

Universal Radial Eversion of Planar Triangulations

Anonymous submission

Abstract

We prove that every finite straight-line planar triangulation in the general-position setting of Problem P3535 admits a radial eversion. The construction keeps each vertex on its prescribed positive ray, exchanges the outer face with the face containing the origin, and yields a nondegenerate crossing-free straight-line drawing of the same maximal planar graph. We also prove the exact ray-order statement, including rays through vertices by treating all edges met at one point as a single tied event. The proof is universal and does not depend on a search, simulation, or finite computation.

1 Problem and exact formulation

The radial-eversion question was stated in Section 5.3 of Dagstuhl Report 25201 [1]; its spherical setting is connected to the work of Erickson and Howard [2, 3]. We use the source contract pinned in the P3535 repository [4] and make the conventions needed by the proof explicit here.

Let $G = (V, E)$ be a finite simple maximal planar graph with $|V| \geq 4$, and let $P = (p_i)_{i \in V}$ be a crossing-free straight-line embedding of G in $\mathbb{R}^2$. Its outer face is a triangle. A distinguished point lies strictly inside one bounded triangular face. After translation we take that point to be the origin and write

$$p_i = u_i = (u_i^x, u_i^y) \in \mathbb{R}^2.$$

For planar points a, b, c , our orientation convention is

$$\text{orient}(a, b, c) = \det(b - a, c - a).$$

For vertices i, j , we abbreviate

$$[ij] = \det(u_i, u_j). \tag{1}$$

Let N denote the bounded face containing the origin and let S denote the outer face. We assume the stated general position condition

$$\det(u_i, u_j) \neq 0 \quad \text{for all distinct } i, j \in V. \tag{2}$$

In particular, no vertex is the origin and no two vertices lie on the same supporting line through the origin. Every bounded face is oriented counterclockwise, while S is oriented clockwise. These are the compatible orientations after the outer face is added to form the abstract sphere triangulation. We use the standard simplicial-sphere convention $|V| \geq 4$. Besides making the tetrahedron the smallest instance, this ensures that distinct face occurrences have distinct triangular boundary cycles. The convention is needed for the word/face equivalence stated in the source: the two face occurrences of K_3 share all three edges and cannot be distinguished by an edge word.

The phrase “ray order” requires care on a ray through a vertex. The following definition covers generic and nongeneric rays uniformly.

Definition 1.1 (Grouped exact-ray events). For a drawing $P = (p_i)$ in which no edge lies on a line through the origin, and for a nonzero direction $d \in \mathbb{R}^2$, put $R_d = \{td : t \geq 0\}$. For each $t \geq 0$, define the abstract edge block

$$\mathcal{E}_P(d, t) = \{ij \in E : td \in \overline{p_i p_j}\}.$$

Delete the empty blocks and list the remaining blocks in strictly increasing order of t . The resulting ordered partition of the edges met by R_d is denoted by $W_P(d)$. Edges in one block are tied and have no internal order. For an ordered list of blocks W , $\text{rev } W$ reverses the order of its blocks and leaves each block unchanged.

Each edge meets a fixed ray in at most one point under the condition in the definition, so the list is finite because G is finite. In an embedding satisfying (2), every event away from a vertex is a singleton, and the event at a vertex v consists of all edges incident to v . Proposition 1.5 proves this assertion and the required one-sided limiting rule.

Definition 1.2 (Radial eversion). A radial eversion of P is a crossing-free nondegenerate straight-line embedding $P' = (p'_i)_{i \in V}$ of the same graph G such that:

- (i) $p'_i = \lambda_i u_i$ for some $\lambda_i > 0$ for every $i \in V$;
- (ii) the original origin-containing face N is the outer face of P' , and the origin lies strictly inside the original outer face S in P' ; and
- (iii) $W_{P'}(d) = \text{rev } W_P(d)$ for every nonzero direction d .

Condition (i) fixes every vertex on its original positive ray. Condition (ii) exchanges the two distinguished faces. Condition (iii) is stated on abstract edge blocks, so it remains meaningful when several incident edges meet a ray at their common endpoint. All inequalities in the construction below are strict; thus “nondegenerate” includes, in particular, that no target face is collinear.

