

The Minimal Uncertainty Product of Noninteracting Fermion Chains Scales as $\log n/n^2$ in the Number of Modes

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In classical Markov dynamics, the thermodynamic uncertainty relation bounds the uncertainty product of every current from below by 2. Quantum coherence can lower the product, but how fast it can fall as a device grows, and what this consumes, has remained unknown. We answer both questions for chains of noninteracting fermionic modes. Precision is limited by how sharply a finite chain can switch its transmission between zero and one in energy. The uncertainty product of n modes cannot fall faster than $\log n/n^2$, and chains reaching this rate exist at every length. The contacts set a second price: their linewidth per transferred particle must grow logarithmically with the precision, which suffices. Contacts of bounded linewidth allow unlimited precision only at a vanishing current. In Hilbert-space dimension N the minimal product therefore falls at least as fast as $\log \log N/(\log N)^2$, proving a conjecture of Menczel et al. (2021). The precision of coherent transport thereby becomes a question of filter design.

Introduction.—Every device that runs out of equilibrium, from a molecular motor to a thermoelectric junction or a clock, delivers a current that fluctuates. In classical Markov dynamics the thermodynamic uncertainty relation bounds the uncertainty product P of any steady current from below by 2 [1–3]. Precision thus has a minimal entropy price, which constrains engines and transport [4] and the accuracy of autonomous clocks [5–7].

Quantum coherence evades this price: coherent conductors and driven quantum systems can produce currents more precise than any classical process at the same dissipation [8–14]. What then limits precision? Menczel, Loisa, Brandner and Flindt conjectured that for coherently driven devices exchanging energy with thermal baths no floor on P survives as the device grows [15]. If so, the questions become how fast P can fall with the size of a device, and which physical resources this consumes.

Energy filtering is the natural route [13, 16, 17]. A conductor transmitting perfectly inside an energy window and not at all outside it has no partition noise, and a large bias removes thermal noise. Chains of quantum dots approach such a window. In one of them P vanishes as the bias and the number of dots grow [18], though only about as fast as the inverse number of dots [19]. Numerically optimized chains do better [19]. What was missing is a proved limit and its price. Bounds that depend on dimension exist when the environment is finite as well [20]. For devices driven by Markovian reservoirs no size-dependent lower bound on P was known, and bounds through the minimal reflection [21] become empty at near-perfect transmission. What near-perfect transmission demands of the contacts was not known either [22], and the one chain shown numerically to make P vanish needs coupling ratios that grow with its length [18].

Here we establish this limit and its price for chains of

noninteracting fermions. A finite chain cannot switch its transmission abruptly in energy, so partition noise survives at the edges of every transmission window. As a result, the product of an n -mode chain can fall no faster than $\log n/n^2$, a rate that chains of every length reach. The contacts set a second price: their linewidth per transferred particle must grow logarithmically with the precision. Contacts of bounded linewidth allow unlimited precision only at a vanishing current. In Hilbert-space dimension N the minimal product therefore falls at least as fast as $\log \log N/(\log N)^2$, which proves the conjecture of Menczel *et al.*

This size limit is the classical question of how sharp a filter of given order can be [23, 24], read as a bound on noise. It predicts that the full-bias Fano factor of an array of n quantum dots cannot fall faster than $1/n^2$, which, to our knowledge, has not been tested.

Thermal current through a chain.—A precise current needs two noise sources removed at once: the randomness of transmission and the reverse jumps of the reservoirs. We set up the class in which this is asked and the chain in which both can be computed. The thermal class, the part of the class of Ref. [15] with positive gaps and thermal occupations, consists of Lindblad generators [25, 26], with $\hbar = k_B = 1$,

$$\mathcal{L}\rho = -i[H_0 + X, \rho] + \sum_{\mu} \gamma_{\mu} [(\bar{n}_{\mu} + 1)\mathcal{D}[L_{\mu}]\rho + \bar{n}_{\mu}\mathcal{D}[L_{\mu}^{\dagger}]\rho], \quad (1)$$

with $\mathcal{D}[L]\rho = L\rho L^{\dagger} - \frac{1}{2}\{L^{\dagger}L, \rho\}$, an arbitrary coherent drive X , and paired channels lowering the bare energy, $[H_0, L_{\mu}] = -\Delta E_{\mu}L_{\mu}$, with Bose occupations $\bar{n}_{\mu} = 1/(e^{\beta_{\mu}\Delta E_{\mu}} - 1)$ at positive finite inverse temperatures β_{μ} . Upper bounds on P in it hold for the full class. The channel affinity is $A_{\mu} = \beta_{\mu}\Delta E_{\mu}$, and the entropy production is $\sigma = \sum_{\mu} A_{\mu}j_{\mu}$. Counting L_{μ} as +1 and

$L_\mu^\dagger$ as -1 in one channel defines a current j and its full variance rate d , without a factor $1/2$. The uncertainty product is $P = \sigma d/j^2$. We write $P_{\min}^N$ for its infimum in Hilbert-space dimension N , over members with a unique attracting stationary state and a positive counted current.

Our device is a chain of n fermionic modes $c_1, \dots, c_n$ with Hamiltonian $H = \sum_{l=1}^{n-1} t_l (c_l^\dagger c_{l+1} + c_{l+1}^\dagger c_l)$, positive hoppings t_l and zero on-site energies. Particles enter at mode 1 and leave at mode n :

$$\begin{aligned} \mathcal{L}\rho = & -i[H, \rho] + 2\Gamma\{f_L \mathcal{D}[c_1^\dagger] + f_R \mathcal{D}[c_1]\}\rho \\ & + 2\Gamma\{f_L \mathcal{D}[c_n] + f_R \mathcal{D}[c_n^\dagger]\}\rho, \end{aligned} \quad (2)$$

with $f_L = 1/(1+e^{-A})$, $f_R = 1-f_L$, and contact linewidth Γ at both ends. We count extraction at mode n positively. With $H_0 = A(I - \hat{n}_1 + \hat{n}_n)$ and $X = H - H_0$, both forward jumps $c_1^\dagger$ and c_n lower H_0 by A at temperature 1, so Eq. (2) is a member of the class (1) in dimension 2^n , with A the affinity of both channels. For every finite $A > 0$ it has a unique stationary state, which is faithful, and every density matrix relaxes to it [27].

Let h be the single-particle hopping matrix and $T(E) = 4\Gamma^2 |(E - h + i\Gamma\Pi_1 + i\Gamma\Pi_n)^{-1}|^2$ the transmission, with $\Pi_l = |l\rangle\langle l|$. Write $I_1 = \int T dE/2\pi$ and $I_2 = \int T(1-T) dE/2\pi$, and let $F_0 = I_2/I_1$ be the Fano factor at full bias. Then $j = qI_1$ with $q = \tanh(A/2)$, every net transfer produces entropy $2A$, and

$$P = 2AqF_0 + \frac{2A}{\sinh A}. \quad (3)$$

These finite-affinity formulas were obtained by Meier et al. in a scattering treatment [19]; we derive them from the full-Fock generator (2), through the tilted generator and the Gaussian stationary state [27]. The first term is partition noise, which needs $F_0 \rightarrow 0$, a transmission close to 0 or 1 at almost every energy. The second is thermal noise from reverse jumps. It needs $A \rightarrow \infty$, but then the first term grows as A . Both must vanish together.

The trigonometric chain of Ref. [18] shows the balance. Its transmission is $1/(1+E^{2n})$ at every length, as we prove in the Supplemental Material [27], so $F_0 = 1/(2n)$. At $A = \log n$ the two terms of Eq. (3) are $(\log n)/n$ and $4(\log n)/n$ asymptotically, so $P \simeq 5 \log n/n$. The entropy per transferred particle grows as $2 \log n$, while the Fano factor d/j falls as $5/(2n)$. The rate of decrease is thus set by how fast F_0 can fall with n .

How sharp can a chain filter be?—Partition noise lives where transmission is neither near zero nor near one, so it is set by how steeply an n -mode chain can switch transmission on and off. Here we allow any real on-site energies and any linewidths Γ_L, Γ_R of the two end contacts; Eq. (3) still holds [27]. The end-to-end cofactor of $E - h + i\Gamma_L\Pi_1 + i\Gamma_R\Pi_n$ is the constant $\prod_l t_l$, whatever the on-site energies and linewidths. Unitarity of the

two-terminal scattering then gives

$$\begin{aligned} T(E) &= \frac{1}{1+Q(E)}, \\ Q(E) &= \frac{|\det(E - h - i\Gamma_L\Pi_1 + i\Gamma_R\Pi_n)|^2}{4\Gamma_L\Gamma_R \prod_l t_l^2}, \end{aligned} \quad (4)$$

where $Q \geq 0$ is a real polynomial of degree exactly $2n$, for every chain with nonzero hoppings [27]. Transmission is near one where Q is small and near zero where Q is large, and the noise $T(1-T) = Q/(1+Q)^2$ is at least $2/9$ wherever $\frac{1}{2} \leq Q \leq 1$.

Take an interval of length ℓ on which $0 \leq Q \leq 1$, with $Q = 1$ at its ends. Markov's inequality bounds the derivative of a polynomial of degree d on an interval by d^2 times its maximum, divided by half the length [28, 29]. Applied to $2Q - 1$, it gives $|Q'| \leq d^2/\ell$. Hence Q stays above $\frac{1}{2}$ within $\ell/(2d^2)$ of each end, and these strips carry noise at least $2\ell/(9d^2)$, against a transmitted weight at most ℓ . Outside $\{Q \leq 1\}$, $T(1-T) \geq T/2$; if $Q > 1$ everywhere, this holds at every energy. Summing over the intervals of $\{Q \leq 1\}$ and the rest of the axis, with $d = 2n$,

$$F_0 \geq \frac{1}{18n^2}. \quad (5)$$

The bound needs no symmetry and holds at any coupling ratio and current. It does not depend on the energy scale either: rescaling h, Γ_L and Γ_R together rescales the window and leaves F_0 unchanged. A chain has n resonances to shape its transmission, and each window edge costs a fraction $1/n^2$ of the window, however the resonances are placed. Inserting Eq. (5) in Eq. (3) and splitting at $A = \log n$ gives $P \geq \log n/(18n^2)$ for $n \geq 3$ and every finite A : above $\log n$ the partition term exceeds it, and below, the thermal term $2A/\sinh A \geq 4n \log n/(n^2 - 1)$ does [27].

A filter at the limit.—The bound (5) asks for a profile whose small values are spread over the window and whose rise to 1 is close to as steep as Markov's inequality permits. Let P_k be the Legendre polynomial with $P_k(1) = 1$, and

$$\begin{aligned} r_n(x) &= \frac{2P'_{n+1}(x)}{(n+1)(n+2)} = \frac{C_n^{(3/2)}(x)}{C_n^{(3/2)}(1)}, \\ T_n(E) &= \frac{1}{1+r_n(E/w)^2}, \end{aligned} \quad (6)$$

a normalized Gegenbauer polynomial, with w the half-width of the window. The same normalized Gegenbauer prototype appears in filter synthesis [24]. Orthogonality gives $\int_{-1}^1 r_n^2 dx = 4/[(n+1)(n+2)]$, so the in-band ripple carries little weight; $|r_n| \leq 1$ on the window, with its maximum 1 at the edges $x = \pm 1$. Beyond the edge every coefficient of the expansion of $r_n(1+s)$ in s is positive, so $r_n(1+s) \geq 1 + n(n+3)s/4$. The slope of $Q = r_n^2$ at

the edge is thus within a factor 4 of the Markov limit. Together these give

$$F_0 \leq \frac{6}{n(n+3)} + \frac{8}{n^2(n+3)^2} < \frac{6}{n^2}, \quad (7)$$

including both infinite tails [27]. Figure 1(a) compares T_8 with the transmission $1/(1+E^{2n})$ of the trigonometric chain [18], whose edges are wider by a factor of order n and whose $F_0 = 1/(2n)$. The chains optimized numerically in Ref. [19], whose full-bias Fano factor falls about as $n^{-1.86}$, are consistent with Eq. (5), which forbids any exponent above 2.

A transmission T_n is useful only if a chain of the form (2) produces it. For every $n \geq 2$ and $w > 0$ it does. Write $r_n(E/w) = R(E)/\kappa$ with R monic, and let D be the monic polynomial whose roots are the roots of $R^2 + \kappa^2$ in the lower half-plane. Write $D = U + iV$ with U and V real polynomials. Then Γ is half the coefficient of E^{n-1} in V . The polynomial $(U + R)/2$ has n simple real roots x_k , which carry positive weights $\kappa/[2\Gamma|(U + R)'(x_k)/2|]$ summing to one. The Jacobi matrix of this discrete measure [30] has zero diagonal, positive mirror-symmetric hoppings, and transmission exactly $T_n(E)$ at every real E [27].

Choosing $A = 2 \log n$ in Eq. (3) makes the thermal term $8 \log n/[n^2(1 - n^{-4})]$. Let $P_{\min}(n)$ be the infimum of P over all chains of n fermionic modes and all finite $A > 0$. The chains have nonzero real nearest-neighbour hoppings, real on-site energies, and Markovian contacts of linewidths $\Gamma_L, \Gamma_R > 0$ on the end modes. Inserting Eqs. (5) and (7) in Eq. (3) gives the size law: for every $n \geq 3$,

$$\frac{\log n}{18n^2} \leq P_{\min}(n) < 33 \frac{\log n}{n^2}. \quad (8)$$

The upper bound is realized, for every $n \geq 2$ and $w > 0$, by the chain with transmission (6) at $A = 2 \log n$, whose current satisfies $|\pi j/w - 1| \leq 4/n^2$ [27].

Thus the product falls as $\Theta(\log n/n^2)$ and no faster, and the order-optimal chain does not buy precision by stopping transport: at fixed w the current tends to w/π . Figure 1(b) shows the product minimized over A inside the proven band; the trigonometric chain falls only as $\log n/n$. The remaining factor between the constants $1/18$ and 33 is not resolved. Equation (7) also shows that the lower bound (5) on F_0 cannot be raised by a factor $\log n$.

What the contacts must supply.—Precision needs near-perfect transmission over a window of energies. A contact of finite linewidth cannot match over a broad window, a limit Fano found for electrical networks [31]. We state it for quadratic networks with two single-mode contacts, into which every mode decays. The single-particle Hamiltonian is an $n \times n$ Hermitian h , with on-site energies and complex hoppings allowed. Contacts of linewidths Γ_L and Γ_R act on modes 1 and n . We require

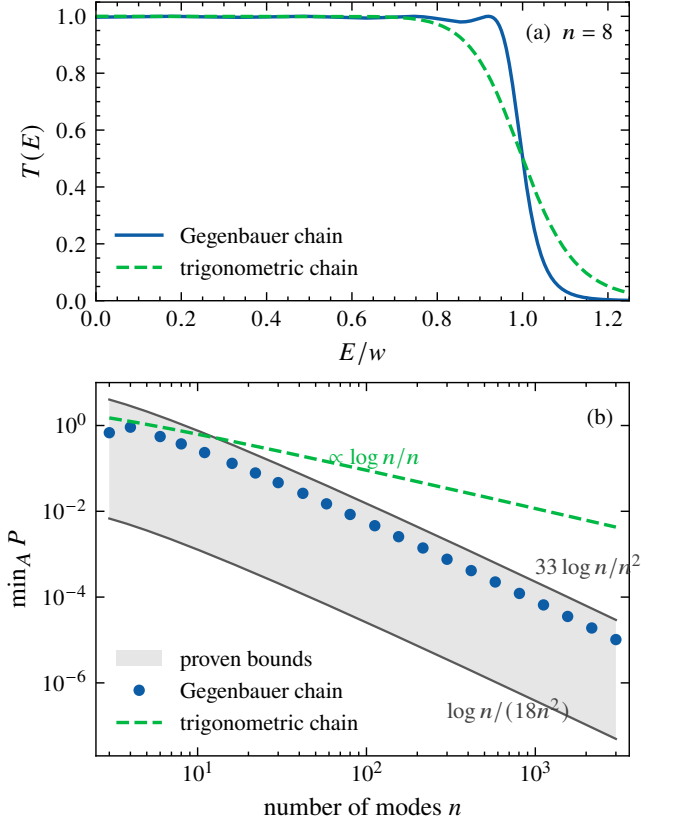


FIG. 1. Size law. (a) Transmission of the Gegenbauer chain, Eq. (6), and of the trigonometric chain [18] with $n = 8$ modes. The ripple of the Gegenbauer chain is small and largest next to the window edge. Its edge slope is within a factor 4 of the Markov limit. (b) Uncertainty product minimized over the affinity, for both chains, against the number of modes. The shaded band lies between the bounds (8); the Gegenbauer chain falls as $\log n/n^2$ inside it, the trigonometric chain as $\log n/n$. Products are computed from Eq. (3) with F_0 integrated over the whole energy axis.

$-ih - \Gamma_L \Pi_1 - \Gamma_R \Pi_n$ to be strictly stable and the transmission to be nonzero. The generator then has a unique attracting stationary state and obeys Eq. (3) [27]. Let z_k be the eigenvalues of $h + i\Gamma_L \Pi_1 - i\Gamma_R \Pi_n$, the zeros of the reflection determinant counted with any factors it shares with the denominator. The poles and these zeros give

$$\int \frac{dE}{2\pi} \log \frac{1}{1 - T(E)} = \Gamma_L + \Gamma_R - \sum_k |\text{Im } z_k| \leq 2\Gamma_*, \quad (9)$$

with $\Gamma_* = \min(\Gamma_L, \Gamma_R)$, because $\sum_k |\text{Im } z_k| \geq |\Gamma_L - \Gamma_R|$. A fixed budget of $\log[1/(1 - T)]$ is thus available. Spending it on high transmission over a window narrows the window: a window of width ΔE with $1 - T \leq \epsilon$ costs $\Delta E \log(1/\epsilon)/2\pi$. Partition noise is the average of $1 - T$ over the transmitted weight, $F_0 = \int T(1 - T)/\int T$. Jensen's inequality for the convex function $\log(1/x)/(1 -$

x), applied with that weight, turns Eq. (9) into [27]

$$I_1 \frac{\log(1/F_0)}{1 - F_0} \leq 2\Gamma_*. \quad (10)$$

A small F_0 at a given current thus requires a contact linewidth that grows as $\log(1/F_0)$. The bound is set by the weaker contact. Here Γ_* is the linewidth that contact gives the end mode, not a rate of reservoir jumps. Unlike bounds through the minimal reflection [21], Eq. (10) stays informative when transmission is near perfect, and it prices in contact linewidth the near-perfect transmission whose loss restores a cost-precision trade-off [22].

Combined with Eq. (3), this budget bounds Γ_*/j from below at a given product p . Write $s = \log(1/p)$, and let $\mathcal{R}(p)$ be the minimum of $\log(1/f)/[2q(1-f)]$ with $f = (p - 2A/\sinh A)/(2Aq)$, over the affinities with $2A/\sinh A < p$. The thermal term $2A/\sinh A > e^{-A}$ forces $A > s$. Partition noise is then allowed only $F_0 \leq p/(2Aq)$, smaller than p by a factor at least of order s , and Eq. (10) turns this factor into the second logarithm. For $0 < p < 2$, every two-contact network as above with $P \leq p$ thus obeys

$$\Gamma_*/j \geq \mathcal{R}(p), \quad (11)$$

and, for $s = \log(1/p) \geq 8$ [27],

$$0 \leq 2\mathcal{R}(p) - [s + \log(2s)] \leq \frac{2 \log s + \log 8 + 5}{s}. \quad (12)$$

The bound is sharp up to an additive constant: for every $0 < p \leq 1$ an equal-contact chain with $P < p$ exists, and as $p \rightarrow 0$ its cost is $\Gamma/j = s/2 + \frac{1}{2} \log s + \log 8 + o(1)$. The infimum of the contact linewidth per transferred particle over two-contact networks of any size, at any finite affinity, with $P \leq p$ is therefore, as $p \rightarrow 0$,

$$\inf \frac{\Gamma_*}{j} = \frac{1}{2} \log \frac{1}{p} + \frac{1}{2} \log \log \frac{1}{p} + O(1), \quad (13)$$

the contact law.

