 1, \quad (11)$$

and there it is the angular condition

$$\left| s - \frac{\kappa}{\tau v} \right| < \frac{\sqrt{(\tau^2 - 1)(v^2 - 1)}}{\tau v}, \quad (12)$$

a window that lies inside $[-1, 1]$ because $(\tau v \mp 1)^2 - (\tau^2 - 1)(v^2 - 1) = (\tau \mp v)^2 \geq 0$. In the cosine variables the nonzero region of (11) is exactly

$$A^2 + C^2 < 1, \quad (13)$$

the *support cone* of the problem, and the window (12) occupies the fraction

$$\lambda = \frac{\sqrt{(1 - A^2 - C^2)(1 - B^2 - D^2)}}{\sqrt{(1 - A^2)(1 - B^2)(1 - C^2)(1 - D^2)}} \quad (14)$$

of the uniform s -distribution.

Assembling the four sign sectors, the sphere integrals, and the window fraction, the total incompatibility volume and probability take the form

$$V_{\text{inc}} = 64\pi^2 J, \quad P_{\text{inc}} = \frac{J}{I^2} = 144J, \quad (15)$$

with the *core integral*

$$J = \int_{0 \leq B \leq A \leq 1, \substack{0 \leq D \leq C \leq 1 \\ A^2 + C^2 < 1}} (A^2 - B^2)(C^2 - D^2) \sqrt{1 - A^2 - C^2} \sqrt{1 - B^2 - D^2} dA dB dC dD. \quad (16)$$

Every subsequent statement about the exact value flows from (16): it is manifestly positive, and all sign sectors, Jacobians, and angular multiplicity factors are already accounted for.

III. THE EXACT VALUE

A. One-dimensionalization

The evaluation of (16) starts by extracting the radial dependence. Substitute

$$A = R \cos \vartheta, \quad C = R \sin \vartheta, \quad B = A\xi, \quad D = C\zeta, \quad (17)$$

with $0 \leq R \leq 1$, $0 \leq \vartheta \leq \pi/2$, $0 \leq \xi, \zeta \leq 1$, and write

$$Q = \cos^2 \vartheta \xi^2 + \sin^2 \vartheta \zeta^2 \in [0, 1]. \quad (18)$$

The volume element is $R^3 \cos \vartheta \sin \vartheta dR d\vartheta d\xi d\zeta$, and J becomes

$$J = \int W \sqrt{1 - R^2 Q} dR d\vartheta d\xi d\zeta, \quad (19)$$

$$W = R^7 \cos^3 \vartheta \sin^3 \vartheta (1 - \xi^2)(1 - \zeta^2) \sqrt{1 - R^2}, \quad (20)$$

all powers and trigonometric factors descending from the original Lebesgue Jacobian, not from any chosen sampling measure.

Expand the last root with the standard binomial series $\sqrt{1 - z} = 1 - \sum_{n \geq 1} c_n z^n$, $c_n = \binom{2n}{n} / (4^n(2n - 1)) > 0$. Each moment

$$T_n = \int W (R^2 Q)^n dR d\vartheta d\xi d\zeta \quad (21)$$

evaluates in closed form by the Beta function, because $\int_0^1 (1 - \xi^2) \xi^{2k} d\xi = 2/((2k + 1)(2k + 3))$ and $\int_0^1 R^{2n+7} \sqrt{1 - R^2} dR = \frac{1}{2} B(n + 4, \frac{3}{2})$. Writing $f_k = (k + 1)/((2k + 1)(2k + 3))$ and $F(t) = \sum_{k \geq 0} f_k t^k$, with the closed form $F(t) = ((1 + t) \operatorname{artanh} \sqrt{t}/\sqrt{t - 1})/(4t)$, the angular integral collapses to the convolution $S_n = [t^n] F(t)^2$ and

$$T_n = B\left(n + 4, \frac{3}{2}\right) \frac{S_n}{(n + 1)(n + 2)(n + 3)}. \quad (22)$$