Our terminal result is positive and universal.

Theorem 1.3 (Universal radial eversion). *Every drawing satisfying the hypotheses above has a radial eversion in the sense of Definition 1.2.*

The proof is entirely exact over $\mathbb{R}$. General position is used through (2), which makes every bracket and every face circuit appearing below nonzero. Maximal planarity supplies the compatible sphere triangulation and the dual paths used at the end of the proof.

1.1 Ray words, face orientations, and the strict system

We first prove the geometric equivalence implicit in the problem statement. Call a direction *generic* if its ray contains no vertex. Under (2), such a ray is transverse to every edge it meets.

Lemma 1.4 (Positive-cone edge incidence). *Let u_i, u_j be linearly independent, let $\lambda_i, \lambda_j > 0$, and put $p_i^\lambda = \lambda_i u_i$ and $p_j^\lambda = \lambda_j u_j$. A ray R_d meets $\overline{p_i^\lambda p_j^\lambda}$ if and only if d belongs to the closed positive cone of u_i, u_j . Thus positive radial scaling preserves the set of abstract edges met by every ray. If the intersection is in the relative interior of the edge, its ray parameter varies continuously with d . Moreover, for the abstractly directed edge $i \rightarrow j$, the sign of its transverse crossing with R_d is independent of the positive radial scales.*

Proof. Because u_i, u_j are independent, write uniquely $d = \alpha u_i + \beta u_j$. If

$$td = (1-s)\lambda_i u_i + s\lambda_j u_j, \quad t \geq 0, \quad 0 \leq s \leq 1,$$

then $\alpha, \beta \geq 0$. Conversely, if $\alpha, \beta \geq 0$, not both zero, set

$$t = \left(\frac{\alpha}{\lambda_i} + \frac{\beta}{\lambda_j} \right)^{-1}, \quad s = \frac{t\beta}{\lambda_j}. \quad (3)$$

Then $0 \leq s \leq 1$ and the displayed intersection equation holds. The intersection is in the edge interior exactly when $\alpha, \beta > 0$. Inside that open cone, α, β , and hence t , vary continuously with d . Finally,

$$\det(p_j^\lambda - p_i^\lambda, d) = -(\lambda_j \alpha + \lambda_i \beta)[ij]. \quad (4)$$

The parenthesized factor is positive, so the sign depends only on the abstractly directed edge and its two original rays, not on the scales. $\square$

Proposition 1.5 (Ray-word and orientation equivalence). *Let $P = (u_i)$ and $P^\lambda = (\lambda_i u_i)$, with all $\lambda_i > 0$, be two crossing-free nondegenerate straight-line embeddings of the same labeled maximal planar triangulation. Let N and S be, respectively, the origin-containing and outer faces of P . The following conditions are equivalent:*

- (a) $W_{P^\lambda}(d) = \text{rev } W_P(d)$ for every nonzero direction d ;
- (b) in P^λ , the face N is outer, the origin lies strictly inside S , and every bounded face of P other than N has the opposite orientation in P^λ .

Proof. The origin remains strictly inside the geometric triangle N under every positive radial scaling. Indeed, a strictly positive dependence $\sum_{i \in N} \rho_i u_i = 0$ becomes the strictly positive dependence $\sum_{i \in N} (\rho_i / \lambda_i) (\lambda_i u_i) = 0$, after normalization of the coefficients. The target triangle N lies in the closed target outer triangle, so the origin lies strictly inside that outer triangle. General position then puts it in the strict interior of a unique target bounded face.

We begin with generic rays. By Lemma 1.4, a generic ray meets exactly the same abstract edges in both realizations. In P , those edges form a simple path in the abstract dual from N to S . Indeed, the intersection of a ray with a Euclidean triangle is connected, so the ray cannot leave and later re-enter a face. If condition (b) holds, the same abstract dual path in P^λ runs from S to N ; hence its edge order is reversed.