The upper side of this law is realized by the Chebyshev transmission $T(E) = [1 + \epsilon^2 C_n(E/w)^2]^{-1}$, with C_n the Chebyshev polynomial of the first kind. Its order n is odd and of order $\log(1/p)/\sqrt{p}$, and its ripple is $\epsilon^2 = p/(16A)$ at the affinity $A = \log(16/p) + \log \log(16/p)$. Its hoppings are the classical prototype of coupled-resonator filters [23]; we prove that they give this transmission exactly with equal Markovian contacts [27]. The trigonometric chain, optimized over length and affinity, needs $\Gamma/j = [s + 2 \log s + \log 4 + 1 + o(1)]/p$, exponentially more (Fig. 2).

No dimension-independent precision floor.—Menczel et al. asked for the minimal product at a fixed Hilbert-space dimension N , over the whole thermal class [15]. The Gegenbauer chain that realizes the upper bound of Eq. (8) has dimension 2^n . Attaching auxiliary levels by

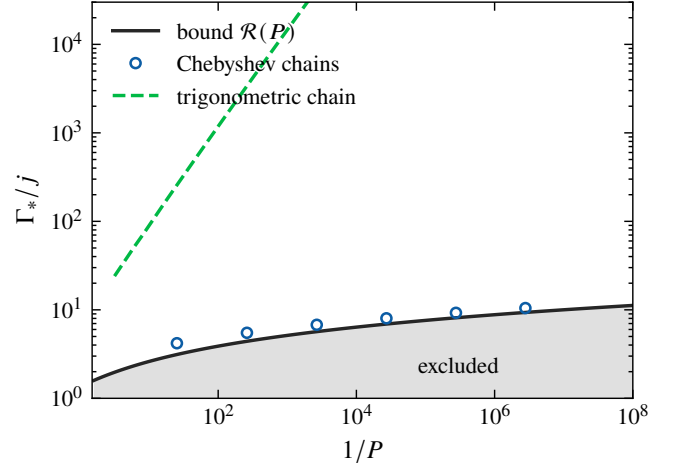


FIG. 2. Contact law. Weaker contact linewidth per transferred particle needed for uncertainty product P . The shaded region below $\mathcal{R}(P)$ is excluded for every two-contact network [Eq. (11)]. Open circles: Chebyshev chains built for $p = 10^{-1}, \dots, 10^{-6}$, plotted at their actual product. Dashed: the trigonometric chain, optimized over length and affinity.

thermal channels of small occupation, as in Ref. [15], reaches all remaining dimensions at no extra entropy production and a current above $w/(4\pi)$ [27]. For every integer $N \geq 4$, with $n = \lfloor \log_2 N \rfloor$,

$$0 \leq P_{\min}^N < 66 \frac{\log n}{n^2}, \quad P_{\min}^N = O\left(\frac{\log \log N}{(\log N)^2}\right). \quad (14)$$

This proves the conjecture of Ref. [15] that $P_{\min}^N \rightarrow 0$. The rate is an upper bound for the full thermal class. The size law (8) is proved only for chains, and no lower bound in N is known for the class.

Precision at bounded couplings.—The two laws price precision in different resources. In the Gegenbauer chain at $A = 2 \log n$ the entropy per transferred particle grows as $4 \log n$ while the relative noise falls as $1/n^2$. In the Chebyshev chain the hoppings in the middle stay near $w/2$, while Γ grows as $w \log(1/P)/(2\pi)$, so its contact-to-bulk ratio grows with the precision. Can precision grow with every coupling bounded? Equation (10) with $j \leq I_1$ gives $F_0 \geq e^{-2\Gamma_*/j}$, so at every affinity

$$P \geq 2 \tanh \frac{1}{2} e^{-2\Gamma_*/j}. \quad (15)$$

A finite current through bounded contacts keeps a positive floor: along every sequence of two-contact networks with bounded Γ_* and $P \rightarrow 0$, the current tends to zero. With a current tending to zero, the floor tends to zero, and suitable chains do drive the product to zero. There are chains with zero on-site energies, every hopping in [1, 6], both linewidths equal to 1, and lengths $m_k \rightarrow \infty$, whose products at finite affinities tend to zero [27].

Bounded couplings can make an arbitrarily sharp filter at the price of bandwidth. A strong bond v followed by a

weak bond 1 passes an amplitude $1/v$ at the band center, so ℓ such pairs act as one hopping $v^{-\ell}$. Such hoppings carry transfer across dimerized chains [32, 33]. Likewise, ν pairs of bonds (3, 1) between a contact and the chain reduce the unit linewidth to $9^{-\nu}$. With $v \leq 6$, such blocks approximate, as their depth grows, any chain with zero on-site energies and equal contacts, scaled down in energy. Here the target is a Chebyshev chain of growing order. Every energy is scaled by $\delta_\nu \propto 9^{-\nu}$, and the current falls by the same factor. The error must be small on the whole energy axis against the shrinking transmitted weight, and narrow resonances outside the window could carry comparable weight. At such resonances either the dressed contact or the effective coupling across the central blocks is exponentially small. Either one charges the transmission against the spectral weight of the end sites, whose integral is fixed [27].

In this construction the current at a fixed target order n falls exponentially with the length, at a rate of order $1/n$. Equation (15) requires only $j \leq 2\Gamma_*/\log(2 \tanh \frac{1}{2}/P)$. Along the whole sequence no rate in the length is known.

Conclusions.—Quantum coherence removes the size-independent floor on the uncertainty product, but precision is not free. In chains of coherent modes the entropy per transferred particle still grows, logarithmically in the length. The relative noise falls faster, as the inverse square of the length, limited by how sharply transmission can switch in energy. The contacts add a second price, set by how far in energy transmission can stay near one; with contacts of bounded linewidth this price is paid in current instead.

For coherent transport, the thermodynamic uncertainty relation thus becomes a question of filter design. The precision a chain of given length can reach is set by the sharpness a filter of given order can reach. The bound is testable in quantum-dot arrays, already proposed as broadband energy filters [13]: their Fano factor at full bias cannot fall faster than $1/n^2$ in the number n of dots. The same accounting may apply to coherent clocks, whose ticks are counted currents. Three questions remain open: whether one device can meet both prices at once, how fast precision can grow with length at bounded couplings, and how fast it can grow beyond noninteracting chains, as in the ring clock of Ref. [34].

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Data availability.—The data and the code that support the findings of this article are available from the author upon reasonable request.

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Supplemental Material for “The Minimal Uncertainty Product of Noninteracting Fermion Chains Scales as $\log n/n^2$ in the Number of Modes”

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This Supplemental Material contains the proofs of the results stated in the main text. Sections I–III establish, for every finite chain and every strictly stable two-contact network at finite affinity, the nonnegativity of the product, a unique faithful attracting stationary state on the full Fock space, and the exact current and variance. Section IV writes the transmission of a chain as $1/(1+Q)$ and derives the lower bound on partition noise from Markov’s inequality. Sections V–VII construct the Gegenbauer chain at every length and assemble the size law. Sections VIII and IX derive the contact budget of two-contact networks and the minimal contact cost, which an equal-contact Chebyshev chain attains. Section X shows that the product can vanish with every coupling bounded, and that it then forces the current to vanish. Section XI extends the upper bound of the size law to every Hilbert-space dimension. Section XII computes the transmission of the trigonometric chain used as a benchmark, and Sec. XIII calibrates the counting conventions against the small-system values of Menczel et al. [1].

We follow the notation of the main text. The number of modes is $n \geq 2$, $f_L = 1/(1+e^{-A})$, $f_R = 1 - f_L$, $q = \tanh(A/2) = f_L - f_R$, and $\Pi_j = |j\rangle\langle j|$ is a projector of the single-particle space $\mathbb{C}^n$. We write $\beta(A) = 2A/\sinh A$ for the thermal term of the product, and $\log$ is the natural logarithm. The Legendre polynomial of degree k is written P_k , normalized by $P_k(1) = 1$, to keep it apart from the product P ; C_k is the Chebyshev polynomial of the first kind, $C_k(\cos z) = \cos(kz)$.

I. NONNEGATIVITY OF THE PRODUCT

The bounds of the main text are bounds on a product that must first be shown never to be negative. This fixes the lower end of the dimension bound of the main text. The class is that of the main text: Lindblad generators with bare Hamiltonian H_0 , drive X , and paired channels L_μ , $L_\mu^\dagger$ with rates $\gamma_\mu(n_\mu+1)$, $\gamma_\mu n_\mu$ and affinities $A_\mu = \log[(n_\mu+1)/n_\mu]$. Every member of the class produces entropy at a nonnegative rate, so the product of every counted channel with $j_\mu \neq 0$ is nonnegative, and so is its infimum in every dimension:

$$\sigma \geq 0, \quad P_\mu \geq 0, \quad P_{\min}^N \geq 0. \quad (\text{S1})$$

To see this, let ρ be the stationary state and v a vector with $\rho v = 0$. The diagonal element of $\mathcal{L}\rho = 0$ at v is $0 = \sum_\mu \gamma_\mu [(n_\mu+1)\langle v, L_\mu \rho L_\mu^\dagger v \rangle + n_\mu \langle v, L_\mu^\dagger \rho L_\mu v \rangle]$, because the Hamiltonian and anticommutator terms vanish at v . Every term is nonnegative, so $\rho^{1/2} L_\mu^\dagger v = \rho^{1/2} L_\mu v = 0$. The kernel of ρ is therefore invariant under every L_μ and $L_\mu^\dagger$, and so is the support of ρ . On the support write $\rho = \sum_i p_i |i\rangle\langle i|$ with $p_i > 0$, and put $w_{ij}^\mu = |\langle i | L_\mu | j \rangle|^2$, $\alpha_\mu = n_\mu + 1$, $\alpha'_\mu = n_\mu$. Inserting the dissipators into $-\text{Tr}(\dot{\rho} \log \rho)$ channel by channel gives the entropy flows and currents

$$\begin{aligned} \dot{S}_\mu &= \gamma_\mu \sum_{ij} w_{ij}^\mu (\alpha_\mu p_j - \alpha'_\mu p_i) \log \frac{p_j}{p_i}, \\ j_\mu &= \gamma_\mu \sum_{ij} w_{ij}^\mu (\alpha_\mu p_j - \alpha'_\mu p_i), \end{aligned} \quad (\text{S2})$$

with sums over the support. The Hamiltonian contributes nothing, since $\text{Tr}([H, \rho] \log \rho) = 0$, and stationarity gives $\sum_\mu \dot{S}_\mu = 0$. Adding this zero to $\sigma = \sum_\mu A_\mu j_\mu$ and using $A_\mu = \log(\alpha_\mu/\alpha'_\mu)$,

$$\sigma = \sum_{\mu, ij} \gamma_\mu w_{ij}^\mu (\alpha_\mu p_j - \alpha'_\mu p_i) \log \frac{\alpha_\mu p_j}{\alpha'_\mu p_i} \geq 0, \quad (\text{S3})$$

because $(x-y)\log(x/y) \geq 0$ for $x, y > 0$. Each channel thus contributes a relative-entropy-like sum of nonnegative terms. The variance rate is nonnegative by definition, so $P_\mu \geq 0$, which is Eq. (S1).

II. UNIQUE STATIONARY STATE AND RELAXATION

The current and its variance are long-time properties of steady operation, so they are defined only if the device forgets how it was prepared. This section shows that every chain relaxes to one stationary state from every initial state, and so does every two-contact network whose single-particle modes all decay. Coherences between Fock sectors of opposite parity are included.

Let $h \in \mathbb{C}^{n \times n}$ be Hermitian, $H = \sum_{ab} h_{ab} c_a^\dagger c_b$ on the full 2^n -dimensional Fock space, $\Gamma_L, \Gamma_R > 0$ and $0 < A < \infty$. The two-contact generator is

$$\mathcal{L}\rho = -i[H, \rho] + 2\Gamma_L(f_L \mathcal{D}[c_1^\dagger] + f_R \mathcal{D}[c_1])\rho + 2\Gamma_R(f_L \mathcal{D}[c_n] + f_R \mathcal{D}[c_n^\dagger])\rho. \quad (\text{S4})$$

Mode 1 is filled at rate $2\Gamma_L f_L$ and emptied at rate $2\Gamma_L f_R$; mode n is emptied at rate $2\Gamma_R f_L$ and filled at rate $2\Gamma_R f_R$, so every rate is positive at finite A . The chain of the main text is the case of real tridiagonal h with zero diagonal, $h_{j,j+1} = h_{j+1,j} = t_j > 0$, and $\Gamma_L = \Gamma_R = \Gamma$. Networks allow complex hoppings, arbitrary real on-site energies and unequal contacts. Put

$$\Sigma = \Gamma_L \Pi_1 + \Gamma_R \Pi_n, \quad K = -ih - \Sigma. \quad (\text{S5})$$

The matrix K is the single-particle drift: as shown in Sec. III, it generates the decay of the two-point functions. We call the network strictly stable when every eigenvalue of K has negative real part.

A. Every mode of a chain decays

A single-particle mode fails to decay only if it never reaches a contact. Precisely, (i) K is strictly stable if and only if no eigenvector of h has vanishing first and last components. Hence (ii) every chain with nonzero hoppings, real on-site energies and $\Gamma_L, \Gamma_R > 0$ is strictly stable.

For (i), if $Kx = \lambda x$ with $x \neq 0$, then

$$\text{Re } \lambda \|x\|^2 = \text{Re} \langle x, Kx \rangle = -\Gamma_L |x_1|^2 - \Gamma_R |x_n|^2 \leq 0, \quad (\text{S6})$$

since $\langle x, -ihx \rangle$ is imaginary. The real part therefore vanishes only when $x_1 = x_n = 0$. Then $\Sigma x = 0$ and $hx = i\lambda x$, so x is an eigenvector of h invisible to both contacts. Conversely such an eigenvector of h with eigenvalue ϵ satisfies $Kx = -i\epsilon x$. For (ii), in a chain an eigenvector with $x_1 = 0$ has, from the first row of $hx = \epsilon x$, $t_1 x_2 = (\epsilon - \epsilon_1)x_1 = 0$, and row by row $x_3 = \dots = x_n = 0$. No eigenvector of h therefore vanishes at the first mode.

Graph connectivity does not imply strict stability. On the diamond with $h_{12} = h_{13} = h_{24} = h_{34} = 1$, the vector $(0, 1, -1, 0)$ is an eigenvector of h with eigenvalue zero that vanishes at both contacts. Strict stability does not imply transmission either. With $h_{12} = h_{13} = h_{24} = 1$, $h_{34} = -1$, the subspaces spanned by $\{e_1, (e_2 + e_3)/\sqrt{2}\}$ and $\{e_4, (e_2 - e_3)/\sqrt{2}\}$ are invariant, each is a damped dimer, and $G_{41} \equiv 0$. The two hypotheses, strict stability and $I_1 > 0$, are therefore stated separately below.

B. A unique attracting stationary state on the full Fock space

Decaying modes are enough for the whole many-body state to relax. Let h be Hermitian, $\Gamma_L, \Gamma_R > 0$, $0 < A < \infty$, and let K be strictly stable. Then the generator (S4) has a unique stationary state ρ_* , which is faithful; zero is an algebraically simple eigenvalue of $\mathcal{L}$ on all $2^n \times 2^n$ matrices, every other eigenvalue has negative real part, and every matrix ρ relaxes as

$$e^{t\mathcal{L}}\rho \rightarrow \rho_* \text{Tr } \rho. \quad (\text{S7})$$

By Sec. II A, this holds for every chain with nonzero real hoppings, real on-site energies and $\Gamma_L, \Gamma_R > 0$. The argument has four steps. Only multiples of the identity commute with the Hamiltonian and all jumps, so every stationary state is faithful. The dissipation identity then excludes undamped oscillations, and contractivity excludes growth.

Write the four jumps as $B \in \{c_1^\dagger, c_1, c_n, c_n^\dagger\}$ with rates $w_B \in \{2\Gamma_L f_L, 2\Gamma_L f_R, 2\Gamma_R f_L, 2\Gamma_R f_R\}$, all positive. For $x \in \mathbb{C}^n$ put $c(x) = \sum_a x_a c_a$ and $c^\dagger(x) = \sum_a x_a c_a^\dagger$. The anticommutation relations give

$$[H, c(x)] = -c(h^\top x), \quad [H, c^\dagger(x)] = c^\dagger(hx). \quad (\text{S8})$$

The commutator with H thus moves a contact mode through the network, along $h^\top$ for annihilators and along h for creators.

Step 1: the commutant is trivial. Let Z commute with H and with the four jumps. The set $V = \{x : [Z, c(x)] = 0\}$ is a subspace containing e_1 and e_n . By the Jacobi identity and Eq. (S8), $[Z, c(h^\top x)] = -[Z, [H, c(x)]] = 0$ for $x \in V$, so V is invariant under $h^\top$. Its orthogonal complement is then invariant under $(h^\top)^\dagger = \bar{h}$ and orthogonal to e_1 and e_n . If it were nonzero, it would contain an eigenvector y of $\bar{h}$ with $y_1 = y_n = 0$, and $\bar{y}$ would be an eigenvector of h with the same real eigenvalue and the same vanishing components, contradicting Sec. II A. Hence $V = \mathbb{C}^n$. The same argument with $V' = \{x : [Z, c^\dagger(x)] = 0\}$, which is invariant under h , gives $V' = \mathbb{C}^n$. So Z commutes with every c_a and $c_a^\dagger$. These generate the full matrix algebra: products of $\hat{n}_a$ and $1 - \hat{n}_a$ give the projections onto Fock basis states, and single c_a or $c_a^\dagger$ flip one occupation. So Z is a multiple of the identity.

Step 2: every stationary state is faithful. A stationary density matrix exists, as a limit point of the time averages $\tau^{-1} \int_0^\tau e^{t\mathcal{L}} \rho dt$. Let ρ be one, with support projection P . As in Sec. I, the diagonal elements of $\mathcal{L}\rho = 0$ on the kernel of ρ force $(I - P)BP = 0$ for all four jumps. The set of jumps is closed under adjoints, so $[P, B] = 0$. The dissipators then have no off-diagonal blocks with respect to P , and the off-diagonal block of $\mathcal{L}\rho = 0$ reads $(I - P)HP\rho = 0$. As ρ is invertible on its support, $(I - P)HP = 0$, so $[P, H] = 0$. By Step 1, $P = I$.

Step 3: no other eigenvalue on the imaginary axis. For any matrix Y , expanding the adjoint generator $\mathcal{L}^\dagger$ gives the dissipation identity

$$\mathcal{L}^\dagger(Y^\dagger Y) - \mathcal{L}^\dagger(Y^\dagger)Y - Y^\dagger \mathcal{L}^\dagger(Y) = \sum_B w_B [B, Y]^\dagger [B, Y]. \quad (\text{S9})$$

The Hamiltonian part cancels on the left, and each dissipator leaves the squared commutator of its jump with Y . Suppose $\mathcal{L}^\dagger Y = i\xi Y$ with real ξ , so that $\mathcal{L}^\dagger Y^\dagger = -i\xi Y^\dagger$. Taking the expectation of Eq. (S9) in a faithful stationary state makes the left side vanish, so $[B, Y] = 0$ for every jump. The dissipative part of $\mathcal{L}^\dagger$ then annihilates Y , and $[H, Y] = \xi Y$. Now, for $x \in V = \{x : [Y, c(x)] = 0\}$, $[Y, [H, c(x)]] = [[Y, H], c(x)] + [H, [Y, c(x)]] = -\xi[Y, c(x)] = 0$, so V is invariant under $h^\top$, and likewise for creators. As in Step 1, Y commutes with every c_a and $c_a^\dagger$, so $Y = zI$, and $[H, Y] = \xi Y$ forces $\xi = 0$ unless $Y = 0$. Thus the kernel of $\mathcal{L}^\dagger$ is spanned by I , the kernel of $\mathcal{L}$ is one-dimensional, and no nonzero imaginary eigenvalue exists.