Tonelli's theorem justifies termwise integration of the nonnegative series, giving

$$J = T_0 - \sum_{n \geq 1} c_n T_n. \quad (23)$$

B. Exact evaluation

The series (23) is a rigorous computation device but not the shortest route to the closed form. With $x = \sqrt{R^2 Q}$, a rearrangement of the integrals (Appendix B) reduces J to

$$J = \frac{1}{240} \left(\frac{1}{9} + K \right), \quad K = \int_0^1 \frac{\frac{1}{9} - (1-x^2)^5 F(x^2)^2}{x^2} dx, \quad (24)$$

whose integrand has an elementary antiderivative structure. The substitution $q = \operatorname{artanh} x$ turns the integrand into a quadratic polynomial in q with rational-function coefficients $h(x) = A_2(x)q^2 + A_1(x)q + A_0(x)$, and an explicit Hermite-type certificate exists: three rational functions r_2, r_1, r_0 of $z = x + x^{-1}$, together with one logarithmic term $\ell(x) = \frac{64}{105} [-\log x + 2 \log \frac{1+x}{2}]$, satisfy the term-by-term differential identities

$$\begin{aligned} r_2' &= A_2, \\ (r_1 + \ell)' &= A_1 - \frac{2r_2}{1-x^2}, \\ r_0' &= A_0 - \frac{r_1}{1-x^2}, \end{aligned} \quad (25)$$

displayed with their coefficients in Appendix B. The combined antiderivative $\Psi = r_2 q^2 + (r_1 + \ell)q + r_0$ has removable singularities at both endpoints, with boundary values $\Psi(0^+) = 0$ and $\Psi(1^-) = 5/9$. The residual integral left by (25) is a single Basel-type logarithmic integral,

$$\int_0^1 \frac{\log x - 2 \log \frac{1+x}{2}}{1-x^2} dx = -\frac{\pi^2}{24}, \quad (26)$$

evaluated by the substitutions $t = (1-x)/(1+x)$, $s = t^2$ reducing it to $\frac{1}{4} \int_0^1 \log(1-s) ds/s$. Combining the pieces,

$$K = \frac{5}{9} - \frac{8\pi^2}{315}, \quad (27)$$

and hence the closed forms.

Theorem 1. *Under the normalized Lebesgue ensemble on $V \times V$,*

$$J = \frac{1}{360} - \frac{\pi^2}{9450} = \frac{105 - 4\pi^2}{37800}, \quad (28)$$

$$P_{\text{inc}} = 144J = \frac{2}{5} - \frac{8\pi^2}{525} = 0.2496060281\dots, \quad (29)$$

and

$$\frac{1}{4} - P_{\text{inc}} = \frac{32\pi^2 - 315}{2100} = 0.0003939718\dots > 0. \quad (30)$$

The value (29) is transcendental.

Proof. The computation above gives (28)–(29), and (30) follows by subtraction. For transcendence: π^2 is transcendental by the Lindemann–Weierstrass theorem; if P_{inc} were algebraic, then $\pi^2 = (105 - 525P_{\text{inc}}/8) \cdot 8/40$ would be algebraic, a contradiction. The same argument applies to J . $\square$

An independent verification of the constant, along a completely different route through uniform Stiefel frames in dimension five, is given in Appendix A; it yields $J = (105 - 4\pi^2)/37800$ again.

C. Rational brackets at every order

The closed form can be checked without any floating-point arithmetic. Because $0 \leq R^2 Q \leq 1$ and all $c_n T_n \geq 0$, the series (23) is positive-term and, with $d_n = \binom{2n}{n}/4^n$ and $c_n = d_{n-1} - d_n$,

$$U_N - d_N T_{N+1} < J < U_N - c_{N+1} T_{N+1}, \quad (31)$$

where $U_N = T_0 - \sum_{n=1}^N c_n T_n$. Every quantity in (31) is rational: S_n obeys the finite formula

$$16S_n = \frac{O_n}{n+1} + \frac{O_n + O_{n+1} - 1}{n+2} + \frac{O_{n+1} - 1}{n+3}, \quad (32)$$

with $O_n = \sum_{j=0}^n (2j+1)^{-1}$, and $B(n+4, \frac{3}{2})$, c_n follow two-term rational recurrences. At second order already,

$$U_2 = \frac{369392}{212837625}, \quad 144U_2 = \frac{5910272}{23648625} < \frac{1}{4}, \quad (33)$$

with gap $\frac{1}{4} - 144U_2 = 7537/94594500 > 0$: a purely rational certificate that the probability is strictly below one quarter. At twentieth order the interval width is below 2.3×10^{-7} ; the exact-fraction implementation, including a Machin-identity rational bound on π for the closed-form endpoint, is archived in the source repository [8].

