

THE MINIMAX DISCRIMINATION RATE OF GENERAL ONE-DIMENSIONAL PERIODIC GRID CODES

QIHANG WANG

ABSTRACT. We determine, up to universal multiplicative constants, the minimax discrimination time of the full admissible class of one-dimensional periodic grid-cell codes with n binary neurons. If $0 < \rho \leq 1/2$ is the required separation on the stimulus circle, then the minimax time is infinite exactly when $\rho < 2^{-n}$. In the finite regime it is comparable to $\log_2(1/\rho)/n$, with explicit constants. The lower bound follows from a finite Abel-summation certificate that is uniform over the number of modules, the allocation of neurons, the integer-nested scales, and the receptive-field placements. Published extreme and balanced dyadic constructions give the matching upper rate. The main results of this paper were obtained through a generative-AI workflow using OpenAI GPT-5.6 Sol and Anthropic Claude Fable 5. Further details appear in the disclosure at the end of the paper.

1. INTRODUCTION

Periodic grid-cell models encode a position by several modules operating at different spatial scales. For fixed module sizes and scales, Ost and Reynaud-Bouret obtained quantitative discrimination bounds and explicit constructions; they also formulated the minimax problem for the union in which the module count, allocation, scales, and receptive fields all vary [5]. We resolve that full-union optimization at the level of universal constants.

The biological motivation comes from the discovery of periodic spatial maps in entorhinal cortex [1]. Subsequent theoretical work viewed grid codes as robust analog error-correcting codes [2] and as economical hierarchical representations with geometrically related scales [3]. These perspectives make the architecture-optimized discrimination problem natural, but they do not supply the uniform minimax theorem proved here. In particular, the statistical-testing analysis of Ost and Reynaud-Bouret explicitly leaves the minimax rate for the general union (and simultaneous adaptation) as an open problem [5]; our contribution is to close the rate question in the stated one-dimensional, integer-nested model while keeping the adaptation question separate.

Recent experimental and synthetic-population analyses further report measurable within-module variability in grid spacing and orientation, with improved local decoding under that heterogeneity [4]. Such variability is outside the equal-scale interval model treated here, so the theorem should be read as a sharp baseline for the idealized nested architecture and as a precise starting point for a future heterogeneous-code minimax problem.

The broader neuroscience picture is also becoming more dynamic. A recent review places grid codes in both physical and non-physical cognitive maps [6], while recordings during path-integration behavior show rapid re-anchoring between task-relevant reference frames [7]. These observations strengthen the value of a transparent benchmark: the theorem isolates the information-theoretic cost of periodic ambiguity before environmental dynamics, reference-frame switching, or within-module heterogeneity are introduced.

The threshold is sharp: no binary code with n neurons can distinguish every pair below distance 2^{-n} , whereas an extreme dyadic grid code reaches that resolution. Above the threshold, the main issue is uniformity over all fixed grid configurations. The key observation is that the bottleneck appearing in the fixed-configuration lower bound has an explicit nonnegative dual certificate. A

Key words and phrases. grid-cell codes, periodic codes, minimax discrimination, neural coding, statistical testing.

finite Abel identity turns all scale constraints into the factor $1 + \log(1/\rho)$ without a continuous relaxation.

Our result concerns the full admissible one-dimensional periodic union and allows the minimizing code to depend on ρ . It does not determine the optimal multiplicative constant, produce one fixed code adaptive for every ρ , or address higher-dimensional models or non-integer scale ratios.