Step 4: no growth. The adjoint semigroup $e^{t\mathcal{L}^\dagger}$ is unital and completely positive, so the Kadison–Schwarz inequality gives $\|e^{t\mathcal{L}^\dagger} Y\| \leq \|Y\|$ for all $t \geq 0$. An eigenvalue with positive real part, or a nontrivial Jordan block on the imaginary axis, would make $e^{t\mathcal{L}^\dagger}$ grow. Hence zero is algebraically simple and every other eigenvalue has negative real part, which gives Eq. (S7).

The argument never restricts Y to even operators, so it excludes persistent coherences between Fock sectors of opposite parity as well. The decay rate it implies depends on n , h , Γ_L , Γ_R and A , and we do not bound it uniformly.

III. EXACT CURRENT AND VARIANCE AT FINITE AFFINITY

The product formula of the main text separates partition noise from thermal noise, and every bound in the paper is a bound on one of these two terms. This section derives that formula from the full-Fock generator at finite affinity. The counting field gives the current and variance as derivatives of one eigenvalue. The two-point functions close, the stationary state is Gaussian, and two Fourier identities reduce the variance to integrals of the transmission.

A. Counting statistics of a channel

For any member of the class and counted channel μ , the tilted generator is $\mathcal{L}_s = \mathcal{L} + (e^s - 1)\mathcal{J}_+ + (e^{-s} - 1)\mathcal{J}_-$, with $\mathcal{J}_+\rho = \gamma_\mu(n_\mu + 1)L_\mu\rho L_\mu^\dagger$ and $\mathcal{J}_-\rho = \gamma_\mu n_\mu L_\mu^\dagger \rho L_\mu$. Put $\mathcal{K}_1 = \mathcal{J}_+ - \mathcal{J}_-$ and $\mathcal{K}_2 = \mathcal{J}_+ + \mathcal{J}_-$. Let zero be an algebraically simple eigenvalue of $\mathcal{L}$, with the rest of the spectrum in the open left half-plane. Then the eigenvalue $\lambda(s)$ of $\mathcal{L}_s$ continuing zero is analytic near $s = 0$, and it is the scaled cumulant generating function of the count. Expanding $\mathcal{L}_s \rho_s = \lambda(s) \rho_s$ with $\text{Tr } \rho_s = 1$ to second order in s gives, with ρ the stationary state,

$$\begin{aligned} j &= \text{Tr}(\mathcal{K}_1 \rho), & \mathcal{L}\eta &= j\rho - \mathcal{K}_1 \rho, & \text{Tr } \eta &= 0, \\ d &= \text{Tr}(\mathcal{K}_2 \rho) + 2 \text{Tr}(\mathcal{K}_1 \eta), \end{aligned} \quad (\text{S10})$$

where $\eta = \partial_s \rho_s|_{s=0}$ is the first-order response of the stationary state to the counting field. The trace of the first-order equation gives $j = \lambda'(0)$, and the trace of the second-order equation gives $d = \lambda''(0)$, the full variance rate with no factor $1/2$.

B. Two-contact networks

For the generator (S4) we count the right contact, extraction minus injection, so $\mathcal{J}_+\rho = 2\Gamma_R f_L c_n \rho c_n^\dagger$ and $\mathcal{J}_-\rho = 2\Gamma_R f_R c_n^\dagger \rho c_n$. Let $G(E) = (E - h + i\Sigma)^{-1}$ and

$$T(E) = 4\Gamma_L \Gamma_R |G_{n1}(E)|^2, \quad I_1 = \int_{\mathbb{R}} T \frac{dE}{2\pi}, \quad I_2 = \int_{\mathbb{R}} T(1-T) \frac{dE}{2\pi}, \quad F_0 = \frac{I_2}{I_1}. \quad (\text{S11})$$

For the chain this is the transmission of the main text, with $4\Gamma_L \Gamma_R = 4\Gamma^2$.

The current and variance are then fixed by the two transmission integrals. Under the hypotheses of Sec. IIB, and if $I_1 > 0$, the transmission obeys $0 \leq T \leq 1$, $0 < F_0 < 1$, and the right-contact count has

$$j = qI_1, \quad d = q^2 I_2 + 2f_L f_R I_1, \quad \sigma = 2Aqj, \quad (\text{S12})$$

so that

$$P = \frac{\sigma d}{j^2} = 2AqF_0 + \beta(A), \quad \beta(A) = \frac{2A}{\sinh A}. \quad (\text{S13})$$

The generator belongs to the coherently driven thermal-jump class with $H_0 = A(I - \hat{n}_1 + \hat{n}_n)$ and $X = H - H_0$. Every chain with nonzero real hoppings, real on-site energies and $\Gamma_L, \Gamma_R > 0$ satisfies these hypotheses, with $I_1 > 0$ by Eq. (S26).

For boundary-driven chains the formulas (S12) were obtained by Meier et al. [2] from the scattering description. The proof below establishes that they give the long-time cumulants of the full-Fock generator (S4), for chains and for networks with complex hoppings and unequal contacts. Equation (S13) shows the two noise sources: partition noise, weighted by $2AqF_0$, and thermal noise, $\beta(A)$, which decays only when the affinity grows.

Thermal class. Take the forward channels $L_L = c_1^\dagger$ and $L_R = c_n$, with $n_A = 1/(e^A - 1)$, $\gamma_\alpha = 2\Gamma_\alpha q$ and temperature one. Then $[H_0, c_1^\dagger] = -Ac_1^\dagger$ and $[H_0, c_n] = -Ac_n$, so both gaps equal A , and $\gamma_\alpha(n_A + 1) = 2\Gamma_\alpha f_L$, $\gamma_\alpha n_A = 2\Gamma_\alpha f_R$. These are the rates of (S4), and the class allows the drive X to be chosen independently of the baths. The total number $\sum_a \hat{n}_a$ commutes with H ; left forward jumps raise it and right forward jumps lower it, so the stationary net currents of the two channels are equal. Each carries affinity A , hence $\sigma = 2Aqj$.

Two-point closure. For any operator Y define its moment matrix $C[Y]_{ab} = \text{Tr}(c_b^\dagger c_a Y)$. Inserting the adjoint dissipators, $\mathcal{D}[c_p]^\dagger(c_b^\dagger c_a) = -\frac{1}{2}(\delta_{pa} + \delta_{pb})c_b^\dagger c_a$ and $\mathcal{D}[c_p^\dagger]^\dagger(c_b^\dagger c_a) = -\frac{1}{2}(\delta_{pa} + \delta_{pb})c_b^\dagger c_a + \delta_{pa}\delta_{pb}I$, together with $[H, c_b^\dagger c_a]$, gives for every operator Y

$$C[\mathcal{L}Y] = KC[Y] + C[Y]K^\dagger + Q_s \text{Tr} Y, \quad Q_s = 2\Gamma_L f_L \Pi_1 + 2\Gamma_R f_R \Pi_n. \quad (\text{S14})$$

The Hamiltonian contributes $-i[h, C[Y]]$, each contact damps the row and column of its mode at the total rate $2\Gamma_\alpha$, and the gain terms feed Q_s . The identity is linear in Y and needs no positivity, Gaussianity or parity of Y . With $U(t) = e^{Kt}$, strict stability makes U decay exponentially, Jordan blocks included. Since $\frac{d}{dt}(UU^\dagger) = -2U\Sigma U^\dagger$,

$$2 \int_0^\infty U\Sigma U^\dagger dt = I. \quad (\text{S15})$$

The stationary covariance $C \equiv C[\rho_*]$ solves Eq. (S14) with left side zero. Writing $Q_s = 2f_R \Sigma + 2q\Gamma_L \Pi_1$ and using Eq. (S15),

$$C = \int_0^\infty UQ_s U^\dagger dt = f_R I + q\Omega, \quad \Omega = 2\Gamma_L \int_0^\infty U\Pi_1 U^\dagger dt. \quad (\text{S16})$$

Equation (S15) also gives $0 \preceq \Omega \preceq I$, so the eigenvalues of C lie in $[f_R, f_L] \subset (0, 1)$: every mode is partially filled at finite affinity, and Ω measures how much of the left bath each mode sees.

Gaussian stationary state. Quadratic Lindblad generators with linear jumps have Gaussian stationary states [3]; for the present generator we prove it directly. Diagonalize $C = O \text{diag}(k_\ell) O^\dagger$ with O unitary and $0 < k_\ell < 1$. With modes $\tilde{c}_\ell = \sum_a \overline{O_{a\ell}} c_a$, define

$$\rho_G = \prod_\ell [k_\ell \tilde{c}_\ell^\dagger \tilde{c}_\ell + (1 - k_\ell) \tilde{c}_\ell \tilde{c}_\ell^\dagger]. \quad (\text{S17})$$

It is positive definite with unit trace, its eigenvalues are products of factors k_ℓ and $1 - k_\ell$, and $\text{Tr}(c_b^\dagger c_a \rho_G) = \sum_\ell O_{a\ell} O_{b\ell} k_\ell = C_{ab}$. Because each normal mode is independent, its four-point functions obey Wick's rule:

$$\text{Tr}(c_b^\dagger c_a c_k^\dagger c_l \rho_G) - C_{ab} C_{lk} = C_{lb}(\delta_{ak} - C_{ak}). \quad (\text{S18})$$

The state ρ_G is proportional to $\exp(\sum_{ab} c_a^\dagger J_{ab} c_b)$ with J Hermitian. Conjugation by such an exponential maps each c_a to a linear combination of the c_b , and each $c_a^\dagger$ to one of the $c_b^\dagger$. Hence every term of $\mathcal{L}\rho_G$ is a polynomial times ρ_G : for instance $c_n \rho_G c_n^\dagger = c_n (\rho_G c_n^\dagger \rho_G^{-1}) \rho_G$. Each polynomial is quadratic and conserves particle number, so after normal ordering

$$\mathcal{L}\rho_G = \left(\sum_{kl} Y_{kl} c_k^\dagger c_l + y \right) \rho_G \quad (\text{S19})$$

for some complex matrix Y and number y . Trace preservation gives $y = -\sum_{kl} Y_{kl} C_{lk}$, and Eq. (S18) then gives $C[\mathcal{L}\rho_G] = (I - C)YC$. Equation (S14) holds for every operator, and its right side vanishes at $C[\rho_G] = C$, so $(I - C)YC = 0$. Both C and $I - C$ are invertible, so $Y = 0$, $y = 0$, and $\mathcal{L}\rho_G = 0$. By the uniqueness of Sec. IIB, $\rho_G = \rho_*$. In particular ρ_* is faithful because every k_ℓ lies strictly between 0 and 1.

Current. Let $v = e_n$, $x = Cv$ and $\omega = \Omega v$. From Eq. (S10), $j = 2\Gamma_R[f_L C_{nn} - f_R(1 - C_{nn})] = 2\Gamma_R(C_{nn} - f_R) = 2\Gamma_R q \Omega_{nn}$. With the Fourier convention $\hat{f}(E) = \int e^{iEt} f(t) dt$, the stable propagator obeys $\int_0^\infty e^{iEt} U(t) dt = iG(E)$. Let $\varphi(t) = \langle n|U(t)|1 \rangle$ for $t \geq 0$ and $\varphi(t) = 0$ for $t < 0$, so $|\hat{\varphi}| = |G_{n1}|$. Plancherel's theorem gives $I_1 = 4\Gamma_L \Gamma_R \int_0^\infty |\varphi|^2 dt = 2\Gamma_R \Omega_{nn}$, hence $j = qI_1$.

Response. Let $\Theta = C[\eta]$. Wick's rule gives the two contractions

$$C[c_n \rho_* c_n^\dagger] = C_{nn} C - x x^\dagger, \quad C[c_n^\dagger \rho_* c_n] = (1 - C_{nn})C + (v - x)(v - x)^\dagger. \quad (\text{S20})$$

In the first, the direct pairing of $\langle c_n^\dagger c_b^\dagger c_a c_n \rangle$ gives $C_{nn} C_{ab}$ and the exchange pairing gives $-C_{an} C_{nb}$. Applying Eq. (S14) to the response equation of (S10), with $\text{Tr} \eta = 0$, the terms proportional to C cancel because $j = 2\Gamma_R(C_{nn} - f_R)$. Substituting $x = f_R v + q\omega$, the cross terms cancel as well, and

$$K\Theta + \Theta K^\dagger = 2\Gamma_R[f_L x x^\dagger + f_R(v - x)(v - x)^\dagger] = 2\Gamma_R[q^2 \omega \omega^\dagger + f_L f_R \Pi_n]. \quad (\text{S21})$$

Its unique solution is $\Theta = -2\Gamma_R \int_0^\infty U[q^2 \omega \omega^\dagger + f_L f_R \Pi_n] U^\dagger dt$. The response is thus driven by two sources at the counted contact: the bias-induced correlation ω and the thermal occupation noise $f_L f_R$.

Variance. Since $\text{Tr} \eta = 0$, $\text{Tr}(\mathcal{K}_1 \eta) = 2\Gamma_R \Theta_{nn}$, and $\text{Tr}(\mathcal{K}_2 \rho_*) = 2\Gamma_R(f_R + qC_{nn}) = 4\Gamma_R f_L f_R + q^2 I_1$. Therefore

$$d = 4\Gamma_R f_L f_R + q^2 I_1 - 8\Gamma_R^2 \int_0^\infty \left[q^2 |\langle n|U(t)\omega\rangle|^2 + f_L f_R |\langle n|U(t)|n\rangle|^2 \right] dt. \quad (\text{S22})$$

Two identities reduce the integrals. The nn element of Eq. (S15) reads

$$8\Gamma_R^2 \int_0^\infty |\langle n|U(t)|n\rangle|^2 dt = 4\Gamma_R - 2I_1. \quad (\text{S23})$$

For the other, let $\psi(t) = \int_{\mathbb{R}} \overline{\varphi(s)} \varphi(s+t) ds$. For $t \geq 0$ the semigroup law and Eq. (S16) give

$$\langle n|U(t)\omega\rangle = 2\Gamma_L \int_0^\infty \langle n|U(t+s)|1\rangle \langle 1|U(s)^\dagger|n\rangle ds = 2\Gamma_L \psi(t). \quad (\text{S24})$$

This step keeps the ordered amplitude $\langle n|U|1\rangle$ throughout and uses no reciprocity, so it holds for complex h . Since $\psi(-t) = \overline{\psi(t)}$ and $\hat{\psi} = |\hat{\varphi}|^2 = |G_{n1}|^2$, Plancherel's theorem on the full line gives

$$\int_{\mathbb{R}} T^2 \frac{dE}{2\pi} = 16\Gamma_L^2 \Gamma_R^2 \int_{\mathbb{R}} |\psi|^2 dt = 8\Gamma_R^2 \int_0^\infty |\langle n|U(t)\omega\rangle|^2 dt. \quad (\text{S25})$$

Inserting Eqs. (S23) and (S25) into Eq. (S22), the terms proportional to Γ_R cancel and $d = q^2(I_1 - \int T^2 dE/2\pi) + 2f_L f_R I_1$, which is Eq. (S12). With $4f_L f_R/q = 2/\sinh A$, the product $P = 2Ad/j$ becomes Eq. (S13). Exponential decay of U justifies every exchange of integrals.

Range of T . Let $W = (\sqrt{\Gamma_L} e_1, \sqrt{\Gamma_R} e_n)$, so $WW^\dagger = \Sigma$, and $\mathcal{S}(E) = I - 2iW^\dagger G(E)W$. For real E the resolvent identity $G - G^\dagger = -2iG\Sigma G^\dagger$ makes $\mathcal{S}$ unitary, and $|\mathcal{S}_{21}|^2 = T$, so $0 \leq T \leq 1$. Strict stability removes real poles, and

$G_{n1}(E) = O(E^{-2})$ because the two contact modes are orthogonal, so $T = O(E^{-4})$ is integrable. Since $1 - T = |\mathcal{S}_{11}|^2$ and $\mathcal{S}_{11}(E) \rightarrow 1$ as $|E| \rightarrow \infty$, the rational function $1 - T$ is not identically zero, so $T < 1$ except at finitely many energies; with $I_1 > 0$ this gives $0 < I_2 < I_1$. For a chain, Eq. (S26) below shows $T > 0$ on the whole real axis, so $I_1 > 0$ holds automatically. This completes Eqs. (S12) and (S13).

As a check of the normalization, the chain with $n = 2$ and $t_1 = \Gamma$ has $I_1 = \Gamma/2$ and $F_0 = 1/4$, so at $A = \log 2$ the formulas give $j = \Gamma/6$, $d = 17\Gamma/72$ and $P = 17 \log 2/6$. Under the common scaling $(h, \Gamma_L, \Gamma_R) \mapsto \lambda(h, \Gamma_L, \Gamma_R)$ with $\lambda > 0$, $T_\lambda(E) = T(E/\lambda)$, so I_1, I_2, j, d scale by λ while F_0 and P are unchanged.

IV. TRANSMISSION OF A CHAIN AND THE LOWER BOUND ON PARTITION NOISE

Partition noise lives where the transmission is neither near zero nor near one, so its size is set by how steeply a finite chain can switch its transmission. This section shows that the transmission of a chain is $1/(1 + Q)$ with Q a polynomial of degree $2n$. Markov's inequality then bounds how fast Q can rise at every window edge. The result is the lower bound on partition noise and on the product used in the main text.

A. The transmission is $1/(1 + Q)$

In this section a chain may carry arbitrary real on-site energies $\varepsilon_1, \dots, \varepsilon_n$, nonzero real hoppings $t_1, \dots, t_{n-1}$ and contact linewidths $\Gamma_L, \Gamma_R > 0$ on modes 1 and n ; equal contacts are the case $\Gamma_L = \Gamma_R = \Gamma$. A chain is a two-contact network whose drift is strictly stable by Sec. II A, so Eqs. (S12) and (S13) apply to it. Put $h_- = h - i\Gamma_L\Pi_1 - i\Gamma_R\Pi_n$, $h_z = h + i\Gamma_L\Pi_1 - i\Gamma_R\Pi_n$, $D(E) = \det(E - h_-)$, $Z(E) = \det(E - h_z)$ and $\tau = \prod_{j=1}^{n-1} t_j$. Deleting the first row and last column of $E - h_-$ leaves an upper triangular matrix: its entry in row l and column l is $-t_l$, and every entry below the diagonal comes from a pair of modes more than one site apart. The on-site energies and the linewidths sit only on the diagonal of $E - h_-$, in positions that are either deleted or above the diagonal of the minor. The cofactor sign cancels the $n - 1$ minus signs. Hence

$$G_{n1}(E) = \frac{\tau}{D(E)}, \quad (\text{S26})$$

with a numerator independent of energy, of the on-site energies and of the linewidths. The left reflection amplitude $\mathcal{S}_{11} = 1 - 2i\Gamma_L G_{11}$ follows from the matrix determinant lemma applied to the rank-one change $E - h_z = E - h_- - 2i\Gamma_L\Pi_1$:

$$\mathcal{S}_{11}(E) = \frac{Z(E)}{D(E)}, \quad |\mathcal{S}_{11}|^2 = 1 - T. \quad (\text{S27})$$

Since $T = 4\Gamma_L\Gamma_R|G_{n1}|^2$, unitarity gives $|D(E)|^2 = 4\Gamma_L\Gamma_R\tau^2 + |Z(E)|^2$ for real E .

The transmission is therefore a rational function with a constant numerator. For every chain with nonzero hoppings, real on-site energies and $\Gamma_L, \Gamma_R > 0$, and every real E ,

$$T(E) = \frac{1}{1 + Q(E)}, \quad Q(E) = \frac{Z(E)\overline{Z(E)}}{4\Gamma_L\Gamma_R\tau^2}, \quad (\text{S28})$$

where Q is a real polynomial of degree exactly $2n$, with positive leading coefficient $1/(4\Gamma_L\Gamma_R\tau^2)$ and $Q \geq 0$ on the real line. Indeed, by Eqs. (S26) and (S27), $T = 4\Gamma_L\Gamma_R\tau^2/(4\Gamma_L\Gamma_R\tau^2 + |Z|^2)$. The product $Z(E)\overline{Z(E)}$ is invariant under conjugation of its coefficients, equals $|Z(E)|^2$ for real E , and is monic of degree $2n$ because Z is.