IV. THE FIXED-BIAS PHASE DIAGRAM

The exact constant answers the original question; the conditional structure explains it. Throughout this section

$$p(u, v) = \Pr\{\text{incompatibility} \mid |a_0| = u, |b_0| = v\}, \quad (34)$$

for $u, v \in [0, 1]$, with the conditional measure the normalized Lebesgue measure on the two balls of radii $1-u$ and $1-v$ (degenerate endpoints read by continuity).

A. Fixed bias is one-sided erasure

The key observation is operational: conditioning on a bias is conditioning on an erasure channel. Write $a = \varepsilon u$ and $a = (1-u)y$ with $\|y\| \leq 1$, $\varepsilon \in \{\pm 1\}$. For the $+$ sector, if the unbiased measurement with first

effect $E = (I - y \cdot \sigma)/2$ has the outcome pair $(E, I - E)$, then the biased measurement is exactly the postprocessed measurement

$$\mathcal{E}_u(E, I - E) = ((1 - u)E, I - (1 - u)E), \quad (35)$$

a classical channel that keeps outcome one with probability $1 - u$ and rewrites it to outcome two otherwise; the $-$ sector differs by an outcome relabeling and $y \mapsto -y$, neither of which affects joint measurability or the distribution. Joint measurability is closed under such classical postprocessing—summing parent elements along the postprocessing pattern exhibits a parent—so biasing can only destroy incompatibility. More precisely, if $0 \leq u_1 \leq u_2 < 1$ and $w = (u_2 - u_1)/(1 - u_1)$, then $\mathcal{E}_{u_2} = \mathcal{E}_w \circ \mathcal{E}_{u_1}$, and a monotone coupling of the erasure channels gives the following.

Proposition 1. $p(u_2, v) \leq p(u_1, v)$ and $p(u, v_2) \leq p(u, v_1)$ whenever $u_1 \leq u_2$ and $v_1 \leq v_2$; $p(u, v) = p(v, u)$; and $p(u, v) \leq p(0, 0) = 3/5$. The monotonicity argument uses only the postprocessing structure of one-sided erasure and holds for binary measurements in arbitrary Hilbert space dimension.

The unbiased value $p(0, 0) = 3/5$ of [7] is thereby the global maximum of the diagram; we recover it from the exact slice formula below.

B. The sharp threshold

Theorem 2. $p(u, v) > 0$ if and only if $u + v < 1$; for $u + v \geq 1$ the conditional incompatibility probability is

$$p(u, v) = \frac{9}{4(1 - u)^3(1 - v)^3} \int_u^{1-v} \int_v^{1-X} \left(1 - \frac{u^2}{X^2}\right) \left(1 - \frac{v^2}{Y^2}\right) \sqrt{(1 - X - Y) \left(1 - \frac{u^2}{X} - \frac{v^2}{Y}\right)} dY dX, \quad (37)$$

where the integration is over the triangle $\mathcal{D}_{u,v} = \{X \geq u, Y \geq v, X + Y \leq 1\}$; for $u + v \geq 1$, $p(u, v) = 0$. In particular $p(0, 0) = 3/5$, and $P_{\text{inc}} = 16 \int_0^1 \int_0^1 (1 - u)^3 (1 - v)^3 p(u, v) du dv$ over the effective domain $u + v < 1$.

Proof. The conditional radial density at bias u is $3r^2/(1 - u)^3$; changing variables $X = A^2$ with $B^2 = u^2/X$, $r^2 = (1 - X)(1 - u^2/X)$ converts it to $\frac{3}{2(1 - u)^3} (1 - u^2/X^2) r dX$, and the factor r cancels the r of the angular window (14) once for each measurement. The remaining integrand is (37). The $p(0, 0)$ value is the Dirichlet beta-type integral $\frac{9}{4} B_3(1, 1, \frac{3}{2}) = 3/5$. $\square$

The boundary behavior follows from (37) by dominated convergence with the scaling $X = u + \varepsilon\xi$, $Y = v + \varepsilon\eta$ at $\varepsilon = 1 - u - v \downarrow 0$.

identically zero, and this implication holds for binary measurements in arbitrary Hilbert space dimension.