Here is the equivalent local orientation check, which will also prove the converse. Traverse the boundary of an abstractly oriented triangle $f = (i, j, k)$ through $i \rightarrow j \rightarrow k \rightarrow i$. In either realization, write q_a, q_b for the geometric endpoints of a directed boundary edge $a \rightarrow b$. When a generic outward ray crosses the triangle, a counterclockwise realization has

$$\det(q_b - q_a, d) > 0$$

at the entrance edge and the opposite sign at the exit edge; for a clockwise realization the two roles are reversed. This follows because the interior of a counterclockwise boundary lies locally to the left of every directed edge. Equation (4) says that the signed crossing of each labeled edge is unchanged by positive radial scaling. Consequently, for two realizations met through the same two boundary edges, the entrance and exit edges are interchanged if and only if the orientation of f is reversed. Thus the ordinary-face orientation clause in (b) locally gives the same reversed traversal as the abstract-dual path argument.

Conversely, suppose (a) holds. On every generic ray, the first source edge is an edge of N , and the last source edge is an edge of S . Hence the first target edge is in S , and the last target edge is in N . Let F_0^λ and F_∞^λ be the target origin-containing and outer faces. Every boundary edge of F_0^λ is the first edge on an open interval of generic target rays, and every boundary edge of F_∞^λ is similarly the last edge on an open interval. It follows that all three edges of F_0^λ belong to the triangular cycle S , and all three edges of F_∞^λ belong to N . A labeled triangular face is determined by its three boundary edges, so

$$F_0^\lambda = S, \quad F_\infty^\lambda = N.$$

This proves the polar-face exchange. Here the fact that a labeled triangular face is determined by its three boundary edges uses $|V| \geq 4$; two face occurrences sharing all three edges would force the entire graph to be K_3 .

Now fix any source bounded face $f \neq N$, and choose a generic ray meeting the interior of f . The ray enters and exits f across two consecutive edges of its dual path. Word reversal makes the target ray traverse the same labeled face between the same two edges in the opposite order. The local entrance/exit rule above and (4) then force the orientation of f to reverse. This completes the equivalence for generic rays.

It remains to derive the exact statement at a vertex from condition (b). Suppose R_d contains v . General position makes v the only vertex on the supporting line. Since both realizations are embeddings, their event at v is the same abstract tied block

$$\delta(v) = \{vw : vw \in E\},$$

and every other event is a singleton from a nonincident edge. A nonincident edge meeting R_d meets it in its relative interior, so its intersection parameter is continuous under small rotations by Lemma 1.4. Distinct nonincident edges have distinct parameters, and none has the vertex parameter τ_v ; otherwise the embedding would have a crossing or a nonincident edge passing through v . Their mutual orders and their positions relative to τ_v are therefore stable under sufficiently small one-sided rotations.

Split the incident block into

$$\delta_+(v) = \{vw : [vw] > 0\}, \quad \delta_-(v) = \{vw : [vw] < 0\}.$$

On a sufficiently small counterclockwise rotation, precisely the edges in $\delta_+(v)$ meet the ray near v ; on a clockwise rotation, precisely the edges in $\delta_-(v)$ do. Their parameters tend to τ_v . This is the infinitesimal form of $\det(u_v, u_w - u_v) = [vw] \neq 0$, and the side is preserved by positive radial scaling.

Apply the already proved generic reversal on either side and pass to the limit. The order of every pair of nonincident events reverses. For a nonincident edge e and an incident edge a , use the side on which a occurs; writing primes for target parameters gives

$$\text{sign}(\tau'_e - \tau'_v) = -\text{sign}(\tau_e - \tau_v).$$

Thus every nonincident event changes sides relative to the entire vertex block, and all nonincident events reverse their mutual order. Exactly as in Definition 1.1, no order is imposed inside $\delta(v)$. Hence

$$W_{P^\lambda}(d) = \text{rev } W_P(d)$$

on the vertex ray. Together with the generic argument this proves (a) from (b), while the reverse implication already used only generic rays. $\square$

We next record the exact strict linear system whose feasibility is equivalent to eversion. This equivalence does not treat local determinant signs as a substitute for global embeddedness; the latter is supplied by Lemma 2.3.