The polynomial Q measures reflection relative to transmission. Its real zeros are the energies of perfect transmission, and they may be degenerate. The coefficients of Z can be complex for a chain without mirror symmetry: for $n = 3$ with hoppings (t_1, t_2) , zero on-site energies and equal contacts, $Z = E^3 + (\Gamma^2 - t_1^2 - t_2^2)E + i\Gamma(t_2^2 - t_1^2)$, so the conjugation in Eq. (S28) is needed. Equation (S28) rests on the constant numerator of Eq. (S26); a general network has an energy-dependent cofactor and need not have this form.

B. Markov's inequality and the window edges

Markov's inequality [4] states that every real polynomial p of degree at most d obeys

$$\max_{[-1,1]} |p'| \leq d^2 \max_{[-1,1]} |p|; \quad (\text{S29})$$

see also the statement in Ref. [5]. A polynomial bounded on an interval cannot rise faster than its degree squared allows.

Applied at the edges of every window where $Q \leq 1$, this bounds the partition noise of every profile $1/(1+Q)$ from below. Let Q be a real polynomial, nonnegative on $\mathbb{R}$, of even degree $d \geq 2$ with positive leading coefficient, and $T = 1/(1+Q)$. Then

$$\frac{\int_{\mathbb{R}} T(1-T) dx}{\int_{\mathbb{R}} T dx} \geq \frac{2}{9d^2}. \quad (\text{S30})$$

Since $Q(x) \geq c|x|^d$ for large $|x|$ and some $c > 0$, both integrals converge, and the denominator is positive. Let $S = \{x : Q(x) \leq 1\}$. The nonzero polynomial $Q - 1$ has finitely many real zeros, so S is a finite union of disjoint closed intervals of positive length and finitely many points; it is compact and possibly empty. Points have measure zero. Each interval $I = [a, b]$ of length $\ell = b - a$ has $Q(a) = Q(b) = 1$, since otherwise it would extend, and $0 \leq Q \leq 1$ on I . Applying Eq. (S29) to $p(y) = 2Q(\frac{a+b}{2} + \frac{\ell}{2}y) - 1$, which satisfies $|p| \leq 1$ on $[-1, 1]$ and $p'(y) = \ell Q'(\frac{a+b}{2} + \frac{\ell}{2}y)$, gives

$$\max_I |Q'| \leq \frac{d^2}{\ell}. \quad (\text{S31})$$

Put $\delta = \ell/(2d^2)$. The strips $[a, a + \delta]$ and $[b - \delta, b]$ are disjoint because $2\delta < \ell$, and by Eq. (S31) each has $\frac{1}{2} \leq Q \leq 1$. There $T(1-T) = Q/(1+Q)^2 \geq 2/9$, since $Q/(1+Q)^2$ increases on $[0, 1]$. Hence

$$\int_I T(1-T) dx \geq \frac{2}{9} \cdot \frac{\ell}{d^2} \geq \frac{2}{9d^2} \int_I T dx, \quad (\text{S32})$$

using $T \leq 1$ in the last step. Each window edge thus carries partition noise proportional to the window length divided by d^2 . Off S , $Q > 1$ and $T(1-T) = TQ/(1+Q) \geq T/2 \geq 2T/(9d^2)$. Summing over the intervals and integrating over the complement proves Eq. (S30).

The argument divides by neither the length of S nor the total transmission, so it covers empty S , touching minima, degenerate zeros, and several disconnected windows.

C. No chain filters faster than $1/n^2$

For a chain this gives the bound on partition noise of the main text, and with the thermal term the bound on the product. Every chain with $n \geq 2$ modes, nonzero hoppings, real on-site energies and contact linewidths $\Gamma_L, \Gamma_R > 0$ obeys

$$F_0 \geq \frac{1}{18n^2}. \quad (\text{S33})$$

For $n \geq 3$ and every finite $A > 0$,

$$P \geq \frac{\log n}{18n^2}. \quad (\text{S34})$$

No mirror symmetry, bound on coupling ratios, or current floor is assumed.

Equation (S28) gives $T = 1/(1+Q)$ with $d = 2n$, and the factor $1/2\pi$ cancels in F_0 , so Eq. (S30) gives Eq. (S33). For the product we split the affinity axis at $A = \log n$ in Eq. (S13). If $A \geq \log n$, then $q \geq (n-1)/(n+1) \geq \frac{1}{2}$ and $P \geq AF_0 \geq \log n/(18n^2)$. If $0 < A \leq \log n$, we use that β is decreasing: its derivative has the sign of $\sinh A - A \cosh A$, which vanishes at zero and has derivative $-A \sinh A < 0$. Hence

$$P \geq \beta(A) \geq \beta(\log n) = \frac{4n \log n}{n^2 - 1} \geq \frac{\log n}{18n^2}, \quad (\text{S35})$$

the last step being $72n^3 \geq n^2 - 1$; this is Eq. (S34).

At small affinity the thermal term alone exceeds the bound, and at large affinity the partition term alone does. Neither term can vanish without the other growing, which is why the bound decays only as $\log n/n^2$.

V. THE GEGENBAUER PROFILE

The lower bound asks for a profile whose small values spread over the window and whose rise at the edges is nearly as steep as Markov's inequality permits. This section shows that the normalized Gegenbauer polynomial has both properties, so that its partition noise falls below $6/n^2$, including both infinite tails.

For $n \geq 1$ let

$$r_n(x) = \frac{2P'_{n+1}(x)}{(n+1)(n+2)}, \quad T_n(x) = \frac{1}{1+r_n(x)^2}, \quad (\text{S36})$$

and $\mu_n = \int_{\mathbb{R}} T_n dx$, $\nu_n = \int_{\mathbb{R}} T_n(1-T_n) dx$. The derivative formula for Jacobi polynomials [6, Eq. 18.9.15] gives $r_n = P_n^{(1,1)}/(n+1)$, and the relation to ultraspherical polynomials [6, Eq. 18.7.1] gives $r_n(x) = C_n^{(3/2)}(x)/C_n^{(3/2)}(1)$, with $C_n^{(\lambda)}$ the Gegenbauer polynomial. This normalized Gegenbauer polynomial is a classical filter prototype [7]; the integral bounds below and their use for the uncertainty product are ours.

The transmitted weight of the profile stays close to the window length 2, and its partition noise is of order n^{-2} : for every integer $n \geq 1$,

$$2 - \frac{4}{(n+1)(n+2)} \leq \mu_n \leq 2 + \frac{8}{n(n+3)}, \quad \frac{\nu_n}{\mu_n} \leq B_G(n) = \frac{6}{n(n+3)} + \frac{8}{n^2(n+3)^2} < \frac{6}{n^2}. \quad (\text{S37})$$

The derivation bounds the band and the tails separately.

Endpoint expansion. The explicit Jacobi series [6, Eq. 18.5.7] with $\alpha = \beta = 1$ and $x = 1 + s$ gives, for every real s ,

$$r_n(1+s) = \sum_{k=0}^n e_{nk} s^k, \quad e_{nk} = \frac{n!(n+k+2)!}{(n-k)!(n+2)!2^k k!(k+1)!} > 0. \quad (\text{S38})$$

Thus $e_{n0} = 1$, $e_{n1} = n(n+3)/4$, and the leading coefficient $e_{nn} = (2n+2)!/[2^n(n+2)!(n+1)!]$ is positive, so r_n^2 has degree exactly $2n$. Parity [6, Table 18.6.1] gives $r_n(-x) = (-1)^n r_n(x)$, so T_n is even. Every coefficient being positive means the profile rises at least linearly outside the window, at rate $n(n+3)/4$, all the way to infinity.

In-band norm. Put $m = n+1$. From $P_m(\pm 1) = (\pm 1)^m$ and $P'_m(\pm 1) = (\pm 1)^{m+1}m(m+1)/2$, and since P''_m has degree below m and is orthogonal to P_m with unit weight, integration by parts gives $\int_{-1}^1 (P'_m)^2 dx = [P_m P'_m]_{-1}^1 = m(m+1)$. Hence

$$\int_{-1}^1 r_n^2 dx = \frac{4}{(n+1)(n+2)} \equiv a_n. \quad (\text{S39})$$

The ripple inside the window has mean square of order n^{-2} . With $(1+Q)^{-1} \geq 1-Q$ and $Q/(1+Q)^2 \leq Q$ pointwise for $Q = r_n^2$, $2 - a_n \leq \int_{-1}^1 T_n \leq 2$ and $\int_{-1}^1 T_n(1-T_n) \leq a_n$.

Both tails. With $b_n = n(n+3)/4$, Eq. (S38) gives $r_n(1+s) \geq 1 + b_n s$ for all $s \geq 0$, so

$$\int_1^\infty T_n dx \leq \int_0^\infty \frac{ds}{(1+b_n s)^2} = \frac{1}{b_n}. \quad (\text{S40})$$

The whole out-of-band mass is thus of order n^{-2} , with no cutoff. By evenness and $T_n(1-T_n) \leq T_n$, both tails together contribute at most $2/b_n = 8/[n(n+3)]$ to μ_n and to ν_n .

Band and tails together. Adding band and tails gives the bounds on μ_n and $\nu_n \leq a_n + 8/[n(n+3)]$. With $z = n(n+3)$ and $(n+1)(n+2) = z+2$, $\nu_n/\mu_n \leq [4/(z+2) + 8/z]/[2 - 4/(z+2)] = (6z+8)/z^2 = B_G(n)$, and $6/n^2 - B_G(n) = (18n+46)/[n^2(n+3)^2] > 0$, which is Eq. (S37).

For a common scale $w > 0$, the transmission $T_n(E/w)$ has $\int T_n(E/w) dE = w\mu_n$, and $F_0 = \nu_n/\mu_n$ is unchanged. At $n=1$, $T_1 = (1+x^2)^{-1}$ gives $\mu_1 = \pi$ and $\nu_1/\mu_1 = 1/2 < B_G(1) = 2$; the next profiles are $r_2 = (5x^2-1)/4$ and $r_3 = (7x^3-3x)/4$.

Where the ripple and the edge are. On the window $|r_n| \leq 1$, with equality at $x = \pm 1$ [6, §18.14], so the ripple reaches its largest value at the edges. Its mean square (S39) is nevertheless of order n^{-2} , so the bulk transmits almost perfectly. At the edge the slope of $Q = r_n^2$ is $Q'(1) = n(n+3)/2$, within a factor 4 of the Markov bound (S31), $d^2/\ell = 2n^2$ for $d = 2n$ and $\ell = 2$. The profile therefore leaves the window within a factor 4 of the steepest slope a polynomial of its degree can have, and its noise has the order n^{-2} of Eq. (S30). The logarithm-free order is sharp for the scalar inequality: $Q = r_n^2$ has degree $d = 2n$ and $d^2\nu_n/\mu_n \leq 4n^2 B_G(n) < 24$, so no bound of the form $d^2 F_0 \geq c \log d$ holds for all nonnegative polynomials.

VI. EXACT REALIZATION BY AN EQUAL-CONTACT CHAIN AT EVERY LENGTH

The profile (S36) becomes physical only if a chain with equal contacts transmits exactly $T_n(E/w)$. The construction below inverts the two-contact scattering problem: the target fixes the denominator D of Eq. (S26) through a spectral factorization, the denominator fixes an n -point spectral measure, and the measure fixes the Jacobi matrix.

Algorithm. Fix $n \geq 2$ and work at $w = 1$.

1. Let $\kappa = 1/e_{nn}$ and $M(z) = \kappa r_n(z)$, the monic polynomial proportional to the target; M has real coefficients and parity n .
2. The polynomial $F = M^2 + \kappa^2$ is positive on the real line and has real coefficients, so it has exactly n roots, with multiplicity, in each open half-plane. Let D be the monic polynomial whose roots are those in the lower half-plane, and write $D = D_1 + iD_2$ with D_1, D_2 of real coefficients.
3. Put $\Gamma = \frac{1}{2}[z^{n-1}]D_2$, $p = (D_1 + M)/2$ and $q_D = D_2/(2\Gamma)$.
4. Let $x_1 < \dots < x_n$ be the roots of p and $\rho_s = \kappa/(2\Gamma|p'(x_s)|)$.
5. Let h be the Jacobi matrix of the measure $\sum_s \rho_s \delta_{x_s}$: Gram–Schmidt orthonormalize $1, x, \dots, x^{n-1}$ in this measure with positive leading coefficients, and take the matrix of multiplication by x .
6. Multiply h and Γ by w .

The algorithm always succeeds, and the chain it returns transmits the target exactly. For every integer $n \geq 2$ and every $w > 0$, the algorithm terminates and outputs a real $n \times n$ Jacobi matrix with zero diagonal and positive hoppings $t_j = t_{n-j}$, and a contact $\Gamma > 0$, such that the equal-contact chain transmits

$$4\Gamma^2 |(E - h + i\Gamma(\Pi_1 + \Pi_n))_{n1}^{-1}|^2 = \frac{1}{1 + r_n(E/w)^2} \quad \text{for all real } E. \quad (\text{S41})$$

Moreover $\det(E - h + i\Gamma(\Pi_1 + \Pi_n)) = D(E)$ and $\prod_j t_j = \kappa/(2\Gamma)$ at $w = 1$.

Steps (a) to (g) below derive this. By the scaling remark of Sec. III, step 6 maps $T(E)$ to $T(E/w)$, so it suffices to take $w = 1$.

(a) *Spectral factor.* The factorization $DD^\# = F$, with $D^\#(z) = \overline{D(\bar{z})} = D_1 - iD_2$, reads

$$D_1^2 + D_2^2 = M^2 + \kappa^2. \quad (\text{S42})$$

The polynomial D_1 is monic of degree n and D_2 has degree at most $n - 1$. If the lower roots are $u_j - iv_j$ with $v_j > 0$, then $[z^{n-1}]D_2 = \sum_j v_j > 0$, so $\Gamma > 0$ and q_D is monic of degree $n - 1$.

(b) *Two Blaschke channels.* Define

$$\mathcal{S}(z) = \frac{1}{D(z)} \begin{pmatrix} M(z) & -i\kappa \\ -i\kappa & M(z) \end{pmatrix}, \quad (\text{S43})$$

the scattering matrix the chain must have. Its eigenvalues on the constant vectors $(1, \mp 1)/\sqrt{2}$ are $s_\pm = N_\pm/D$ with $N_\pm = M \pm i\kappa$. The monic polynomials $N_\pm$ have no real roots, no common roots, and $N_- = N_+^\#$. Let L_+ and U_+ be the lower and upper roots of N_+ ; the lower roots of N_- are then the conjugates of U_+ , so $D = \prod_{L_+} (z - \lambda) \prod_{U_+} (z - \bar{\alpha})$ and

$$s_+(z) = \prod_{\alpha \in U_+} \frac{z - \alpha}{z - \bar{\alpha}}, \quad (\text{S44})$$

and likewise s_- with U_- the upper roots of N_- . The two sets have $|U_+| + |U_-| = n$. Each channel is thus a finite Blaschke product, of modulus one on the real axis. On the real axis write $s = e^{i\theta}$ with $\theta(-\infty) = 0$; differentiating each factor gives

$$\theta'(x) = \sum_{\alpha \in U} \frac{2 \operatorname{Im} \alpha}{(x - \operatorname{Re} \alpha)^2 + (\operatorname{Im} \alpha)^2} > 0, \quad \theta(+\infty) = 2\pi|U|. \quad (\text{S45})$$

The phase winds monotonically, so $s = -1$ at exactly $|U|$ distinct real points, where $\theta = (2j - 1)\pi$. There are no other solutions: after clearing denominators, $1 + s$ has a numerator of degree $|U|$ with leading coefficient 2.

The Cayley function $k = (1/i)(1 - s)/(1 + s) = -\tan(\theta/2)$ has a simple pole at each such point x_0 ; near it, $\theta = \pi + \theta'(x_0)(x - x_0) + \dots$, so

$$\text{Res}_{z=x_0} k = \frac{2}{\theta'(x_0)} > 0. \quad (\text{S46})$$

The two channels never share a pole, because $s_+ - s_- = 2i\kappa/D \neq 0$ on the real axis.

(c) *The Cayley matrix.* The matrix $\mathbf{K} = (1/i)(I - \mathcal{S})(I + \mathcal{S})^{-1}$ has eigenvalues $k_{\pm}$ on the same vectors, hence n distinct simple real poles, and its diagonal residues are half the scalar residues (S46). By Eq. (S42) and direct 2×2 multiplication,

$$\mathbf{K}(z) = \frac{1}{D_1(z) + M(z)} \begin{pmatrix} D_2(z) & \kappa \\ \kappa & D_2(z) \end{pmatrix} = \frac{1}{2p(z)} \begin{pmatrix} 2\Gamma q_D(z) & \kappa \\ \kappa & 2\Gamma q_D(z) \end{pmatrix}. \quad (\text{S47})$$

For instance, the eigenvalue on $(1, 1)/\sqrt{2}$, $(D_2 + \kappa)/(D_1 + M)$, equals k_- precisely because $(D_1 - M)(D_1 + M) = \kappa^2 - D_2^2$. The off-diagonal entry $\kappa/(2p)$ has a pole at every root of p , and p is monic of degree n , so p has exactly n simple real roots $x_1 < \dots < x_n$, the poles of $\mathbf{K}$.

(d) *A positive spectral measure.* The entry $\mathbf{K}_{11} = \Gamma q_D/p$ has positive residues by step (c), so $\rho_s = q_D(x_s)/p'(x_s) > 0$. Since $q_D/p = 1/z + O(z^{-2})$, partial fractions give $\sum_s \rho_s = 1$. At a root of p , $D_1 = -M$, and Eq. (S42) gives $D_2(x_s)^2 = \kappa^2$, so

$$\rho_s = \frac{\kappa}{2\Gamma|p'(x_s)|} > 0, \quad \sum_s \rho_s = 1. \quad (\text{S48})$$

The target has thus produced a probability measure on n distinct real points.

(e) *The Jacobi matrix.* A probability measure on n distinct real points with positive weights determines a unique real Jacobi matrix h with positive off-diagonal entries such that $\det(z - h) = \prod_s (z - x_s)$ and $(z - h)_{11}^{-1} = \sum_s \rho_s / (z - x_s)$ [8]. Step 5 is that construction. No polynomial of degree below n vanishes at all n support points, so Gram-Schmidt never breaks down. The isometry from $L^2(\rho)$ onto $\mathbb{C}^n$ that sends the orthonormal polynomials to the basis maps multiplication by x to h and the constant 1 to e_1 . Here $\det(z - h) = p(z)$ and $G_{11}^0 = q_D/p$, where $G^0(z) = (z - h)^{-1}$. The parity of p is n : the lower roots of the even polynomial F are invariant under $\lambda \mapsto -\lambda$, so $D(-z) = (-1)^n D^\#(z)$, and $D_1(-z) = (-1)^n D_1(z)$. Hence the roots of p are symmetric, opposite roots carry equal weights by Eq. (S48), the orthonormal polynomials alternate in parity, and every diagonal entry $\int x \phi_j(x)^2 d\rho$ vanishes.