2. CODES AND DISCRIMINATION TIME

Identify the stimulus circle with $\mathbb{S}^1 = [0, 1)$, with endpoints identified, and use the normalized geodesic distance

$$d(s, t) = \min\{|s - t|, 1 - |s - t|\}.$$

A binary code with n neurons is a vector $f = (f_1, \dots, f_n)$ whose coordinates take two firing-rate values. Only the active set matters for the normalized discrimination quantity, so write

$$I_s^f = \{r \in \{1, \dots, n\} : f_r(s) \text{ is at the high rate}\}$$

and

$$\Delta_{s,t}^f = \max\{|I_s^f \setminus I_t^f|, |I_t^f \setminus I_s^f|\}.$$

With the convention $1/0 = \infty$, define

$$T(f, \rho) = \max_{d(s,t) \geq \rho} \frac{1}{\Delta_{s,t}^f}, \quad T(\mathcal{F}, \rho) = \inf_{f \in \mathcal{F}} T(f, \rho).$$

This is the combinatorial normalization of discrimination time used in [5]; the statistical time differs by factors depending only on the fixed firing rates and error level.

Fix an integer $m \geq 1$, positive integers $n_1, \dots, n_m$ with $\sum_i n_i = n$, and scales

$$1 = \lambda_1 > \lambda_2 > \dots > \lambda_m > 0$$

such that λ_i^{-1} and λ_i/λ_{i+1} are positive integers. In module i , each of its n_i neurons is the periodic lift to $\mathbb{S}^1$ of the indicator of an arbitrary circular interval on the circle of length λ_i . Denote this fixed-configuration class by

$$\mathcal{G}((n_i, \lambda_i)_{i=1}^m).$$

The general class is the union over every admissible choice,

$$\bar{\mathcal{G}}_n = \bigcup_{m=1}^n \bigcup_{n_1 + \dots + n_m = n} \bigcup_{(\lambda_i)} \mathcal{G}((n_i, \lambda_i)_{i=1}^m).$$

We use the following results from [5]. They are collected here in the precise form needed below.

Proposition 2.1 (Published fixed-class bounds and constructions). *The following statements hold.*

- (1) *Every binary code with n neurons has $T(f, \rho) = \infty$ when $0 < \rho < 2^{-n}$.*
- (2) *For a one-module class, either the time is infinite or*

$$T \geq \frac{1}{\lfloor 4n\rho \rfloor}.$$

- (3) *Suppose $m \geq 2$, and either λ_1/λ_2 is even and $\rho \leq 1/2$, or λ_1/λ_2 is odd and $\rho \leq \frac{1}{2} - \frac{\lambda_2}{2}$; this is the hypothesis of Theorem 4.2 in [5]. Put $\lambda_{m+1} = 0$ and*

$$A = \min_{1 \leq k \leq m} \left(\sum_{i=1}^k \frac{n_i}{\lambda_i} \right) \max\{\rho, \lambda_{k+1}\}.$$

Then

$$T(\mathcal{G}((n_i, \lambda_i)_{i=1}^m), \rho) \geq \frac{1}{\lfloor 6A \rfloor}.$$

- (4) *The extreme dyadic code is admissible and has time one whenever $\rho \geq 2^{-n}$.*

- (5) Let $1 \leq m \leq n$, take $\lambda_i = 2^{-(i-1)}$, and distribute the neurons as evenly as possible among the m modules. If $n \geq 2m$ and $2^{-m} \leq \rho \leq 1/2$, an explicit receptive-field placement has

$$T \leq \frac{16}{\lfloor n/m \rfloor}.$$

3. A DISCRETE ABEL CERTIFICATE

The following lemma is the uniform optimization step.

Lemma 3.1 (Abel certificate). *Fix an admissible configuration with $m \geq 2$. If $0 < \rho \leq 1/4$, then the quantity A in Proposition 2.1 satisfies*

$$A \leq \frac{n}{1 + \log(1/\rho)}.$$

Consequently, unless the fixed-class time is infinite,

$$T(\mathcal{G}((n_i, \lambda_i)_{i=1}^m), \rho) \geq \frac{1 + \log(1/\rho)}{6n}.$$

Proof. The fixed-class lower bound is applicable on this range. Indeed, if λ_1/λ_2 is even, this is one of the alternatives in Proposition 2.1. If it is odd, admissibility and $\lambda_1 = 1$ imply $\lambda_2 \leq 1/2$, so

$$\rho \leq \frac{1}{4} \leq \frac{1}{2} - \frac{\lambda_2}{2},$$

which is the other alternative.