Proof. For the vanishing side let E, F be the first effects of two binary measurements in any dimension and $E_u = (1 - u)E$, $F_v = (1 - v)F$ their biased versions. If $u + v \geq 1$ then $E_u + F_v \leq (2 - u - v)I \leq I$, so the four operators

$$G_{11} = 0, \quad G_{12} = E_u, \quad G_{21} = F_v, \quad G_{22} = I - E_u - F_v \quad (36)$$

are positive semidefinite with the correct row and column marginals: a parent. For the positivity side, suppose $u + v < 1$ and choose $X > u$, $Y > v$ with $X + Y < 1$; set $A = \sqrt{X}$, $B = u/\sqrt{X}$, $C = \sqrt{Y}$, $D = v/\sqrt{Y}$. The induced Bloch radii satisfy $(1 - u)^2 - r^2 = (X - u)^2/X > 0$, so these are interior points of the conditional balls, and in the reduced criterion (10)–(12) one has $\tau > 1$, $v > 1$ with the window (12) of positive length; since s is uniform, a positive-measure set of direction pairs is incompatible. A positive-measure neighborhood of these parameters stays incompatible by continuity, so the conditional probability is positive. $\square$

The support of p is thus exactly the open diamond $\{u + v < 1\}$; the numerically observed diamond phase boundary [7] is a theorem, and the mechanism it reflects—total erasure exceeding one outcome’s worth of information—is dimension-independent. Only the interior positivity uses the qubit structure.

C. Exact slice formula and critical laws

Theorem 3. For $u + v < 1$,

Theorem 4. At ordinary boundary points $u_0, v_0 > 0$ with $u_0 + v_0 = 1$,

$$p(u, v) \sim \frac{(1 - u - v)^5}{5uv(1 - u)^3(1 - v)^3} \sim \frac{(1 - u - v)^5}{5u_0^4v_0^4}, \quad (38)$$

while along the axes into the corners,

$$p(0, 1 - \varepsilon) \sim \left(1 - \frac{3\pi}{16}\right)\varepsilon. \quad (39)$$

Proof sketch. The quintic exponent decomposes geometrically as $1 + 1 + 1 + 2$: two radial slack factors, one from each square root in (37), and a two-dimensional shrinking domain; the constant comes from $\int_{\xi+\eta<1} \xi\eta\sqrt{1 - (\xi + \eta)^2} d\xi d\eta = 1/45$. The axial law (39) rescales only the Y variable; the constant is $\frac{2}{3} \int_{\xi+\eta<1} \eta\sqrt{(1 - \xi - \eta)(1 + \eta)} d\xi d\eta = \frac{2}{9} - \frac{\pi}{24}$. The uniform convergence of the expansions on the fixed scaling

triangles is what dominated convergence supplies; the full computation is in Appendix B. $\square$

Two critical classes thus meet at the corners of the diamond: a quintic law along generic boundary approach and a linear law along the axes. The diagonal slice carries an operational meaning. Define the *common-erasure robustness*

$$R(A, B) = \inf\{u \in [0, \tfrac{1}{2}] : \mathcal{E}_u(A), \mathcal{E}_u(B) \text{ jointly measurable}\}, \quad (40)$$

zero for already compatible pairs. The set of u with $\mathcal{E}_u(A), \mathcal{E}_u(B)$ jointly measurable is closed (finite dimension: the parent feasibility set is compact and varies continuously), so the incompatibility region is the interval $[0, R)$, and monotonicity identifies its probability at fixed equal bias with the survival function:

$$p(u, u) = \Pr\{R > u\}, \quad \mathbb{E}R = \int_0^{1/2} p(u, u) du. \quad (41)$$

Theorem 4 gives the exact tail of the robustness distribution, $p(u, u) \sim \frac{256}{5}(1 - 2u)^5$ as $u \uparrow 1/2$. Whether $\mathbb{E}R$ admits an elementary closed form is open (Section VI).

V. WHY THE PROBABILITY NEARLY EQUALS ONE QUARTER

The gap $(32\pi^2 - 315)/2100 \approx 3.94 \times 10^{-4}$ is small enough that the numerical conjecture was plausible. Two independent mechanisms account for it, and both are exact statements.