(f) *The full boundary resolvent.* Let v_s be the real orthonormal eigenvectors of h . Then $\rho_s = v_s(1)^2$, and the residues of $G_{n1}^0 = \tau/p$, from Eq. (S26) at $\Gamma = 0$, give $v_s(1)v_s(n) = \tau/p'(x_s)$. Hence, by Eq. (S48),

$$v_s(n)^2 = \frac{\tau^2}{p'(x_s)^2 \rho_s} = \left(\frac{2\Gamma\tau}{\kappa} \right)^2 \rho_s. \quad (\text{S49})$$

Summing over s , $1 = (2\Gamma\tau/\kappa)^2$, so $\tau = \kappa/(2\Gamma)$ and $v_s(n)^2 = \rho_s$. The spectral measures of h at e_n and e_1 coincide, so $G_{nn}^0 = G_{11}^0$. The reversed matrix JhJ , with J the reversal of the basis, is a Jacobi matrix with positive off-diagonal entries whose measure at e_1 is ρ ; by uniqueness $JhJ = h$, that is, $t_j = t_{n-j}$. With Eq. (S47), the undamped boundary block is

$$G_b^0(z) = \begin{pmatrix} G_{11}^0 & G_{1n}^0 \\ G_{n1}^0 & G_{nn}^0 \end{pmatrix} = \frac{1}{2\Gamma p(z)} \begin{pmatrix} D_2(z) & \kappa \\ \kappa & D_2(z) \end{pmatrix} = \frac{\mathbf{K}(z)}{\Gamma}. \quad (\text{S50})$$

(g) *Equal contacts.* Let $W_1 = (e_1, e_n)$ and $\tilde{G} = (z - h + i\Gamma W_1 W_1^\dagger)^{-1}$. The Woodbury identity gives $\tilde{G}_b = G_b^0(I + i\Gamma G_b^0)^{-1}$, hence

$$I - 2i\Gamma \tilde{G}_b = (I - i\mathbf{K})(I + i\mathbf{K})^{-1} = \mathcal{S}. \quad (\text{S51})$$

The damped chain therefore has exactly the scattering matrix (S43). Its off-diagonal entry gives $\tilde{G}_{n1} = \kappa/(2\Gamma D) = \tau/D$. The damped cofactor is still τ , so Eq. (S26) gives $\det(z - h + i\Gamma(\Pi_1 + \Pi_n)) = D(z)$, and for real E

$$4\Gamma^2 |\tilde{G}_{n1}(E)|^2 = \frac{\kappa^2}{|D(E)|^2} = \frac{\kappa^2}{M(E)^2 + \kappa^2} = \frac{1}{1 + r_n(E)^2}. \quad (\text{S52})$$

This proves Eq. (S41).

The two contacts sit on distinct modes because $n \geq 2$; the contact linewidth Γ is fixed by the target as half the subleading coefficient of D_2 , not chosen. For $n = 2$, $r_2(E) = (5E^2 - 1)/4$, $M = E^2 - 1/5$ and $\kappa = 4/5$. The two-site chain has $D = (E + i\Gamma)^2 - t_1^2$, and $|D|^2 = M^2 + \kappa^2$ requires $t_1^2 - \Gamma^2 = 1/5$ and $t_1^2 + \Gamma^2 = \sqrt{17}/5$. Restoring w ,

$$t_1^2 = \frac{w^2(\sqrt{17} + 1)}{10}, \quad \Gamma^2 = \frac{w^2(\sqrt{17} - 1)}{10}, \quad (\text{S53})$$

and indeed $2\Gamma t_1 = \kappa w^2$. The shortest even chain is included; the construction is not confined to odd lengths.

VII. THE SIZE LAW

The lower bound and the exact Gegenbauer chain together fix the minimal product of an n -mode chain up to a constant factor, at a current that stays finite. Let $P_{\min}(n)$ be the infimum of P over all chains with n modes, nonzero real hoppings, real on-site energies and contact linewidths $\Gamma_L, \Gamma_R > 0$, and over all finite $A > 0$. The lower bound of Sec. IV C covers this class, and the chain of Sec. VI, with zero on-site energies and equal contacts, lies inside it.

For every $n \geq 2$ and $w > 0$, the chain of Sec. VI at affinity $A_n = 2 \log n$ has

$$P_n \leq V_n \equiv 4q_n(\log n)B_G(n) + \frac{8 \log n}{n^2(1 - n^{-4})} < 33 \frac{\log n}{n^2}, \quad q_n = \frac{n^2 - 1}{n^2 + 1}, \quad (\text{S54})$$

and its current obeys

$$\frac{w}{\pi} q_n \left(1 - \frac{2}{(n+1)(n+2)}\right) \leq j_n \leq \frac{w}{\pi} q_n \left(1 + \frac{4}{n(n+3)}\right), \quad j_n \geq \frac{w}{2\pi}, \quad \left|\frac{\pi j_n}{w} - 1\right| \leq \frac{4}{n^2}. \quad (\text{S55})$$

Consequently, for every $n \geq 3$,

$$\frac{1}{18n^2} \leq \inf F_0 < \frac{6}{n^2}, \quad \frac{\log n}{18n^2} \leq P_{\min}(n) < 33 \frac{\log n}{n^2}, \quad (\text{S56})$$

the first pair holding also at $n = 2$, with the infimum over the same chains.

The chain belongs to the class, so Eq. (S13) applies. At $A_n = 2 \log n$, $q_n = \tanh(\log n) = (n^2 - 1)/(n^2 + 1)$ and $\sinh(2 \log n) = (n^2 - n^{-2})/2$, so Eq. (S13) reads $P_n = 4q_n(\log n)F_0 + 8 \log n/[n^2(1 - n^{-4})]$, the thermal term being exact. Equation (S37) gives $F_0 \leq B_G(n) < 6/n^2$, hence $P_n \leq V_n$. With $0 < q_n < 1$ and $1 - n^{-4} \geq 15/16$, $V_n < (24 + 128/15) \log n/n^2 = (488/15) \log n/n^2 < 33 \log n/n^2$. For the current, $I_1 = (w/2\pi)\mu_n$ and $j_n = q_n I_1$, so the bounds on μ_n give the first pair in Eq. (S55). Both factors of the lower bound increase with n , and at $n = 2$ their product is $(3/5)(5/6) = 1/2$. The deficit of the lower bound is $2/(n^2 + 1) + 2q_n/[(n+1)(n+2)] \leq 4/n^2$, and the excess of the upper bound is at most $4/[n(n+3)] \leq 4/n^2$. Equation (S56) combines these with Eqs. (S33) and (S34).

The chain reaches the order $\log n/n^2$ of the lower bound while its current stays near w/π . The upper bound itself behaves as $V_n \sim 32 \log n/n^2$, which is not a statement about the asymptotic coefficient of the true product, and neither constant is claimed optimal. Every coefficient of the chain is finite, positive and computable exactly for each n , but we prove no bound on its coupling ratios $\max(\Gamma, t_j)/\min(\Gamma, t_j)$ as n grows. The common scaling preserves F_0 , P and every coupling ratio, so it cannot supply such a bound.

VIII. THE CONTACT BUDGET OF TWO-CONTACT NETWORKS

Near-perfect transmission over a window spends a resource that the contacts supply in fixed amount. This section shows that the weaker contact linewidth caps the integrated logarithm of the reflection probability. It then converts that cap into a bound on partition noise at given current. Let h be Hermitian, $\Gamma_L, \Gamma_R > 0$, K strictly stable and $I_1 > 0$, and put $\Gamma_* = \min(\Gamma_L, \Gamma_R)$. Let $z_1, \dots, z_n$ be the eigenvalues, with algebraic multiplicity, of $h_z = h + i\Gamma_L \Pi_1 - i\Gamma_R \Pi_n$. Then

$$\mathcal{B} \equiv \int_{\mathbb{R}} -\log(1 - T(E)) \frac{dE}{2\pi} = \Gamma_L + \Gamma_R - \sum_{k=1}^n |\operatorname{Im} z_k| \leq 2\Gamma_*, \quad (\text{S57})$$

and

$$I_1 \frac{\log(1/F_0)}{1 - F_0} \leq \mathcal{B} \leq 2\Gamma_*. \quad (\text{S58})$$

The logarithm of the reflection probability is the budget that near-perfect transmission spends. It is fixed by the positions of the poles and zeros of the reflection amplitude, the logarithmic method of Bode and Fano [9], and it is capped by the weaker contact. The left inequality converts the budget into a bound on partition noise at given transmitted current.

Reflection determinant. Let $h_- = h - i\Sigma$, whose eigenvalues p_k have $\text{Im } p_k < 0$ by strict stability. The matrix determinant lemma applied to $E - h_z = E - h_- - 2i\Gamma_L\Pi_1$ gives the left reflection amplitude

$$\mathcal{S}_{11}(E) = 1 - 2i\Gamma_L G_{11}(E) = \frac{\det(E - h_z)}{\det(E - h_-)} = \frac{\prod_k (E - z_k)}{\prod_k (E - p_k)}, \quad (\text{S59})$$

and unitarity of $\mathcal{S}$ (Sec. III) gives $|\mathcal{S}_{11}|^2 = 1 - |\mathcal{S}_{21}|^2 = 1 - T$. Both determinants are monic of degree n . Common factors, real zeros and repeated roots are kept with their multiplicities; no minimal realization is needed.

Convergence. At large $|E|$, $T = O(E^{-4})$, so $-\log(1-T) = O(E^{-4})$. The numerator of (S59) is a monic polynomial, so its real zeros are finitely many, and near a real zero of order m , $-\log |\mathcal{S}_{11}|^2 = -2m \log |E - a| + O(1)$ is integrable. Thus $\mathcal{B}$ is an ordinary Lebesgue integral with a nonnegative integrand.

Pole-zero identity. For real a, b , the antiderivative of $\log[(E - a)^2 + b^2]$ gives

$$\int_{-L}^L \log |E - a - ib|^2 dE = 4L \log L - 4L + 2\pi|b| + o(1) \quad (L \rightarrow \infty), \quad (\text{S60})$$

including $b = 0$. Applied to the n zeros and n poles of (S59), the divergent terms cancel, and monotone convergence identifies the symmetric limit with $\mathcal{B}$: $\mathcal{B} = \sum_k |\text{Im } p_k| - \sum_k |\text{Im } z_k|$. All p_k lie in the lower half-plane and $\text{Im Tr } h_- = -(\Gamma_L + \Gamma_R)$, so the first sum is $\Gamma_L + \Gamma_R$ for any finite matrix. For the zeros,

$$\sum_k |\text{Im } z_k| \geq \left| \text{Im} \sum_k z_k \right| = |\text{Im Tr } h_z| = |\Gamma_L - \Gamma_R|, \quad (\text{S61})$$

so $\mathcal{B} \leq \Gamma_L + \Gamma_R - |\Gamma_L - \Gamma_R| = 2\Gamma_*$. This step replaces the total linewidth by the weaker one: a strong contact cannot pay for a weak one.

Jensen step. The measure $d\mu = T dE / (2\pi I_1)$ is a probability measure, and $x = 1 - T$ has mean $\int x d\mu = F_0$. The function

$$g(x) = \frac{\log(1/x)}{1-x} = \int_0^1 \frac{ds}{x + (1-x)s}, \quad 0 < x \leq 1, \quad (\text{S62})$$

is decreasing and convex, as its integral representation shows, with $g(1) = 1$. Its singular point $x = 0$ occurs only at the finitely many energies of perfect transmission, a μ -null set. Jensen's inequality gives $g(F_0) \leq \int g(1-T) d\mu = \mathcal{B}/I_1$, which is Eq. (S58).

Among measurable transmissions with given I_1 and $\mathcal{B}$, Eq. (S58) is the best possible relation: the flat transmission $T = 1 - f$ on an interval of length $2\pi I_1 / (1 - f)$ and zero elsewhere has $F_0 = f$ and $\mathcal{B} = I_1 g(f)$. Such a transmission is a relaxation, not a network: a finite chain has $T > 0$ everywhere. For the chain the bound reads $\mathcal{B} = 2\Gamma - \sum_k |\text{Im } z_k|$. For the two-site chain with $t_1 < \Gamma$, $Z = E^2 + \Gamma^2 - t_1^2$ has zeros at $\pm i\sqrt{\Gamma^2 - t_1^2}$, so $\mathcal{B} = 2\Gamma - 2\sqrt{\Gamma^2 - t_1^2} < 2\Gamma$; the budget is saturated only when the reflection zeros are real.

IX. THE MINIMAL CONTACT COST

The budget prices partition noise in contact linewidth, and the thermal term fixes how much partition noise a target product allows. This section combines the two into the minimal contact linewidth per transferred particle, and shows that an equal-contact Chebyshev chain attains it up to an additive constant.

A. The scalar bound

For $0 < p < 2$ let $a_p > 0$ solve $\beta(a_p) = p$; it exists uniquely because β decreases strictly from 2 to 0 (Sec. IV). For $A > a_p$ define

$$f_p(A) = \frac{p - \beta(A)}{2Aq(A)}, \quad H_p(A) = \frac{g(f_p(A))}{2q(A)}, \quad \mathcal{R}(p) = \inf_{A > a_p} H_p(A), \quad (\text{S63})$$

with $q(A) = \tanh(A/2)$ and g from Eq. (S62). The partition-noise allowance left by the thermal term at affinity A is $f_p(A)$, and $H_p(A)$ is the contact cost that the budget charges for it.

No network can pay less than $\mathcal{R}(p)$. Every network satisfying the hypotheses of Eq. (S57), at any finite $A > 0$, with $P \leq p < 2$, obeys

$$\frac{\Gamma_*}{j} \geq \mathcal{R}(p). \quad (\text{S64})$$

The infimum in Eq. (S63) is attained at a finite A , and for $s = \log(1/p) \geq 8$ it is squeezed as

$$0 \leq 2\mathcal{R}(p) - [s + \log(2s)] \leq \frac{2 \log s + \log 8 + 5}{s}. \quad (\text{S65})$$

The bound and its attainment. By Eq. (S13), $P \leq p$ requires $\beta(A) < p$, so $A > a_p$, and then $0 < F_0 \leq f_p(A)$. Also $f_p(A) < 1$, because $\beta(A) + 2Aq(A) = 2A \coth A > 2 > p$. The function g decreases, $g'(f) = [1 - 1/f - \log f]/(1-f)^2 < 0$, so Eq. (S58) and $j = qI_1$ give

$$\frac{\Gamma_*}{j} \geq \frac{g(F_0)}{2q} \geq \frac{g(f_p(A))}{2q} = H_p(A) \geq \mathcal{R}(p). \quad (\text{S66})$$

As $A \downarrow a_p$, $f_p \rightarrow 0$ and $H_p \rightarrow \infty$; as $A \rightarrow \infty$, $f_p \sim p/(2A)$ and $H_p \rightarrow \infty$ again. By continuity the infimum is a minimum.

Lower half of Eq. (S65). Let $s \geq 8$ and $A > a_p$. Since $\beta(1) > 1 > p$ and β decreases, $A > 1$; then $\beta(A) = 4Ae^{-A}/(1 - e^{-2A}) > e^{-A}$, so $A > s$. Hence $q > 1/2$ and $f = f_p(A) < p/A < e^{-1}$. With $Y = \log(1/f) > 1$ and $\log(1/q) \leq 1/q - 1$,

$$\frac{g(f)}{q} \geq \frac{Y}{q} \geq Y + \log \frac{1}{q} = \log \frac{1}{fq} \geq s + \log(2A) > s + \log(2s), \quad (\text{S67})$$

since $fq \leq p/(2A)$. Minimizing over A gives the lower half. Small precision forces an affinity at least s , and dividing the partition-noise allowance by that affinity is what produces the $\log s$ correction.

Upper half of Eq. (S65). Take $A_t = s + 2 \log s + \log 8$, so $e^{-A_t} = p/(8s^2)$ and $s < A_t \leq 2s$ for $s \geq 8$. Then $\alpha \equiv \beta(A_t)/p = A_t/[2s^2(1 - e^{-2A_t})] \leq 2/s \leq 1/4$, and $f_t = f_p(A_t) = p(1 - \alpha)/(2A_t q_t)$. With $X = s + \log(2s)$,

$$\log \frac{1}{f_t} = s + \log(2A_t) + \log q_t - \log(1 - \alpha) \leq X + \frac{2 \log s + \log 8 + 4}{s} \equiv X + \Delta, \quad (\text{S68})$$

using $\log(A_t/s) \leq (2 \log s + \log 8)/s$, $\log q_t \leq 0$ and $-\log(1 - \alpha) \leq \alpha/(1 - \alpha) \leq 4/s$. Next $q_t \geq 1 - 2e^{-A_t}$ and $f_t \leq p/s$ give $q_t(1 - f_t) \geq 1 - 2p/s$, so $1/[q_t(1 - f_t)] \leq 1 + 4p/s$. Since $X + \Delta \leq 2s$ and $8p \leq 1/s$ for $s \geq 8$,

$$2\mathcal{R}(p) \leq 2H_p(A_t) \leq (X + \Delta) \left(1 + \frac{4p}{s}\right) \leq X + \Delta + 8p \leq X + \frac{2 \log s + \log 8 + 5}{s}. \quad (\text{S69})$$

Equation (S65) gives $\mathcal{R}(p) = \frac{1}{2} \log(1/p) + \frac{1}{2} \log \log(1/p) + \frac{1}{2} \log 2 + O(\log s/s)$. It is the exact optimum of the scalar relaxation: assigning the flat transmission of Sec. VIII, with $f = f_p(A)$ and an interval of normalized length $2\Gamma_*/\log(1/f)$, to the formulas (S12) gives $P = p$ and $\Gamma_*/j = H_p(A)$. That witness is not a network, so for physical devices only the lower bound (S64) is asserted here; the matching physical upper bound comes from the Chebyshev chain below.

B. A chain with exact Chebyshev transmission

The upper side of the contact law needs a chain whose transmission ripples by a prescribed small amount on the window and rises exponentially outside it. Let $n \geq 3$ be odd, $\epsilon, w > 0$, $u = \text{asinh}(1/\epsilon)$, $\theta = \pi/n$ and $\eta = \sinh(u/n)$. The chain with zero diagonal and

$$t_j^2 = \frac{w^2 [\eta^2 + \sin^2(j\theta)]}{4 \sin((j - \frac{1}{2})\theta) \sin((j + \frac{1}{2})\theta)} \quad (1 \leq j < n), \quad \Gamma = \frac{w\eta}{2 \sin(\theta/2)}, \quad (\text{S70})$$

transmits, for every real E ,

$$T(E) = \frac{1}{1 + \epsilon^2 C_n(E/w)^2}. \quad (\text{S71})$$

The coefficients (S70) are the equal-contact form of the classical Chebyshev filter prototype [10]; the proof below establishes the exact transmission with Markovian contacts at every odd order, without a narrow-band approximation. Every hopping is positive because the sines in the denominator have arguments in $(0, \pi)$. The transmission ripples between $1/(1 + \epsilon^2)$ and 1 on the window $|E| \leq w$, and $C_n(E/w) = \cosh(n \operatorname{acosh}(E/w))$ grows exponentially outside it.

Similarity to a solvable operator. Take $w = 1$ and put $a = u/n$. Index the sites by $j = 0, \dots, n-1$, and let $y_j = (j + \frac{1}{2})\theta$ and $b = \theta/2 + ia$. Define

$$A_j = \frac{\sin(y_j + b)}{2 \sin y_j}, \quad B_j = \frac{\sin(y_j - b)}{2 \sin y_j}, \quad (\mathbf{L}v)_j = A_j v_{j+1} + B_j v_{j-1}, \quad (\text{S72})$$

with reflecting ghost values $v_{-1} = v_0$ and $v_n = v_{n-1}$. The interior diagonal of $\mathbf{L}$ vanishes, and its end entries are $B_0 = \sin(-ia)/(2 \sin(\theta/2)) = -i\Gamma$ and $A_{n-1} = \sin(\pi + ia)/(2 \sin(\theta/2)) = -i\Gamma$. With $|\sin(x + iy)|^2 = \sin^2 x + \sinh^2 y$,

$$A_{j-1}B_j = \frac{\sin(j\theta + ia)\sin(j\theta - ia)}{4 \sin((j - \frac{1}{2})\theta)\sin((j + \frac{1}{2})\theta)} = t_j^2. \quad (\text{S73})$$

No A_j with $j < n-1$ vanishes, so a diagonal similarity $\mathbf{D}^{-1}\mathbf{L}\mathbf{D}$ with $d_{j+1}/d_j = t_{j+1}/A_j$ produces equal off-diagonal entries t_j and leaves the diagonal unchanged. Hence $\mathbf{L}$ is similar to $h_{\text{eff}} = h - i\Gamma(\Pi_1 + \Pi_n)$ and has the same characteristic polynomial.