Let

$$j = \max\{i : \lambda_i \geq \rho\}, \quad S_k = \sum_{i=1}^k \frac{n_i}{\lambda_i}.$$

For $k > j$, the term defining A is $\rho S_k \geq \rho S_j$. For $k < j$ it is $\lambda_{k+1} S_k$, while for $k = j$ it is ρS_j . Thus

$$A = \min(\{\lambda_{k+1} S_k : 1 \leq k < j\} \cup \{\rho S_j\}),$$

and hence

$$S_k \geq \frac{A}{\lambda_{k+1}} \quad (1 \leq k < j), \quad S_j \geq \frac{A}{\rho}.$$

Since $S_i - S_{i-1} = n_i/\lambda_i$, with $S_0 = 0$, finite Abel summation gives

$$\sum_{i=1}^j n_i = \lambda_j S_j + \sum_{i=1}^{j-1} (\lambda_i - \lambda_{i+1}) S_i.$$

All coefficients are nonnegative. Substituting the displayed constraints therefore yields

$$n \geq A \left[\frac{\lambda_j}{\rho} + \sum_{i=1}^{j-1} \left(\frac{\lambda_i}{\lambda_{i+1}} - 1 \right) \right].$$

For $x \geq 1$ and $r > 1$, the elementary bounds $x \geq 1 + \log x$ and $r - 1 \geq \log r$ give

$$\begin{aligned} \frac{\lambda_j}{\rho} + \sum_{i=1}^{j-1} \left(\frac{\lambda_i}{\lambda_{i+1}} - 1 \right) &\geq 1 + \log \frac{\lambda_j}{\rho} + \sum_{i=1}^{j-1} \log \frac{\lambda_i}{\lambda_{i+1}} \\ &= 1 + \log \frac{1}{\rho}, \end{aligned}$$

because $\lambda_1 = 1$. This also covers $j = 1$, when the sum is empty. The asserted bound on A follows.

Finally, Proposition 2.1 gives $T \geq 1/\lfloor 6A \rfloor$. If the floor vanishes, the time is infinite. Otherwise

$$T \geq \frac{1}{6A} \geq \frac{1 + \log(1/\rho)}{6n},$$

as required. $\square$

4. THE FULL MINIMAX RATE

Theorem 4.1 (General one-dimensional periodic grid codes). *For every integer $n \geq 1$ and every $0 < \rho \leq 1/2$,*

$$T(\bar{\mathcal{G}}_n, \rho) = \infty \iff 0 < \rho < 2^{-n}.$$

For $2^{-n} \leq \rho \leq 1/2$,

$$\frac{\log 2}{6} \frac{\log_2(1/\rho)}{n} \leq T(\bar{\mathcal{G}}_n, \rho) \leq 64 \frac{\log_2(1/\rho)}{n}.$$

Proof. Every member of $\bar{\mathcal{G}}_n$ is a binary code. Proposition 2.1(1) therefore makes the full-union infimum infinite for $\rho < 2^{-n}$. At and above the endpoint, Proposition 2.1(4) supplies an admissible code of finite time. This proves the equivalence, including $\rho = 2^{-n}$.

We next prove the lower bound. Suppose first that $0 < \rho \leq 1/4$. For every fixed configuration with $m \geq 2$, Lemma 3.1 gives, unless the time is infinite,

$$T \geq \frac{\log(1/\rho)}{6n} = \frac{\log 2}{6} \frac{\log_2(1/\rho)}{n}.$$

For $m = 1$, Proposition 2.1(2) gives, in the finite case,

$$T \geq \frac{1}{4n\rho} \geq \frac{\log(1/\rho)}{4n},$$

since $\rho \log(1/\rho) \leq 1/e < 1$. The same lower bound is therefore uniform over the entire union.