A. Low-order cancellation

The positive series (23) for the probability starts at

$$P_0 = 144T_0 = \frac{256}{945} = 0.270899 \dots \quad (42)$$

and subtracts the strictly positive terms $\delta_n = 144c_nT_n$. The first five are

n	1	2	3	4	5
δ_n	$\frac{1024}{51975}$	$\frac{30208}{23648625}$	$\frac{1024}{4606875}$	$\frac{5397248}{92868150375}$	$\frac{12670976}{655383804075}$

(43)

whose sum rapidly approaches $P_0 - P_{\text{inc}} = 0.0212934 \dots$. After one term the partial sum is $U_1 = 4352/17325 > 1/4$; after two it is $U_2 = 5910272/23648625 < 1/4$; all further terms are positive and only lower the value. Any estimate that captures the leading structure of the problem but truncates the series near its crossing therefore sits close to $1/4$ without any exact symmetry being present.

B. Support cone and angular window

The second mechanism is geometric. The incompatibility event requires first that the two measurements' sharpness parameters enter the cone (13), $A^2 + C^2 < 1$, and then that the relative direction falls in the window (12). Integrating the window out of the cone integral defines the *cone mass*

$$K_{\text{cone}} = \int_{\substack{A, C \geq 0 \\ A^2 + C^2 < 1}} W(A)W(C) dA dC, \quad (44)$$

with $W(A) = \int_0^A w(A, B) dB$ the marginal radial weight, which has the closed form $W(A) = \frac{A(1+2A^2)(1-A^2)+(4A^2-1)\sqrt{1-A^2}\arcsin A}{8}$ and antiderivative G with $G(1) = 1/12$. The cone integral evaluates exactly (the four trigonometric moment blocks are tabulated in Appendix B):

$$K_{\text{cone}} = \frac{101}{21600} - \frac{15\pi^2}{65536}, \quad (45)$$

so the two factors of the event have probabilities

$$P_{\text{cone}} = \frac{K_{\text{cone}}}{I^2} = \frac{101}{150} - \frac{135\pi^2}{4096} = 0.34804 \dots, \quad (46)$$

$$\bar{\lambda} = \frac{J}{K_{\text{cone}}} = \frac{8192(105 - 4\pi^2)}{7(206848 - 10125\pi^2)} = 0.71717 \dots, \quad (47)$$

and

$$P_{\text{inc}} = P_{\text{cone}} \bar{\lambda}. \quad (48)$$

Roughly a third of random pairs are sharp enough to be incompatible in principle, and of those, about 72% have an adverse enough relative direction; the product is the probability. The factorization also identifies where the π^2 lives: in both factors, through the same arcsin-type endpoint integrals as in the exact evaluation.

C. A probabilistic representation

Deleting the second square root in (19) and normalizing by T_0 exhibits the series structure as a moment problem. The variables $X = R^2$, $H = \cos^2 \vartheta$, $U = \xi^2$, $V = \zeta^2$ are independent with $X \sim \text{Beta}(4, \frac{3}{2})$, $H \sim \text{Beta}(2, 2)$, $U, V \sim \text{Beta}(\frac{1}{2}, 2)$, and

$$Z = Z_{\text{rad}} = X(HU + (1 - H)V) \in [0, 1] \quad (49)$$

satisfies

$$P_{\text{inc}} = \frac{256}{945} \mathbb{E}\sqrt{1 - Z}. \quad (50)$$

Every moment $\mathbb{E}Z^n$ is rational, e.g. $\mathbb{E}Z = \frac{8}{55}$ and $\mathbb{E}Z^2 = \frac{944}{25025}$, which explains why the bracketing computation (31) never leaves the rationals: the Dirichlet structure, not an accident of simplification, keeps the series arithmetic exact.

VI. SCOPE AND OPEN PROBLEMS

What is established here splits cleanly into qubit-specific statements and universal ones. The exact value (29), the slice formula (37), and the interior positivity in Theorem 2 use the Bloch representation and the binary-qubit criterion (6). The monotonicity of p (Proposition 1) and the vanishing half of the threshold theorem—every pair of binary measurements in any dimension with total bias $u + v \geq 1$ is jointly measurable, by the explicit parent (36)—use only the postprocessing structure of one-sided erasure. The problem’s measure is the plain Lebesgue measure on the parameter domain; other ensembles (Haar-type samplings of effects, fixed-radius families) define different questions and can give different typicality values [9].