The spectrum follows from a polynomial flag. A polynomial in $\cos y$ takes equal values at $y = \pm\theta/2$ and at $y = \pi \pm \theta/2$, so the ghost prescription is its ordinary continuation, and $\mathbf{L}$ acts on node values of $f(y)$ as $A(y)f(y + \theta) + B(y)f(y - \theta)$. For $f_k(y) = \cos(ky) = C_k(\cos y)$, addition formulas give

$$(\mathbf{L}f_k)(y) = \cos b \cos(k\theta) \cos(ky) - \sin b \sin(k\theta) \cot y \sin(ky). \quad (\text{S74})$$

For $k \geq 1$, $\cot y \sin(ky) = x U_{k-1}(x)$ with $x = \cos y$ and U_{k-1} the Chebyshev polynomial of the second kind; it has the same degree k and leading coefficient 2^{k-1} as C_k . Hence $\mathbf{L}C_k = \cos(b + k\theta)C_k + (\text{degree} < k)$, and $\mathbf{L}1 = \cos b$ since $A_j + B_j = \cos b$. The nodes $\cos y_j$ are distinct, so the evaluated $C_0, \dots, C_{n-1}$ form a basis of $\mathbb{C}^n$ in which $\mathbf{L}$ is triangular. The eigenvalues of h_{eff} are therefore

$$z_k = \cos((k + \frac{1}{2})\theta + ia), \quad k = 0, \dots, n-1, \quad (\text{S75})$$

distinct, with $\operatorname{Im} z_k = -\sinh a \sin((k + \frac{1}{2})\theta) < 0$. At each pole $C_n(z_k) = \cos((k + \frac{1}{2})\pi + iu) = (-1)^{k+1}i \sinh u$, so the n poles and their conjugates are the $2n$ roots of $C_n^2 + \sinh^2 u$, and matching leading coefficients,

$$|D(E)|^2 = 2^{2-2n}(C_n(E)^2 + \sinh^2 u), \quad D(E) = \det(E - h_{\text{eff}}). \quad (\text{S76})$$

For the numerator of Eq. (S26), the product $\prod_{j=0}^{n-1} \sin(z + j\pi/n) = 2^{1-n} \sin(nz)$, which follows from the factorization of $\zeta^n - 1$, gives at $z = ia$ and at $z = \theta/2$

$$\prod_{j=1}^{n-1} (\sinh^2 a + \sin^2(j\theta)) = 2^{2-2n} \frac{\sinh^2 u}{\sinh^2 a}, \quad \prod_{j=1}^{n-1} 4 \sin((j - \frac{1}{2})\theta) \sin((j + \frac{1}{2})\theta) = \frac{1}{\sin^2(\theta/2)}. \quad (\text{S77})$$

With $4\Gamma^2 = \sinh^2 a / \sin^2(\theta/2)$ these give $4\Gamma^2 \tau^2 = 2^{2-2n} \sinh^2 u$. Hence $T = \sinh^2 u / (C_n^2 + \sinh^2 u) = 1/(1 + \epsilon^2 C_n^2)$, since $\sinh u = 1/\epsilon$. Restoring w multiplies both sides of each identity by w^{2n} , which gives Eq. (S71).

C. Resource scaling and the contact law

Tuned to a target product, the Chebyshev chain reaches it at a contact cost that exceeds the lower bound only by an additive constant, while its current stays near w/π . For $0 < p \leq 1$ and any $w > 0$ prescribe

$$S_0 = \log \frac{16}{p}, \quad A = S_0 + \log S_0, \quad \epsilon^2 = \frac{p}{16A}, \quad \ell_\epsilon = \log \frac{2}{\epsilon}, \quad (\text{S78})$$

and let n be the smallest odd integer with $n \geq \max\{3, \ell_\epsilon, [256A(\ell_\epsilon + 1)/p]^{1/2}\}$. The Chebyshev chain of Sec. IX B with these n, ϵ, w at affinity A has $P < p$, and

$$\frac{wq}{\pi(1+\epsilon^2)} \leq j \leq \frac{wq}{\pi} \left[1 + \frac{2(\ell_\epsilon + 1)^2}{n^2} \right]. \quad (\text{S79})$$

As $p \downarrow 0$, with $s = \log(1/p)$, $n \sim 8\sqrt{2}s/\sqrt{p}$, $j = (w/\pi)[1 + O(p)]$ and

$$\frac{\Gamma}{j} = \frac{s}{2} + \frac{\log s}{2} + \log 8 + o(1). \quad (\text{S80})$$

The derivation first bounds the transmission integrals of any Chebyshev chain, then inserts the prescription.

Window integrals. Let $0 < \epsilon \leq 1$ and $n \geq \max(3, \ell_\epsilon)$, and put $J_k = \pi I_k/w$. Outside the window substitute $E/w = \cosh(s'/n)$, so $C_n = \cosh s'$; the out-of-band mass and noise are

$$M_o = \int_0^\infty \frac{\sinh(s'/n)}{n} \mathcal{Q}(s') ds', \quad N_o = \int_0^\infty \frac{\sinh(s'/n)}{n} \mathcal{Q}(s')(1 - \mathcal{Q}(s')) ds', \quad \mathcal{Q} = \frac{1}{1 + \epsilon^2 \cosh^2 s'}, \quad (\text{S81})$$

and evenness gives $J_1 = \int_0^1 T(wx) dx + M_o$ and $J_2 = \int_0^1 [T(1-T)](wx) dx + N_o$. From $e^{s'}/2 \leq \cosh s' \leq e^{s'}$ and $e^{-\ell_\epsilon} = \epsilon/2$, $z = \epsilon^2 \cosh^2 s'$ lies between $e^{2(s'-\ell_\epsilon)}$ and $4e^{2(s'-\ell_\epsilon)}$, so $\mathcal{Q} \leq \min(1, e^{-2(s'-\ell_\epsilon)})$ and $\mathcal{Q}(1 - \mathcal{Q}) \leq 4e^{-2|s'-\ell_\epsilon|}$. With $\sinh(s'/n)/n \leq s'e^{s'/n}/n^2$ and $e^{s'/n} \leq e e^{|s'-\ell_\epsilon|/3}$, splitting at $s' = \ell_\epsilon$ and using $\cosh y - 1 \leq ey^2/2$ for $y \leq 1$,

$$M_o \leq \frac{e}{n^2} \left(\frac{\ell_\epsilon^2}{2} + \frac{3\ell_\epsilon}{5} + \frac{9}{25} \right) \leq \frac{2(\ell_\epsilon + 1)^2}{n^2}, \quad N_o \leq \frac{4e}{n^2} \int_{\mathbb{R}} (\ell_\epsilon + |v|) e^{-5|v|/3} dv \leq \frac{16(\ell_\epsilon + 1)}{n^2}. \quad (\text{S82})$$

The leakage beyond the window costs noise of order ℓ_ϵ^2/n^2 and mass of order ℓ_ϵ^2/n^2 ; neither is dropped. Inside the window $|C_n| \leq 1$ and $\int_0^1 C_n^2 dx = \frac{1}{2} - 1/[2(4n^2 - 1)] \leq \frac{1}{2}$, by the substitution $x = \cos \phi$. From $1 - z \leq (1 + z)^{-1} \leq 1$ and $z/(1 + z)^2 \leq z$ with $z = \epsilon^2 C_n^2$, $1 - \epsilon^2/2 \leq J_1 \leq 1 + M_o$ and $J_2 \leq \epsilon^2/2 + N_o$, hence

$$\frac{w}{\pi(1+\epsilon^2)} \leq I_1 \leq \frac{w}{\pi} \left[1 + \frac{2(\ell_\epsilon + 1)^2}{n^2} \right], \quad F_0 \leq (1 + \epsilon^2) \left[\epsilon^2 + \frac{16(\ell_\epsilon + 1)}{n^2} \right]. \quad (\text{S83})$$

The prescription. The prescription has $\epsilon^2 \leq 1$ and $16(\ell_\epsilon + 1)/n^2 \leq p/(16A) = \epsilon^2$, so $F_0 \leq 2\epsilon^2(1 + \epsilon^2) \leq p/(4A)$, and the partition term of Eq. (S13) is at most $p/2$. Since $S_0 > 2$, $\log S_0 \leq S_0/2$ and $e^{-A} = p/(16S_0) \leq 1/16$, $\beta(A) = 4Ae^{-A}/(1 - e^{-2A}) \leq (3p/8)/(1 - 1/256) < p/2$. Hence $P < p$, and Eq. (S79) is $j = qI_1$ with Eq. (S83). As $p \downarrow 0$, $A = s + \log s + \log 16 + o(1)$, $\epsilon \sim \frac{1}{4}\sqrt{p/s}$, $\ell_\epsilon = s/2 + \frac{1}{2}\log s + \log 8 + o(1)$, $u = \ell_\epsilon + O(\epsilon^2)$, and $n \sim (256A\ell_\epsilon/p)^{1/2} \sim 8\sqrt{2}s/\sqrt{p}$; rounding to the next odd integer changes none of these limits. Then $q = 1 - O(p/s)$ and $J_1 = 1 + O(p)$, so $j = (w/\pi)[1 + O(p)]$. Since $u/n = O(\sqrt{p})$, $\Gamma = w \sinh(u/n)/[2 \sin(\pi/2n)] = (wu/\pi)[1 + O(p)]$, and $\Gamma/j = u + O(pu)$, which is Eq. (S80).

The coefficients of the family follow from Eq. (S70), rewritten as $t_j^2 = \frac{w^2}{4} \{1 + [\eta^2 + \sin^2(\theta/2)]/[\sin^2(j\theta) - \sin^2(\theta/2)]\}$. The largest hopping is at the ends and the smallest in the middle, so $t_{\min} = \frac{w}{2}[1 + O(p)]$, $t_{\max} = w\sqrt{u^2 + \pi^2}/(\pi\sqrt{3})[1 + O(p)]$ and $\Gamma/t_{\min} \sim 2u/\pi$. The contact-to-bulk ratio grows as $\log(1/p)$, and no common rescaling changes it.

The two sides together give the contact law of the main text. We call $\mathcal{C}(p)$ the infimum of Γ_*/j over all n , all Hermitian h , all $\Gamma_L, \Gamma_R > 0$ with K strictly stable and $I_1 > 0$, and all finite $A > 0$, subject to $P \leq p$. Then, as $p \downarrow 0$,

$$\mathcal{C}(p) = \frac{1}{2} \log \frac{1}{p} + \frac{1}{2} \log \log \frac{1}{p} + O(1). \quad (\text{S84})$$

The Chebyshev chain is a member of this class with $\Gamma_* = \Gamma$, so Eq. (S80) bounds $\mathcal{C}(p)$ from above. Equations (S64) and (S65) bound it from below by $s/2 + \frac{1}{2}\log s + \frac{1}{2}\log 2$.

The centered infimum thus lies between $\frac{1}{2}\log 2$ and $\log 8$ in the limit; the additive constant and attainment are not decided. The law holds over all network sizes, so it is not a fixed-size statement. Bounded Γ_* with $P \rightarrow 0$ forces $j \rightarrow 0$. Here Γ_* is the single-particle linewidth of the weaker contact, not its stationary jump activity.

D. The trigonometric chain

For comparison, the trigonometric chain of Ref. [11] at half-bandwidth w has $T = 1/[1 + (E/w)^{2n}]$, $\Gamma_{\text{tr}} = w/[2 \sin(\pi/2n)]$, $I_1 = \Gamma_{\text{tr}}/n$ and $F_0 = 1/(2n)$, by Sec. XII and the common scaling. At finite affinity,

$$P_{\text{tr}} = \frac{Aq}{n} + \beta(A), \quad \frac{\Gamma_{\text{tr}}}{j_{\text{tr}}} = \frac{n}{q}. \quad (\text{S85})$$

The partition noise falls only as $1/n$, so the length, and with it the contact cost, must grow as $1/p$. For $s \geq 8$, $P_{\text{tr}} \leq p$ forces $A > s$ as in the lower half of Eq. (S65), and $n \geq Aq/(p - \beta(A))$, so $\Gamma_{\text{tr}}/j_{\text{tr}} \geq A/(p - \beta(A)) > s/p$. Conversely, at A_t and $n = \lceil A_t q_t / (p - \beta(A_t)) \rceil$, $\Gamma_{\text{tr}}/j_{\text{tr}} \leq A_t / [p(1 - 2/s)] + 2$. The cost is therefore $\Theta(\log(1/p)/p)$, against $\Theta(\log(1/p))$ for the Chebyshev chain. More precisely, the continuous relaxation $A/(p - \beta(A))$ is minimal where $p = \beta - A\beta' = 2A^2 \cosh A / \sinh^2 A$. For small p every feasible affinity exceeds $a_p > 2$, and there the right side decreases, because its logarithmic derivative $2/A + \tanh A - 2 \coth A$ is below $2/A - 1$. This gives $A_* = s + 2 \log s + \log 4 + o(1)$ and $\beta(A_*)/p = \tanh A_*/A_*$, and rounding n up adds at most $1/q \leq 2$, so

$$\inf_{n, A: P_{\text{tr}} \leq p} \frac{\Gamma_{\text{tr}}}{j_{\text{tr}}} = \frac{s + 2 \log s + \log 4 + 1 + o(1)}{p}. \quad (\text{S86})$$

X. VANISHING PRODUCT AT BOUNDED COUPLINGS

This section proves the bounded-coupling result of the main text. Its first half is a floor: at a bounded contact linewidth, a product tending to zero forces the current to zero. Its second half is a construction. Chains whose hoppings all lie in $[1, 6]$ and whose two linewidths both equal 1 reach every target chain with zero on-site energies and equal contacts, scaled down in energy. The error is small on the whole energy axis relative to the transmitted weight. Applied to Chebyshev targets of growing order, this drives the product to zero.

A. A floor at finite current

At finite current through a bounded contact the budget leaves a floor on the product. Every two-contact network satisfying the hypotheses of Eq. (S57), at every finite affinity $A > 0$, obeys

$$P \geq 2 \tanh \frac{1}{2} \exp \left(- \frac{2\Gamma_*}{j} \right). \quad (\text{S87})$$

Indeed, by Eq. (S58), $\log(1/F_0) \leq 2\Gamma_*(1 - F_0)/I_1 \leq 2\Gamma_*/j$, since $j = qI_1 \leq I_1$. Hence $F_0 \geq e^{-2\Gamma_*/j}$. In Eq. (S13), for $A \geq 1$ the partition term is $2A \tanh(A/2)F_0 \geq 2 \tanh(\frac{1}{2})F_0$; for $A < 1$ the thermal term is $2A/\sinh A > 2/\sinh 1 > 2 \tanh \frac{1}{2} \geq 2 \tanh(\frac{1}{2})F_0$.

Along a sequence with $\Gamma_* \leq R$ and $P \rightarrow 0$ the right side of Eq. (S87) must vanish, so $j \rightarrow 0$. The floor uses no structure of the network beyond the budget.

B. The embedding construction

Fix a target: an n -site chain, $n \geq 2$, with zero on-site energies, hoppings $t_j^* > 0$, both linewidths $\Gamma^* > 0$, and transmission $T^*(x)$. Its moments $I_1^* = \int T^* dx / 2\pi > 0$ and $F_0^* = \int T^*(1 - T^*) dx / (2\pi I_1^*)$ are finite. Put

$$\zeta_1 = \zeta_n = \frac{323}{280}, \quad \zeta_j = \frac{37}{35} \quad (1 < j < n), \quad Z = \text{diag}(\zeta_1, \dots, \zeta_n). \quad (\text{S88})$$

For an integer depth ν define

$$\tau_\nu = 9^{-\nu}, \quad \delta_\nu = \frac{\tau_\nu}{\zeta_1 \Gamma^*}, \quad \alpha_j = \delta_\nu t_j^* \sqrt{\zeta_j \zeta_{j+1}}, \quad \ell_\nu = \left\lceil \max_j \frac{\log(1/\alpha_j)}{\log 6} \right\rceil, \quad v_j = \alpha_j^{-1/\ell_\nu}. \quad (\text{S89})$$

For ν large enough $0 < \alpha_j < 1$ and $\ell_\nu \geq 1$, so $1 < v_j \leq 6$. The physical chain has zero on-site energies, both linewidths equal to 1, and the bonds, read from left to right,

$$(3, 1)^\nu \prod_{j=1}^{n-1} [(1, v_j)^{\ell_\nu} 1] (1, 3)^\nu, \quad (\text{S90})$$

where the product is concatenation. Every bond lies in $[1, 6]$, and the chain has $m(n, \nu) = 4\nu + (n - 1)(2\ell_\nu + 1) + 1$ modes. With $\vartheta = \log 9 / \log 6$, writing $\log(1/\alpha_j) = \nu \log 9 + O_*(1)$ gives

$$\ell_\nu = \vartheta \nu + O_*(1), \quad v_j = 6 + O_*(\nu^{-1}). \quad (\text{S91})$$

A star on O or C marks dependence on the fixed target; no constant is claimed uniform in the target. In particular $v_j \geq 23/4$ for ν large.

The mechanism is visible at zero energy. A pair of a strong bond v and a weak bond 1 passes an amplitude $1/v$, so ℓ such pairs act as one weak hopping $v^{-\ell}$. The arms $(3, 1)^\nu$ likewise weaken the unit contact to a linewidth τ_ν at the first retained site. The retained sites are the first site after the left arm and the ends of successive central words, n in all. The weights ζ_j are their inverse quasiparticle weights. Effective weak hoppings from dimerized segments and their decimation are used for transfer across topological domain walls [12, 13]. What is added here is the constant Markovian contact and the error control on the whole energy axis.

The embedded chain reproduces the scaled target transmission on the whole energy axis, with an error small against the transmitted weight. Let T_ν be the transmission of the chain (S90). There are target-dependent C_* , ν_* and a target-independent $\lambda < 1$ such that, for $\nu \geq \nu_*$,

$$\frac{\mathcal{E}_\nu}{\delta_\nu} \leq C_* \left[\nu^{-3/8} + \delta_\nu \nu^{3/8} + (\nu + 1)^2 \lambda^\nu \right], \quad \mathcal{E}_\nu = \int_{\mathbb{R}} |T_\nu(E) - T^*(E/\delta_\nu)| \frac{dE}{2\pi}. \quad (\text{S92})$$

Pointwise convergence of $T_\nu(\delta_\nu x)$ to $T^*(x)$ on compact sets would not suffice. The product needs the error small against the transmitted weight $\delta_\nu I_1^*$, which shrinks, and narrow resonances far outside the window could carry weight comparable to it. The derivation below controls them through the spectral mass of the end sites.

Exact blocks. For the closed $2s$ -site block $\mathbf{B}_{v,s}$ with bonds $(v, 1)^{s-1}v$ put $x_v(E) = (E^2 - v^2 - 1)/(2v)$ and $\mathcal{P}_{v,s} = U_s(x_v) + v^{-1}U_{s-1}(x_v)$, with U_s the Chebyshev polynomial of the second kind and $U_{-1} = 0$. The continuant recurrence, with initial values 1, E , $E^2 - v^2$, gives

$$\det(E - \mathbf{B}_{v,s}) = v^s \mathcal{P}_{v,s}, \quad b_{v,s} \equiv (E - \mathbf{B}_{v,s})_{1,2s}^{-1} = \frac{1}{\mathcal{P}_{v,s}}, \quad d_{v,s} \equiv (E - \mathbf{B}_{v,s})_{11}^{-1} = \frac{E}{v} \frac{U_{s-1}(x_v)}{\mathcal{P}_{v,s}}. \quad (\text{S93})$$

The left arm, with its unit contact at the outer end, gives the first retained site the self-energy $\sigma_\nu = u_\nu - i\gamma_\nu$ with

$$\gamma_\nu(E) = \left| U_\nu(x_3) + \frac{1 + iE}{3} U_{\nu-1}(x_3) \right|^{-2}, \quad (\text{S94})$$

and the right arm, its mirror image, gives the last retained site the same self-energy. Eliminating the arms and the central blocks, with $d_j = d_{v_j, \ell_\nu}$ and $b_j = b_{v_j, \ell_\nu}$, leaves the $n \times n$ retained matrix $\mathbf{M}(E)$: off-diagonal entries $-b_j$, interior diagonal entries $E - d_{j-1} - d_j$, and end entries $E - \sigma_\nu - d_1$ and $E - \sigma_\nu - d_{n-1}$. Tracing the two end-to-retained amplitudes gives

$$T_\nu(E) = 4\gamma_\nu(E)^2 |(\mathbf{M}(E)^{-1})_{n1}|^2. \quad (\text{S95})$$

At $E = 0$ the hyperbolic form of U_s gives $\mathcal{P}_{v,s}(0) = (-1)^s v^s$, so $b_j(0) = (-1)^{\ell_\nu} \alpha_j$, and $d_j(0) = 0$, $\sigma_\nu(0) = -i\tau_\nu$. With $S_s(v) = (1 - v^{-2s})/(v^2 - 1)$, the derivatives at zero are $d'_j(0) = -S_{\ell_\nu}(v_j)$ and $\sigma'_\nu(0) = -S_\nu(3)(1 - \tau_\nu)$, from the arm recurrence $\sigma_0 = -i$, $\sigma_l = (E - \sigma_{l-1})/[E(E - \sigma_{l-1}) - 9]$. As $S_s(6) \rightarrow 1/35$ and $S_\nu(3) \rightarrow 1/8$, the diagonal of $\mathbf{M}'(0)$ is $Z + O_*(1/\nu)$, which is where Eq. (S88) comes from.