If $1/4 < \rho \leq 1/2$, then $\Delta_{s,t}^f \leq n$ for every binary code. Thus every finite time is at least $1/n$. Since $1 \leq \log_2(1/\rho) < 2$, this is at least $(\log 2/6) \log_2(1/\rho)/n$. The lower bound now holds throughout the finite regime.

For the upper bound, put $L = \log_2(1/\rho)$ and $m = \lceil L \rceil$. In the finite regime, $1 \leq L \leq n$, so $1 \leq m \leq n$ and $\rho \geq 2^{-m}$. If $n \geq 2m$, Proposition 2.1(5) yields

$$T(\bar{\mathcal{G}}_n, \rho) \leq \frac{16}{\lfloor n/m \rfloor} \leq \frac{32m}{n} \leq 64 \frac{L}{n},$$

using $\lfloor n/m \rfloor \geq n/(2m)$ and $m \leq L + 1 \leq 2L$. If $n < 2m$, the extreme dyadic code instead gives $T = 1$, while

$$n < 2\lceil L \rceil \leq 2(L + 1) \leq 4L.$$

Hence $T < 4L/n \leq 64L/n$. These two cases exhaust the finite regime. $\square$

Remark 4.2 (Scope). The upper construction may depend on ρ . Theorem 4.1 does not assert an exact leading constant or simultaneous adaptation of one fixed code, and its scale hypothesis is the integer-nested one-dimensional model used in the definition of $\bar{\mathcal{G}}_n$.

ACKNOWLEDGEMENTS

The author acknowledges Gewu Intelligence Lab for providing the collaborative research environment in which this project was developed.

DISCLOSURE OF AUTOMATED ASSISTANCE

OpenAI GPT-5.6 Sol was used in a generative-AI workflow to assist with proof-structure exposition, notation organization, LaTeX drafting, and revision of the mathematical narrative in Sections 1–4. Anthropic Claude Fable 5 was used in a manuscript-specific editorial review to identify presentation and metadata corrections, including clarification of a cited theorem’s hypotheses, completion of a parameter range, tightening of a proof justification, and correction of a bibliography author name. The results await further verification by domain experts.

REFERENCES

- [1] T. Hafting, M. Fyhn, S. Molden, M.-B. Moser, and E. I. Moser, *Microstructure of a spatial map in the entorhinal cortex*, *Nature* **436** (2005), 801–806, doi:10.1038/nature03721.
- [2] S. Sreenivasan and I. Fiete, *Grid cells generate an analog error-correcting code for singularly precise neural computation*, *Nature Neuroscience* **14** (2011), 1330–1337, doi:10.1038/nn.2901.
- [3] X.-X. Wei, J. Prentice, and V. Balasubramanian, *A principle of economy predicts the functional architecture of grid cells*, *eLife* **4** (2015), e08362, doi:10.7554/eLife.08362.
- [4] W. T. Redman, S. Acosta-Mendoza, X.-X. Wei, and M. J. Goard, *Robust variability of grid cell properties within individual grid modules enhances encoding of local space*, *eLife* **13** (2025), RP100652, doi:10.7554/eLife.100652.
- [5] G. Ost and P. Reynaud-Bouret, *Neural coding as a statistical testing problem*, *Mathematical Neuroscience and Applications* **3** (2023), article 4, doi:10.46298/mna.10002; arXiv:2209.00950.
- [6] D. Chen, N. Axmacher, and L. Wang, *Grid codes underlie multiple cognitive maps in the human brain*, *Progress in Neurobiology* **233** (2024), article 102569, doi:10.1016/j.pneurobio.2024.102569.
- [7] J.-J. Peng, B. Throm, M. Najafian Jazi, T.-Y. Yen, R. Pizzarelli, H. Monyer, and K. Allen, *Grid cells accurately track movement during path integration-based navigation despite switching reference frames*, *Nature Neuroscience* **28** (2025), 2092–2105, doi:10.1038/s41593-025-02054-6.

SCHOOL OF MATHEMATICAL SCIENCES, PEKING UNIVERSITY, BEIJING 100871, CHINA
 Email address: 2501110049@stu.pku.edu.cn