Three open problems arise directly from the present results. First, the expectation $\mathbb{E}R = \int_0^{1/2} p(u, u) du$ of the common-erasure robustness is defined exactly by (41) and (37); whether it admits a closed form in the style of (29)—a rational combination with π^2 —is unknown, and the same question can be asked for the full distribution. Second, the critical classification at the corners is path-dependent: Theorem 4 gives the quintic law at ordinary boundary points and the linear law along axial approach, but a two-parameter scaling analysis for non-axial corner approach is absent, and we make no claim that a single exponent governs all paths. Third, the template transfers: the postprocessing monotonicity and the threshold construction do not use the qubit structure, so the fixed-bias phase diagram is a well-posed question for binary qutrit measurements and beyond, where both the interior positivity and the exact constants are open.

The exact-fraction certificate script, the raw derivation history, and the numerical spot-checks mentioned for positioning are archived in the source repository [8]; the proofs above do not depend on any of the numerics.

Appendix A: Independent cross-check via Stiefel frames

This appendix records an independent evaluation of the core integral J in a different geometry. It is a cross-check on Theorem 1, not a load-bearing part of the argument.

Change variables in (16) to squares, $U_1 = A^2$, $U_2 = C^2$, $X_1 = B^2$, $X_2 = D^2$. The Jacobian of the four square roots turns J into

$$J = \frac{1}{16} \int_{\Delta_2^2} w(U_1, U_2) w(X_1, X_2) (U_1 - X_1) \times (U_2 - X_2) \mathbf{1}_{\{X_i < U_i\}} dU dX, \quad (\text{A1})$$

with $w(u, v) = u^{-1/2} v^{-1/2} (1 - u - v)^{1/2}$ on the triangle $\Delta_2 = \{u, v \geq 0, u + v \leq 1\}$. Normalizing by the Dirichlet partition function, the pairs $U = (S_1^2, S_2^2, S_3^2 + S_4^2 + S_5^2)$ and $X = (T_1^2, T_2^2, T_3^2 + T_4^2 + T_5^2)$, for S, T independent uniform on the sphere $S^4 \subset \mathbb{R}^5$, are independent $\text{Dir}(\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$

vectors, and

$$J = \frac{\pi^2}{36} \mathbb{E}[\Delta_1 \Delta_2 \mathbf{1}_{\{\Delta_1 > 0, \Delta_2 > 0\}}], \quad \Delta_i = U_i - X_i. \quad (\text{A2})$$

The expectation has a spherical interpretation. Given the inner product $c = S \cdot T$, whose density on $[-1, 1]$ is $\frac{3}{4}(1 - c^2)$, the normalized combinations $P = (S + T)/\sqrt{2(1 + c)}$ and $Q = (S - T)/\sqrt{2(1 - c)}$ form, conditionally on c , a Haar-distributed orthonormal two-frame in $\mathbb{R}^5$, and

$$\Delta_1 \Delta_2 = 4(1 - c^2) P_1 P_2 Q_1 Q_2. \quad (\text{A3})$$

Since $\mathbb{E}(1 - c^2) = \frac{4}{5}$, the target expectation reduces to frame moments of $Z = P_1 P_2 Q_1 Q_2$ over the Stiefel manifold $V_2(\mathbb{R}^5)$: with $M_S = \mathbb{E}[Z \mathbf{1}_{\{P_1 Q_1 > 0, P_2 Q_2 > 0\}}]$,

$$\mathbb{E}[\Delta_1 \Delta_2 \mathbf{1}_{\{\Delta_1 > 0, \Delta_2 > 0\}}] = \frac{16}{5} M_S. \quad (\text{A4})$$

The second moments of Q given P follow from the uniform sphere on the four-dimensional orthogonal complement, $\mathbb{E}(Q_i Q_j | P) = \frac{1}{4}(\delta_{ij} - P_i P_j)$, so $\mathbb{E}Z = -1/140$. The absolute moment requires the disk integral

$$\mathbb{E}|Z| = \frac{1}{8\pi^2} + \frac{1}{420}, \quad (\text{A5})$$

obtained by fixing P , parametrizing Q in the two relevant directions with uniform disk coordinates, evaluating $\mathbb{E}[|Y_1(\rho Y_1 + \sqrt{1 - \rho^2} Y_2)|] = (\sqrt{1 - \rho^2} + \rho \arcsin \rho)/(2\pi)$, and integrating against the two-dimensional marginal of P ; the arcsin-to-arctan identity converts the last piece to elementary primitives (the full chain, with the three antiderivatives and the endpoint evaluation $\text{Li}_2(1) = \pi^2/6$, is in the source repository [8]).