Compact window. After a diagonal sign change removing $(-1)^{\ell_\nu}$ from the links,

$$\delta_\nu^{-1} Z^{-1/2} \mathbf{M}(\delta_\nu x) Z^{-1/2} = x - h^* + i\Gamma^*(\Pi_1 + \Pi_n) + O_*\left(\frac{|x|}{\nu} + \delta_\nu x^2\right), \quad (\text{S96})$$

since $\alpha_j/(\delta_\nu \sqrt{\zeta_j \zeta_{j+1}}) = t_j^*$ and $\tau_\nu/(\delta_\nu \zeta_1) = \Gamma^*$. The remainder is uniform on a fixed small disk of energies. Let g_o and b_o be the end diagonal and cross entry of the closed arm. The arm's self-energy has the rank-one form $\sigma_\nu = g_o - ib_o^2/(1 + ig_o)$; since $g_o(0) = 0$ its denominator is 1 at zero, so σ_ν is analytic with uniformly bounded derivatives on the disk. The closed outer and central inverses at zero have norms at most $1/2$ and $1/(v_j - 1) \leq 4/19$, so a Neumann expansion bounds their derivatives uniformly in ν . Also $\gamma_\nu(\delta_\nu x)/\tau_\nu = 1 + O_*(\delta_\nu x^2)$. The target matrix $x - h^* + i\Gamma^*(\Pi_1 + \Pi_n)$ is invertible for real x : a real eigenvalue would force an eigenvector vanishing at an end, hence everywhere by the recurrence. Its inverse tends to zero at infinity and is therefore bounded on the real axis by a target constant. With Eq. (S95), for $M = \nu^{1/8}$ and ν large,

$$\sup_{|x| \leq M} |T_\nu(\delta_\nu x) - T^*(x)| \leq C_* \left(\frac{M}{\nu} + \delta_\nu M^2 \right). \quad (\text{S97})$$

Block estimates. Inside the gap of the dimerized block, $0 \leq E < v - 1$, let

$$\varrho_v(E) = \frac{v^2 + 1 - E^2 - \sqrt{(v^2 + 1 - E^2)^2 - 4v^2}}{2v}, \quad (\text{S98})$$

which increases from $1/v$ to 1. For $E \geq 4$ define ϱ_3 by the same formula with $|10 - E^2|$ in place of $10 - E^2$, decreasing from 1. The hyperbolic form of U_s gives $|\mathcal{P}_{v,s}| \geq \varrho_v^{-s}$ and $|U_s| \leq (s+1)\varrho_v^{-s}$, hence from Eq. (S94)

$$\gamma_\nu^{-1} \leq C(\nu+1)^2 \varrho_3^{-2\nu} \quad (1 \leq E \leq 2), \quad \gamma_\nu^{-1} \leq C(\nu+1)^2 \quad (2 \leq E \leq 4), \quad \gamma_\nu \leq \varrho_3^{2\nu} \quad (0 \leq E \leq 1 \text{ or } E \geq 4), \quad (\text{S99})$$

where on $[2, 4]$, the band of the arm, $|U_s| \leq s+1$. Since $v_j \geq 23/4$, every central block stays inside its gap for $E \leq 9/2$, and Eq. (S91) with the smoothness of $\log \varrho_v(E)$ gives

$$\mathbf{b}_\nu(E) \equiv \max_j |b_j(E)| \leq C_* \varrho_6(E)^{\vartheta\nu} \quad (0 \leq E \leq 9/2). \quad (\text{S100})$$

Each block is a disjoint set of strong dimers plus weak bonds of norm at most 1, so its spectrum has $|\lambda| \geq v-1$, and bipartite symmetry gives $d_j(E) = E[(E^2 - \mathbf{B}^2)^{-1}]_{11} < 0$ for $0 < E \leq 9/2$. For the arm, the rank-one form gives $\text{Re } \sigma_\nu = g_o[1 - b_o^2/(1 + g_o^2)]$. On $(0, 1]$ the closed arm is bipartite with spectrum outside $(-2, 2)$, so $g_o < 0$ as for d_j , and $|b_o| \leq \varrho_3(1)^\nu < 1$; hence $\text{Re } \sigma_\nu \leq 0$. For $E \geq 4$ the recurrence $g_0 = 0$, $g_l = (E - g_{l-1})/[E(E - g_{l-1}) - 9]$ keeps $0 \leq g_o \leq 1$; hence $\text{Re } \sigma_\nu \leq 1$.

Spectral domination. Let a and b be the first and last retained sites and $\mathcal{S}(E) = -\text{Im}[G_{aa}(E) + G_{bb}(E)]$, with G the resolvent of the whole physical chain. All poles of G lie below the real axis by the recurrence argument above, and $G_{jj}(z) = 1/z + O(z^{-2})$, so closing the contour in the upper half plane gives

$$\int_{\mathbb{R}} \mathcal{S}(E) \frac{dE}{2\pi} = 1. \quad (\text{S101})$$

Eliminating only the arms, the dissipation identity gives $-\text{Im } G_{aa} = \gamma_\nu(|G_{aa}|^2 + |G_{ab}|^2)$ and its mirror, whose sum is at least $2\gamma_\nu|G_{ab}|^2$, while $T_\nu = 4\gamma_\nu^2|G_{ab}|^2$. Hence $T_\nu \leq 2\gamma_\nu\mathcal{S}$ at every energy, with every central resonance retained. On $1 \leq E \leq 9/2$, eliminate in addition the internal retained sites, whose real matrix has diagonal at least E and off-diagonal entries at most $\mathbf{b}_\nu \rightarrow 0$. This leaves

$$\begin{pmatrix} y_1 + i\gamma_\nu & -\kappa \\ -\kappa & y_2 + i\gamma_\nu \end{pmatrix}, \quad y_1, y_2, \kappa \in \mathbb{R}, \quad |\kappa| \leq C_* \mathbf{b}_\nu, \quad (\text{S102})$$

with $y_1, y_2 \geq E - 1 - 2\mathbf{b}_\nu^2/E \geq 5/2$ on $[4, 9/2]$. Direct inversion gives $T_\nu/\mathcal{S} = 4\gamma_\nu\kappa^2/(y_1^2 + y_2^2 + 2\gamma_\nu^2 + 2\kappa^2)$, so altogether

$$T_\nu \leq \min\left(2\gamma_\nu, \frac{2\kappa^2}{\gamma_\nu}\right)\mathcal{S}. \quad (\text{S103})$$

Arbitrarily narrow resonances are thus charged against the total spectral mass, with no assumption on their widths or multiplicities.

Leakage outside the window. The transmission is even in E . On $[1, 2]$ and $[2, 4]$, Eqs. (S101)–(S103) with Eqs. (S99) and (S100) bound the integrated transmission by $C_*(\nu+1)^2\lambda_1^\nu$ and $C_*(\nu+1)^2\lambda_2^\nu$, with $\lambda_1 = \sup_{[1,2]} \varrho_6^{2\vartheta}/\varrho_3^2$ and $\lambda_2 = \varrho_6(4)^{2\vartheta}$. On $[4, 9/2]$ the real parts $y_1, y_2 \geq 5/2$ bound the determinant away from zero, so $T_\nu \leq C\gamma_\nu^2\kappa^2$; splitting at $17/4$ gives the bases $\lambda_3 = \varrho_6(17/4)^{2\vartheta}$ and $\lambda_4 = \varrho_3(17/4)^4\varrho_6(9/2)^{2\vartheta}$. Above $9/2$, $T_\nu \leq 2\gamma_\nu\mathcal{S}$ and Eq. (S101) give at most $2\varrho_3(9/2)^{2\nu}$, base $\lambda_5 = \varrho_3(9/2)^2$. Every base is below $1/9$. For λ_1 , the logarithmic derivative of $\varrho_6^{2\vartheta}/\varrho_3^2$, $4E\{\vartheta[(37 - E^2)^2 - 144]^{-1/2} - [(10 - E^2)^2 - 36]^{-1/2}\}$, is negative on $[1, 2)$ because $\vartheta < 5/4$, so the supremum sits at $E = 1$, where $\varrho_6(1) = 3 - \sqrt{8} < 7/40$, $\varrho_3(1) = (3 - \sqrt{5})/2 > 3/8$ and $\vartheta > 6/5$ give $\lambda_1 < (7/40)^{12/5}(8/3)^2 < 1/9$. Since $6^\vartheta = 9$, $\varrho_6(4), \varrho_6(17/4) < 1/\sqrt{6}$ give $\lambda_2, \lambda_3 < 1/9$; $\varrho_3(17/4), \varrho_6(9/2) < 1/2$ give $\lambda_4 < 1/9$; and $x_3(9/2) = 41/24 > 5/3$ gives $\varrho_3(9/2) < 1/3$. With $\lambda < 1$ the largest ratio $9\lambda_i$,

$$\int_{|E|>1} T_\nu(E) \frac{dE}{2\pi} \leq C_* \tau_\nu (\nu+1)^2 \lambda^\nu. \quad (\text{S104})$$

Leakage inside the window, outside the target. Put $\eta_\nu = \nu^{-1/2}$. On $0 < E \leq 1$ every real diagonal entry of $\mathbf{M}$ is at least E , since $d_j < 0$ and $\text{Re } \sigma_\nu \leq 0$. Write $\mathbf{M} = \mathbf{D} - \mathbf{K}$ with $\mathbf{D}$ diagonal; where $\|\mathbf{K}\| \leq 2\mathbf{b}_\nu \leq E/2$, a Neumann expansion gives $|(\mathbf{M}^{-1})_{n1}| \leq 2\|\mathbf{K}\|/E^2$, because $\mathbf{D}^{-1}$ has no $(n, 1)$ entry. On $[0, \eta_\nu]$, analyticity in E^2 of $\log(v\varrho_v)$ and $\log(3\varrho_3)$ gives $\gamma_\nu, \mathbf{b}_\nu \leq C_*\tau_\nu$, so for $E \geq M\delta_\nu$ with a target constant in M ,

$$T_\nu(E) \leq C_* \frac{\tau_\nu^4}{E^4}, \quad \frac{1}{\delta_\nu} \int_{M\delta_\nu < |E| \leq \eta_\nu} T_\nu \frac{dE}{2\pi} \leq \frac{C_*}{M^3}. \quad (\text{S105})$$

On $[\eta_\nu, 1]$, $\mathbf{b}_\nu \leq C_* \varrho_6(1)^{\vartheta_\nu}$ and $\gamma_\nu \leq \varrho_3(1)^{2\nu}$ give a contribution at most $C_* \nu^2 \varrho_3(1)^{4\nu} \varrho_6(1)^{2\vartheta_\nu}$, which is $o(\tau_\nu)$ because $9\varrho_3(1)^4 \varrho_6(1)^{2\vartheta} < 9(2/5)^4/5^2 < 1$.

The error on the whole axis. The target transmission is $1/(1+Q^*)$ with Q^* of degree $2n \geq 4$, so $\int_{|x|>M} T^* dx \leq C_* M^{-3}$. Integrating Eq. (S97) over $|x| \leq M = \nu^{1/8}$ contributes $C_*(M^2/\nu + \delta_\nu M^3) \leq C_*(\nu^{-3/4} + \delta_\nu \nu^{3/8})$ to $\mathcal{E}_\nu/\delta_\nu$; the two tails contribute $C_* \nu^{-3/8}$ by Eq. (S105); and Eq. (S104) with the estimate on $[\eta_\nu, 1]$ contributes $C_*(\nu+1)^2 \lambda^\nu$ after enlarging $\lambda < 1$. This is Eq. (S92).

The constants C_* and ν_* depend on the target through its hoppings, its linewidth, and the supremum of its real-axis inverse, each finite. The threshold therefore exists for every fixed target; it is not claimed uniform in the target order.

C. Chebyshev targets and the vanishing product

Embedding Chebyshev targets of growing order drives the product to zero with every coupling bounded. There are chains with zero on-site energies, every hopping in $[1, 6]$, both linewidths equal to 1, and lengths $m_k \rightarrow \infty$, and finite affinities $A_k = k$, such that

$$P_k \leq \frac{4k}{k^2 - 1} + \frac{2k}{\sinh k} \rightarrow 0. \quad (\text{S106})$$

Their currents tend to zero, as Eq. (S87) requires.

An error in transmission small against the transmitted weight changes the current and the partition noise only by its relative size. Let T, T' be functions of E with values in $[0, 1]$, $\int T' dE/2\pi = \delta J > 0$, $\int T'(1-T')/\int T' = f$, and $\int |T - T'| dE/2\pi \leq \epsilon \delta J$ with $0 \leq \epsilon < 1$. Then the moments I_1, F_0 of T obey

$$\left| \frac{I_1}{\delta J} - 1 \right| \leq \epsilon, \quad F_0 \leq \frac{f + \epsilon}{1 - \epsilon}. \quad (\text{S107})$$

The triangle inequality gives $|I_1 - \delta J| \leq \epsilon \delta J$. Since $|x(1-x) - y(1-y)| = |x-y||1-x-y| \leq |x-y|$ on $[0, 1]$, the numerator of F_0 is at most $\delta J(f + \epsilon)$, while $I_1 \geq \delta J(1 - \epsilon)$.

For Eq. (S106), take as targets the Chebyshev chains of Sec. IX B with $w = 1$, odd order n and $\epsilon = 1/n$. They have zero on-site energies, positive hoppings and equal linewidths, and by Eq. (S83), since $n \geq \log(2n)$,

$$I_1^* \geq \frac{1}{\pi(1+n^{-2})}, \quad F_0^* \leq (1+n^{-2}) \frac{1+16[\log(2n)+1]}{n^2} \rightarrow 0. \quad (\text{S108})$$

For each $k \geq 2$ choose odd n_k with $F_0^* \leq k^{-2}$, and then, by Eq. (S92), a depth ν_k with $\mathcal{E}_{\nu_k} \leq k^{-2} \delta_{\nu_k} I_1^*$ and $m(n_k, \nu_k) > m(n_{k-1}, \nu_{k-1})$; both conditions hold for every large enough ν . Equation (S107) with $\epsilon = k^{-2}$ gives $F_{0,k} \leq 2/(k^2 - 1)$, and Eq. (S13) at $A_k = k$ gives Eq. (S106). Every member is a finite chain with positive hoppings, so Secs. II and III apply: it has a unique attracting stationary state in the thermal class, and Eq. (S13) holds. Since both linewidths equal 1, Eq. (S58) gives $j_k \leq I_{1,k} \leq 2(1 - F_{0,k})/\log(1/F_{0,k}) \rightarrow 0$.

The order of limits matters: for each target the depth is taken large first, and only then is the target sharpened. At a fixed target order the current falls as $9^{-\nu}$ while the length grows as $[4 + 2\vartheta(n-1)]\nu$. The current is thus exponentially small in the length, at a rate that falls as $1/n$. Along the sequence, where n_k grows, no rate in the length m_k is claimed.

XI. EVERY HILBERT-SPACE DIMENSION

The chains of Sec. VII have dimension 2^n . Auxiliary levels fill in the other dimensions, following the dimension-monotonicity argument of Ref. [1, Sec. 3.1]. Fix a base chain with $n \geq 2$ modes and finite $A > 0$, with generator $\mathcal{L}$, and an integer $k \geq 1$. Let $|\mathbf{F}\rangle = c_1^\dagger |\text{vac}\rangle$, so $H_0|\mathbf{F}\rangle = 0$. Add orthonormal states $|e_1\rangle, \dots, |e_k\rangle$ and set

$$H_N = H \oplus 0, \quad H_{0N} = H_0 \oplus I, \quad X_N = H_N - H_{0N}, \quad L' = L \oplus 0, \quad (\text{S109})$$

for both chain jumps L , together with one channel per auxiliary level,

$$V_a = |\mathbf{F}\rangle\langle e_a|, \quad \gamma_a = 1, \quad n_a = \delta > 0, \quad \Delta E_a = 1, \quad T_a = \frac{1}{\log(1+1/\delta)}. \quad (\text{S110})$$

The dimension is $N = 2^n + k$. Then $[H_{0N}, L'] = -AL'$ and $[H_{0N}, V_a] = -V_a$, every occupation, gap and temperature is positive and finite, and the padded generator $\mathcal{L}_\delta$ is the base generator with H_N and L' plus $\sum_a [(1+\delta)\mathcal{D}[V_a] + \delta\mathcal{D}[V_a^\dagger]]$. No Hamiltonian term touches an auxiliary level, and only the right base channel is counted.

A. Padding at small occupation leaves the product unchanged

The auxiliary levels carry no current, and at small occupation they leave the counting statistics of the chain unchanged. There is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the padded generator has a unique stationary state that attracts every density matrix. Each auxiliary current vanishes exactly, $\sigma_\delta = 2Aj_\delta$, and

$$j_\delta \rightarrow j, \quad d_\delta \rightarrow d, \quad P_\delta \rightarrow P \quad (\delta \rightarrow 0), \quad (\text{S111})$$

where j, d, P are the values of the base chain.

The unpadded limit. At $\delta = 0$, write an $N \times N$ matrix in blocks $Z = \begin{pmatrix} Z_{cc} & Z_{ca} \\ Z_{ac} & Z_{aa} \end{pmatrix}$ over the chain and the auxiliary levels. Absorb the chain rates into jump matrices F_b , so the chain dissipator is $\sum_b \mathcal{D}[F_b]$, and put $R_c = \sum_b F_b^\dagger F_b \succeq 0$. Since $\sum_a V_a Z V_a^\dagger = \text{Tr}(Z_{aa})|F\rangle\langle F|$ and $\sum_a V_a^\dagger V_a$ is the auxiliary projector, direct substitution gives

$$\begin{aligned} \mathcal{L}_0 Z &= \begin{pmatrix} \mathcal{L}Z_{cc} + \text{Tr}(Z_{aa})|F\rangle\langle F| & \Phi Z_{ca} \\ Z_{ac}\Phi^\dagger & -Z_{aa} \end{pmatrix}, \\ \Phi &= -iH - \frac{1}{2}R_c - \frac{1}{2}I. \end{aligned} \quad (\text{S112})$$

Auxiliary populations decay into the anchor state $|F\rangle$, and coherences with the auxiliary levels decay at rate at least $\frac{1}{2}$: if $\Phi u = \varpi u$, then $\text{Re } \varpi \|u\|^2 = -\frac{1}{2}\langle u, R_c u \rangle - \frac{1}{2}\|u\|^2 \leq -\frac{1}{2}\|u\|^2$. The map is block triangular in $(Z_{cc}, Z_{aa}, Z_{ca}, Z_{ac})$. Its spectrum, with multiplicities, is that of $\mathcal{L}$, the value -1 repeated k^2 times, k copies of the spectrum of Φ , and k copies of its conjugate. By Sec. II B, zero is algebraically simple and every other eigenvalue has negative real part; the stationary state is $\rho_* \oplus 0$.

The entries of $\mathcal{L}_\delta$ are affine in δ . The eigenvalues depend continuously on δ , and zero stays an eigenvalue because $\mathcal{L}_\delta$ preserves the trace. For small δ , zero therefore remains algebraically simple and the rest of the spectrum stays in the open left half-plane. Since $\mathcal{L}_\delta$ generates a completely positive trace-preserving semigroup for $\delta > 0$, its unique stationary state ρ_δ is a density matrix attracting every initial state, and $\rho_\delta \rightarrow \rho_* \oplus 0$.

No Hamiltonian term and no other channel touches $|e_a\rangle$, so for any state

$$\frac{d}{dt}\langle e_a | \rho | e_a \rangle = -(1 + \delta)\langle e_a | \rho | e_a \rangle + \delta\langle F | \rho | F \rangle = -j_a. \quad (\text{S113})$$

Stationarity gives $j_a = 0$ exactly, so the auxiliary channels produce no entropy although $A_a = \log(1 + 1/\delta)$ is large: $\sigma_\delta = A(j_L + j_R)$. The operator $Q_N = (\sum_j \hat{n}_j) \oplus I$ commutes with H_N and with every V_a , because $|F\rangle$ and each $|e_a\rangle$ carry the value one, so its stationary balance gives $j_L = j_R$ and $\sigma_\delta = 2Aj_\delta$.

Tilting the right channel does not change the coherence blocks or the auxiliary block, so at $\delta = 0$ the tilted generator has the block form (S112) with $\mathcal{L}$ replaced by its tilt. The characteristic polynomial of the tilted padded generator is jointly analytic in (δ, s) , and its root at $(0, 0)$ is simple. The implicit-function theorem gives a root $\lambda(\delta, s)$ jointly analytic near $(0, 0)$, with $\lambda(\delta, 0) = 0$ and $\lambda(0, s)$ equal to the chain's. For fixed small $\delta > 0$ the other roots stay to its left, so $\lambda(\delta, s)$ is the scaled cumulant generating function of the padded device. Its derivatives in s give $j_\delta \rightarrow j$ and $d_\delta \rightarrow d$; hence $P_\delta = 2Ad_\delta/j_\delta \rightarrow P$, which is Eq. (S111).

B. The size-law product in every dimension

Padding the size-law chain reaches every Hilbert-space dimension at no extra entropy production. For every integer $N \geq 4$ and $w > 0$ let $n = \lfloor \log_2 N \rfloor \geq 2$ and $k = N - 2^n$. There is an N -dimensional member of the thermal-jump class with positive finite temperatures, a unique attracting stationary state, and

$$P_N < 2V_n < 66 \frac{\log n}{n^2}, \quad j_N > \frac{w}{4\pi}, \quad \sigma_N = 2A_n j_N. \quad (\text{S114})$$

Consequently $0 \leq P_{\min}^N = O(\log \log N / (\log N)^2)$ along all integers $N \rightarrow \infty$.

To build the device, take the size-law chain with n modes at $A_n = 2 \log n$. If $k = 0$, it is the device, with $P_N \leq V_n < 2V_n$ and $j_N \geq w/(2\pi)$. If $k \geq 1$, pad it with k levels. By Sec. XI A choose $\delta > 0$ small enough for unique attraction, $|P_\delta - P_n| < V_n$ and $|j_\delta - j_n| < w/(4\pi)$; finitely many positive thresholds have a positive minimum. Then $P_N < P_n + V_n \leq 2V_n$ and $j_N > j_n - w/(4\pi) \geq w/(4\pi)$, strict even if the base bounds hold with equality. Finally $\log_2 N - 1 < n \leq \log_2 N$, so $\log n = \log \log N + O(1)$ and $n^{-2} = O((\log N)^{-2})$, and Eq. (S1) gives $P_{\min}^N \geq 0$.

The decomposition covers every integer: $N = 4, 5, 8, 16, 17$ correspond to $(n, k) = (2, 0), (2, 1), (3, 0), (4, 0), (4, 1)$. The occupation δ may depend on N and w ; we prove no uniform bound on it, on the auxiliary temperatures, or on relaxation. The padded devices are not chains, and the lower bound of Sec. IV does not apply to them.

XII. TRANSMISSION OF THE TRIGONOMETRIC CHAIN AT EVERY LENGTH

This section proves that the trigonometric chain of Ref. [11] at half-bandwidth one, with $\Gamma = 1/(2s_0)$ and $t_j = 1/(2\sqrt{s_{j-1}s_j})$, where $s_r = \sin((2r+1)\theta)$ and $\theta = \pi/2n$, transmits $T(E) = 1/(1 + E^{2n})$ at every length $n \geq 2$, with $I_1 = \Gamma/n$ and $I_2 = \Gamma/(2n^2)$. It also gives the roots of $\chi_n(E) = \det(E - h_{\text{eff}})$, $h_{\text{eff}} = h - i\Gamma(\Pi_1 + \Pi_n)$. Each s_r is positive and $s_{n-1-r} = s_r$.

Normalization. Evaluating $z^n + 1 = \prod_r (z - e^{i(2r+1)\pi/n})$ at $z = 1$ and taking moduli, with $|1 - e^{i\phi}| = 2\sin(\phi/2)$, gives

$$\prod_{r=0}^{n-1} s_r = 2^{1-n}. \quad (\text{S115})$$

With $D = \text{diag}(\sqrt{2s_0}, \dots, \sqrt{2s_{n-1}})$, the matrix $D(E - h_{\text{eff}})D$ has off-diagonal entries -1 , interior diagonal entries $2Es_r$, and end diagonal entries $2Es_0 + i$, because $2s_0\Gamma = 1$. By Eq. (S115), $\det D^2 = 2$, so

$$\tilde{\chi}_n \equiv \det[D(E - h_{\text{eff}})D] = 2\chi_n. \quad (\text{S116})$$

We show that

$$\chi_n(E) = \prod_{r=1}^n (E - e^{-i(2r-1)\theta}). \quad (\text{S117})$$

A matching identity. Consider a path of ℓ consecutive vertices $0, \dots, \ell-1$ with vertex weights s_{k+r} . Let $S(k, \ell, p)$ be the sum, over all sets of p disjoint edges, of the product of the weights of the uncovered vertices ($S = 1$ for the empty product). Here θ is arbitrary. For $0 \leq p \leq \lfloor \ell/2 \rfloor$,

$$\begin{aligned} S(k, \ell, p) &= \left(\prod_{r=p}^{\ell-1-p} s_{k+r} \right) g(\ell, p), \\ g(\ell, p) &= \prod_{r=1}^p \frac{\sin(2(\ell - p - r + 1)\theta)}{\sin(2r\theta)}, \end{aligned} \quad (\text{S118})$$

wherever the denominators are nonzero. For $p = 0$ and for $\ell = 2p$ (one perfect matching, $g(2p, p) = 1$) this is immediate. Otherwise, splitting on whether the last vertex is covered gives $S(k, \ell, p) = s_{k+\ell-1}S(k, \ell-1, p) + S(k, \ell-2, p-1)$. Inserting Eq. (S118) for the two shorter paths and dividing by the common factor, the induction step reduces to

$$\begin{aligned} s_{k+\ell-1-p} \sin(2(\ell-p)\theta) &= s_{k+\ell-1} \sin(2(\ell-2p)\theta) \\ &\quad + s_{k+p-1} \sin(2p\theta), \end{aligned}$$

which is the identity

$$\sin x \sin(u+v) = \sin(x+u) \sin v + \sin(x-v) \sin u$$

at $x = (2(k+\ell-1-p)+1)\theta$, $u = 2p\theta$, $v = 2(\ell-2p)\theta$. The division is legitimate on a dense set of θ and extends by continuity. In every use below $\ell \leq n$ and $p \leq n/2$, so $0 < 2r\theta \leq \pi/2$ and all denominators are positive.

Expansion at the ends. Let $F(k, \ell)$ be the determinant of the $\ell \times \ell$ tridiagonal matrix with off-diagonal entries -1 and diagonal entries $2Es_k, \dots, 2Es_{k+\ell-1}$, with $F(k, 0) = 1$. Expanding over matchings, $F(k, \ell) = \sum_p (-1)^p (2E)^\ell {}^{\ell-2p} S(k, \ell, p)$. Expanding $\tilde{\chi}_n$ in the two end entries i and using the reflection $s_{n-1-r} = s_r$, which makes the two $(n-1)$ -dimensional principal minors equal,

$$\tilde{\chi}_n = F(0, n) + 2iF(1, n-1) - F(1, n-2). \quad (\text{S119})$$

Writing $\tilde{\chi}_n = \sum_{j=0}^n i^j \kappa_j E^{n-j}$, Eq. (S119) gives $\kappa_0 = 2^n \prod_r s_r = 2$, and

$$\begin{aligned}\kappa_{2p} &= 2^{n-2p} [S(0, n, p) + S(1, n-2, p-1)], \\ \kappa_{2p+1} &= 2^{n-2p} S(1, n-1, p).\end{aligned}\tag{S120}$$

Closed form of the coefficients. Put $u_p = \prod_{r=p}^{n-1-p} s_r$, $o_p = \prod_{r=0}^{p-1} s_r$, and $g_p = \prod_{v=p+1}^{2p-1} \sin(2v\theta) / \prod_{r=1}^{p-1} \sin(2r\theta)$. The reflection $\sin(2(n-v)\theta) = \sin(2v\theta)$ turns Eq. (S118) into $g(n, p) = g(n-2, p-1) = g_p$, and both matchings in κ_{2p} leave the vertices $p, \dots, n-1-p$ uncovered, so $\kappa_{2p} = 2^{n-2p+1} u_p g_p$; this includes $n = 2p$, where $\kappa_n = 2$. By reflection and Eq. (S115), $o_p^2 u_p = 2^{1-n}$, so $\kappa_{2p} = 2^{2-2p} g_p / o_p^2$. Define

$$a_j = \frac{\prod_{v=0}^{j-1} \cos(v\theta)}{\prod_{u=1}^j \sin(u\theta)}, \quad a_0 = 1.\tag{S121}$$

Using $\cos(v\theta) = \sin(2v\theta) / (2 \sin(v\theta))$ for $1 \leq v \leq 2p-1$ and separating odd and even indices gives $a_{2p} = g_p / (2^{2p-1} o_p^2)$, so $\kappa_{2p} = 2a_{2p}$, and $\kappa_0 = 2a_0$. For odd orders, Eq. (S118) gives $S(1, n-1, p) = (u_p / s_p) \prod_{v=p+1}^{2p} \sin(2v\theta) / \prod_{r=1}^p \sin(2r\theta)$, hence

$$\frac{\kappa_{2p+1}}{\kappa_{2p}} = \frac{\sin(4p\theta)}{2s_p \sin(2p\theta)} = \frac{\cos(2p\theta)}{\sin((2p+1)\theta)} = \frac{a_{2p+1}}{a_{2p}},$$

and $\kappa_1 = 2^n \prod_r s_r / s_0 = 2/s_0 = 2a_1$. Therefore $\tilde{\chi}_n = 2 \sum_{j=0}^n i^j a_j E^{n-j}$.

The roots. With $\zeta = e^{-i\theta}$, the right side of Eq. (S117) is $\prod_{r=0}^{n-1} (E - \zeta^{2r+1})$. Its coefficient of E^{n-j} is $(-\zeta)^j$ times the elementary symmetric polynomial of $1, \zeta^2, \dots, \zeta^{2(n-1)}$,

$$\mathfrak{e}_j = \zeta^{j(j-1)} \prod_{u=1}^j \frac{1 - \zeta^{2(n-u+1)}}{1 - \zeta^{2u}},$$

which follows by induction on the number ν of variables, at fixed ζ , from $\mathfrak{e}_j(\nu) = \mathfrak{e}_j(\nu-1) + \zeta^{2(\nu-1)} \mathfrak{e}_{j-1}(\nu-1)$ and $(1 - \zeta^{2(\nu-j)}) + \zeta^{2(\nu-j)}(1 - \zeta^{2j}) = 1 - \zeta^{2\nu}$. Since $n\theta = \pi/2$, $(1 - \zeta^{2(n-u+1)}) / (1 - \zeta^{2u}) = e^{-i(n+1-2u)\theta} \cos((u-1)\theta) / \sin(u\theta)$, and the phases combine to $(-1)^j e^{-ijn\theta} = i^j$. The coefficient is $i^j a_j$, which proves Eq. (S117) through Eq. (S116).

Transmission. Every root in Eq. (S117) lies strictly in the lower half-plane. These n roots and their conjugates are exactly the $2n$ roots of $E^{2n} + 1$, so $|\chi_n(E)|^2 = 1 + E^{2n}$ for real E . By Eq. (S115) and $s_{n-1} = s_0$, $\prod_j t_j^2 = s_0^2 / (4^{n-1} \prod_r s_r^2) = s_0^2$, so $4\Gamma^2 \prod_j t_j^2 = 1$, and Eq. (S26) gives $T = 1/(1 + E^{2n})$.

Integrals. A contour integral over the sector of opening angle π/n , which encloses only the pole at $e^{i\theta}$, gives $\int_0^\infty dx / (1 + x^{2n}) = \pi / (2n \sin \theta)$, so $I_1 = 1 / (2n \sin \theta) = \Gamma/n$. Since $ET' = -2nT(1 - T)$ and $ET \rightarrow 0$ at both ends, integration by parts gives $I_2 = I_1 / (2n)$.

For $n = 2$, $\chi_2 = E^2 + i\sqrt{2}E - 1$, $t_1 = \Gamma = 1/\sqrt{2}$ and $I_1 = 1/(2\sqrt{2})$; for $n = 3$, $\chi_3 = (E + i)(E^2 + iE - 1)$, $I_1 = 1/3$ and $I_2 = 1/18$. Since $F_0 = 1/(2n)$, Eq. (S13) gives $P_{\text{tr}} = Aq/n + \beta(A)$. Splitting the affinity at $\log n$ as in Sec. IV, and evaluating at $A = \log n$, gives $\log n / (2n) \leq \min_A P_{\text{tr}} \leq 6 \log n / n$ for $n \geq 3$: the window edges of this profile carry noise of order $1/n$ rather than $1/n^2$.

XIII. FINITE-DIMENSIONAL CALIBRATION

This section checks that our counting conventions reproduce the finite-dimensional values of Ref. [1], and records a four-level member below the reported four-level value. The models of Ref. [1] have one channel, $\gamma = 1$, a ladder jump $L = \sum_k l_k |k\rangle\langle k+1|$, and a total Hamiltonian with zero diagonal. We write $\bar{n}$ for the occupation of their single channel. With $H_0 = \text{diag}(0, 1, \dots, N-1)$, $A = \log[(1 + \bar{n})/\bar{n}]$, $T = 1/A$, and $X = H - H_0$, every such model belongs to the class with $[H_0, L] = -L$, and with one channel $\sigma = Aj$ and $P = Ad/j$.

Replay of the printed parameters. Table S1 lists the printed parameters. Evaluating Eq. (S10) at these values, taken as exact, gives the products in the table. Two independent implementations, one solving the stationary and response equations and one differentiating the tilted eigenvalue numerically at 50-digit precision, agree to 3×10^{-15} . For $N = 2$, $j = 1/4$ and $P = A[(2\bar{n} + 1) - 3/(4(2\bar{n} + 1))]$ exactly at $H_{12} = (2\bar{n} + 1)/(2\sqrt{2})$. The values for $N = 2$ and $N = 3$ agree with Ref. [1] to the printed digits. The four-level value is 0.262667, against the reported 0.2625.

TABLE S1. Printed parameters of Ref. [1] (upper triangle of H ; the lower triangle is conjugate) and the products they give in our conventions, compared with the reported values.

N	$\bar{n}; l_k$	H	P (this work)	reported
2	0.02779; 1	$H_{12} = (2\bar{n} + 1)/(2\sqrt{2})$	1.245872	1.2459
3	0.01096; 1, 0.03357	$H_{12} = 0.1681, H_{23} = 0.3347, H_{13} = 0.2164i$	0.472421	0.47242
4	0.00560; 1, 0.0280, 0	$H_{12} = 0.101, H_{23} = 0.343 - 0.025i, H_{34} = -0.206 - 0.012i,$ $H_{13} = 0.008 + 0.132i, H_{24} = -0.002 - 0.059i, H_{14} = 0.139$	0.262667	0.2625

A. Exact enclosures of the four-level values

Whether the difference from the reported four-level value is ours, and whether a four-level member does better, is settled by a computer-assisted evaluation in exact arithmetic. It shows three things. Part (i): at the printed four-level parameters, $P > 0.26255$. Part (ii): the products at $\text{Re } H_{23} = 0.34251$ and at 0.34349 , both of which round to the printed 0.343 , differ by more than 10^{-4} . Part (iii): the four-level member with $\bar{n} = 0.005399090$, $l = (1, 0.030296333, 0.004094694)$, zero diagonal, and $H_{12} = 0.099280080$, $H_{23} = 0.348078039 + 0.000002033i$, $H_{34} = -0.208739471 + 0.000015192i$, $H_{13} = -0.000000614 + 0.132090823i$, $H_{24} = -0.000004446 - 0.067632678i$, $H_{14} = 0.147669178 - 0.000010122i$ (all decimals exact) has a unique, faithful, attracting stationary state and $P = 0.2611508361761\dots$; hence $P_{\min}^4 < 0.26116$.

Exact arithmetic. Each case is one 16×16 matrix M representing the generator on 4×4 matrices, with rational complex entries. The test applied is exact. Replacing the first row of M by the trace functional gives a matrix M' , inverted in exact rational arithmetic. The stationary state ρ , the image under M'^{-1} of the first unit vector, and the response η of Eq. (S10) are then exact rationals, verified by substitution with zero residual, and j and d are exact positive rationals. In all four cases the leading principal minors of the exact ρ are positive, so $\rho \succ 0$. The only irrational quantity is $A = \log[(1 + \bar{n})/\bar{n}]$. We write $(1 + \bar{n})/\bar{n} = 2^k x$ with $1 \leq x < 2$, so $A = k \log 2 + \log x$; in (i) and (ii), $A = 7 \log 2 + \log(1257/896)$. Each logarithm is enclosed between rational bounds by the series $\log x = 2 \sum_{j \geq 0} z^{2j+1}/(2j+1)$, $z = (x-1)/(x+1) \leq 1/3$, truncated after ν terms. The positive tail is at most $2z^{2\nu+1}/[(2\nu+1)(1-z^2)]$. We take $\nu = 40$ in (i) and (ii), and $\nu = 80$ in (iii). Multiplying by the exact d/j encloses P in a rational interval narrower than 10^{-40} in (i) and (ii) and 10^{-79} in (iii). These intervals are decisive because they are rigorous and far narrower than the margins. The lower end in (i) exceeds 0.26255 , the gap between the two intervals in (ii) exceeds 10^{-4} , and the upper end in (iii) is below 0.26116 . The cases are finite in number and each is evaluated completely. In (iii), $L^\dagger L = \text{diag}(0, 1, l_2^2, l_3^2)$ has distinct entries, so a matrix commuting with L and $L^\dagger$ is diagonal and then scalar. In (i) and (ii), $l_3 = 0$, but the pairs of diagonal entries of $L^\dagger L$ and $LL^\dagger$ still separate the four levels. A commuting matrix is therefore diagonal, and commuting with L makes it $a\Pi_{123} + b\Pi_4$, with Π_{123} and Π_4 the projections on levels 1–3 and on level 4. In Step 3 of Sec. II, $[H, Y] = \xi Y$ for such a Y gives, in the entry $(1, 4)$, $H_{14}(b - a) = 0$; since $H_{14} = 0.139 \neq 0$, Y is scalar and $\xi = 0$. The argument of Steps 3 and 4, with this commutant in place of Step 1, then applies in all four cases. Zero is algebraically simple, every other eigenvalue has negative real part, and every density matrix relaxes to ρ , so Eq. (S10) gives the long-time cumulants. For (iii), a separate evaluation of the tilted eigenvalue in 60-digit arithmetic agrees with the exact cumulants to 10^{-17} .

Part (i) shows that the gap between 0.262667 and the reported 0.2625 is not a rounding error of our computation. Part (ii) shows that the printed parameters cannot fix the fourth digit. Inputs indistinguishable at the printed precision give products differing by more than twice the half-width 5×10^{-5} of the reported last digit. The cause of the difference is therefore not decidable from the published table; both values agree with the summary value 0.26 of Ref. [1]. Part (iii) shows that the four-level ladder model of Ref. [1] reaches below its reported optimum, so $P_{\min}^4 < 0.26116 < 0.2625$. These values calibrate the conventions; none of them enters the proofs of the main text.

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