Sign symmetry of the Haar frames gives the quadrant moment from the two computed moments: the identity $4Z \mathbf{1}_{\{P_1 Q_1 > 0, P_2 Q_2 > 0\}} = Z + |P_1 Q_1| |P_2 Q_2| + P_1 Q_1 |P_2 Q_2| + |Z|$ and the sign-flip invariance of the middle terms yield

$$M_S = \frac{1}{4} (\mathbb{E}Z + \mathbb{E}|Z|) = \frac{1}{32\pi^2} - \frac{1}{840}. \quad (\text{A6})$$

Assembling (A2), (A4), and (A6),

$$J = \frac{\pi^2}{36} \cdot \frac{16}{5} \left(\frac{1}{32\pi^2} - \frac{1}{840} \right) = \frac{105 - 4\pi^2}{37800}, \quad (\text{A7})$$

in exact agreement with (28). The appearance of $1/\pi^2$ in (A5) and its cancellation against the prefactor in (A2) explains why the five-dimensional geometry also produces only rationals and π^2 .

Appendix B: Supporting derivations

1. The angular window

With $\tau = \tan \alpha \tan \gamma$ and $v = \tan \beta \tan \delta$, the relations

$$\tau^2 - 1 = \frac{1 - A^2 - C^2}{A^2 C^2}, \quad v^2 - 1 = \frac{1 - B^2 - D^2}{B^2 D^2} \quad (\text{B1})$$

follow from $1 - \tan^2 \alpha \tan^2 \gamma = (\cos^2 \alpha \cos^2 \gamma - \sin^2 \alpha \sin^2 \gamma) / (\cos^2 \alpha \cos^2 \gamma)$ and the angle addition formula. If $\tau, v < 1$ then $|\tau v s - \kappa| \geq 1 - \tau v \geq \sqrt{(1 - \tau^2)(1 - v^2)}$, the second inequality being equivalent to $(\tau - v)^2 \geq 0$; with both left factors of (10) negative, the product exceeds the square, so the pair is compatible. If the two left factors have opposite signs the product is nonpositive and again compatible. Hence incompatibility requires $\tau > 1$, whence (11)–(12); solving the quadratic (10) for s gives the window centered at $\kappa/(\tau v)$ with half-width $\sqrt{(\tau^2 - 1)(v^2 - 1)}/(\tau v)$, and the containment in $[-1, 1]$ is $(\tau v \mp 1)^2 - (\tau^2 - 1)(v^2 - 1) = (\tau \mp v)^2 \geq 0$. In cosine variables the condition $\tau > 1$ is $A^2 + C^2 < 1$ by (B1).

2. The one-dimensional reduction of J

Starting from (19), the substitution $y = R^2 Q$ inside the binomial expansion and Tonelli's theorem give (23) with (22); the identity

$$c_n \frac{B(n+4, \frac{3}{2})}{(n+1)(n+2)(n+3)} = \frac{1}{480} B\left(n - \frac{1}{2}, 6\right) \quad (n \geq 1) \quad (\text{B2})$$

turns the tail into an integral against $t^{-3/2}(1-t)^5$, and moving T_0 to the same side produces

$$T_0 - J = \frac{1}{480} \int_0^1 t^{-3/2}(1-t)^5 \left(F(t)^2 - \frac{1}{9}\right) dt. \quad (\text{B3})$$

With $t = x^2$ and the elementary polynomial integral $\int_0^1 (1 - (1 - x^2)^5)/(9x^2) dx = 193/567$, the rearrangement (24) follows.

3. The differential certificate

In the variable $q = \operatorname{artanh} x$, $z = x + x^{-1}$, the integrand of K is $h(x) = A_2 q^2 + A_1 q + A_0$ with

$$\begin{aligned} A_2(x) &= -\frac{(1-x^2)^5(1+x^2)^2}{16x^8}, \\ A_1(x) &= \frac{(1-x^2)^5(1+x^2)}{8x^7}, \\ A_0(x) &= \frac{1}{9x^2} - \frac{(1-x^2)^5}{16x^6}, \end{aligned} \quad (\text{B4})$$

and the three certificate functions are

$$\begin{aligned} r_2(x) &= \frac{(z-2)^3(15z^4 + 90z^3 + 192z^2 + 192z + 128)}{1680}, \\ r_1(x) &= -\frac{(z-2)(z+2)(15z^4 - 123z^2 + 284)}{840}, \\ r_0(x) &= \frac{z(15z^4 - 198z^2 + 1112)}{1680} - \frac{1}{9x}, \end{aligned} \quad (\text{B5})$$

with $\ell(x) = \frac{64}{105} [-\log x + 2 \log \frac{1+x}{2}]$. Term-by-term differentiation verifies (25): the three identities become rational-function equalities after clearing denominators, each checkable by coefficient comparison. The apparent Laurent poles at $x = 0$ cancel in the combination $\Psi = r_2 q^2 + (r_1 + \ell)q + r_0$: the coefficients of x^{-5} , x^{-3} , x^{-1} in the three contributions sum to zero column by column. The endpoint expansions are

$$\Psi(x) = \frac{x}{105} (-64 \log x + 113 - 128 \log 2) + O(x^2 |\log x|), \quad (\text{B6})$$

at $x \rightarrow 0$ and $\Psi(1 - \varepsilon) = \frac{5}{9} + O(\varepsilon |\log \varepsilon|)$ at $x \rightarrow 1$, so $\Psi(0^+) = 0$, $\Psi(1^-) = 5/9$, and $K = \Psi(1^-) - \Psi(0^+) - \frac{8\pi^2}{315}$ with the logarithmic residual (26) times $\frac{64}{105}$; this is (27).

4. Boundary asymptotics

For Theorem 4 at an ordinary boundary point, insert $X = u + \varepsilon \xi$, $Y = v + \varepsilon \eta$ with $\varepsilon = 1 - u - v$ into (37). The three nontrivial factors expand as

$$1 - \frac{u^2}{X^2} = \frac{2\varepsilon \xi}{u_0} + o(\varepsilon), \quad 1 - \frac{v^2}{Y^2} = \frac{2\varepsilon \eta}{v_0} + o(\varepsilon), \quad (\text{B7})$$

$$\begin{aligned} 1 - \frac{u^2}{X} - \frac{v^2}{Y} &= \varepsilon(1 + \xi + \eta) + O(\varepsilon^2), \\ 1 - X - Y &= \varepsilon(1 - \xi - \eta), \end{aligned} \quad (\text{B8})$$

uniformly on compact scaling triangles bounded away from $u_0, v_0 = 0$; the square roots contribute ε and the domain ε^2 , for the total ε^5 of (38). The constant is $\int_{\xi+\eta<1} \xi \eta \sqrt{1 - (\xi + \eta)^2} d\xi d\eta = 1/45$ against the prefactor $9/(4u_0^4 v_0^4)$.

For the axial law at $(0, 1 - \varepsilon)$, rescale only $Y = 1 - \varepsilon + \varepsilon \eta$ and $X = \varepsilon \xi$. Then $1 - v^2/Y^2 = 2\varepsilon \eta + O(\varepsilon^2)$ and $1 - v^2/Y = \varepsilon(1 + \eta) + O(\varepsilon^2)$, while $1 - u^2/X^2 \rightarrow 1$ and the X -integration contributes its full ε from the domain; the surviving constant is $\frac{2}{3} \int_{\xi+\eta<1} \eta \sqrt{(1 - \xi - \eta)(1 + \eta)} d\xi d\eta = \frac{2}{9} - \frac{\pi}{24}$, and the prefactor $9/(4(1 - \varepsilon)^3)$ of (37) supplies $1 - 3\pi/16$ as claimed in (39).

5. Cone-mass moments

For (45) the substitution $A = \sin \theta$ converts all four product blocks of $W(A)G(\sqrt{1 - A^2})$ into finitely many integrals $\int_0^{\pi/2} \theta^p \sin^m \theta \cos^n \theta d\theta$ with $p \in \{0, 1, 2\}$, each elementary by parts; the four block values are $121/46080$, $49/76800 - 3\pi^2/32768$, $467/230400 - 11\pi^2/65536$, and $-17/27648 + \pi^2/32768$, summing to (45).

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