For positive radial scales λ_i , define the reciprocal radii

$$x_i = \frac{1}{\lambda_i}. \quad (5)$$

For every compatibly oriented face $f = (i, j, k)$, define the raw face form

$$L_f(x) = [jk]x_i + [ki]x_j + [ij]x_k. \quad (6)$$

Let $\mathcal{F}_{\text{ord}}$ be the set of source bounded faces other than N , and let B_{ev} be the matrix with rows

$$-L_f \quad (f \in \mathcal{F}_{\text{ord}}), \quad e_i \quad (i \in V), \quad (7)$$

where e_i is the i -th standard coordinate row. Thus

$$B_{\text{ev}}x > 0 \iff x_i > 0 \quad (i \in V) \quad \text{and} \quad L_f(x) < 0 \quad (f \in \mathcal{F}_{\text{ord}}). \quad (8)$$

Here a face form is identified with its coefficient row. We call $B_{\text{ev}}x > 0$ the strict eversion system.

Proposition 1.6 (Strict-system equivalence). *A positive radial realization $p'_i = \lambda_i u_i$ is a radial eversion if and only if its reciprocal-radius vector x satisfies $B_{\text{ev}}x > 0$. Conversely, every vector x satisfying $B_{\text{ev}}x > 0$ produces the radial eversion*

$$p'_i = \frac{u_i}{x_i}.$$

Proof. For $p'_i = \lambda_i u_i$, direct determinant expansion gives the exact identity

$$\frac{\text{orient}(p'_i, p'_j, p'_k)}{\lambda_i \lambda_j \lambda_k} = [jk]x_i + [ki]x_j + [ij]x_k = L_f(x). \quad (9)$$

If P' is an eversion, then $x_i > 0$, and Proposition 1.5 says that every $f \in \mathcal{F}_{\text{ord}}$ reverses from its source counterclockwise orientation. Equation (9) gives $L_f(x) < 0$, so $B_{\text{ev}}x > 0$.

Conversely, suppose $B_{\text{ev}}x > 0$ and lift the source rays to

$$q_i = (u_i, -x_i) \in \mathbb{R}^3.$$

The raw form in (6) is the face circuit $r_f x$ defined in (11) below, and expansion in the third column gives

$$\det(q_i, q_j, q_k) = -L_f(x). \quad (10)$$

For every ordinary bounded face, this determinant is positive by (8). The barycentric coordinates of the origin in the geometric triangles N and S are strictly positive. Since the compatible oriented area is positive for N and negative for S , all three coefficients of L_N are positive and all three coefficients of L_S are negative. Hence $L_N(x) > 0$, so the lifted north determinant is negative, whereas $L_S(x) < 0$, so the lifted south determinant is positive. Finally, every north-face height $-x_i$ is negative.

These strict signs by themselves are only local data; we do not infer crossing-freeness from them. Instead, all hypotheses of the pole-anchored projective disk Lemma 2.3 now hold. That lemma proves that central projection to $z = -1$ is a crossing-free nondegenerate straight-line embedding of the same triangulation, with N outer and S containing the origin. The projected vertices are precisely $p'_i = u_i/x_i$. Equation (9) proves that every ordinary source bounded face reverses orientation, and Proposition 1.5 then proves the grouped ray-word reversal on every ray. Hence this projection is a radial eversion. $\square$

2 Universal construction and proof

2.1 Face circuits

Recall the bracket $[ij] = \det(u_i, u_j)$ from (1). By (2), $[ij] \neq 0$ whenever $i \neq j$. For every compatibly oriented face $f = (i, j, k)$, define the row vector

$$r_f = [jk]e_i + [ki]e_j + [ij]e_k \in \mathbb{R}^V \quad (11)$$

and the nonzero scalar

$$\kappa_f = [ij][jk][ki]. \quad (12)$$

Here e_i is the i -th coordinate row. Put $a_f = r_f \mathbf{1}$. Direct expansion gives

$$a_f = \det \begin{pmatrix} u_i^x & u_i^y & 1 \\ u_j^x & u_j^y & 1 \\ u_k^x & u_k^y & 1 \end{pmatrix}. \quad (13)$$

Thus $a_f > 0$ for every bounded face and $a_S < 0$.

The three nonzero entries of r_f/a_f are the barycentric coordinates of the origin with respect to the planar triangle f . There are four face types, with circuit signs as follows:

face type	signs of the three entries of r_f	sign κ_f
N	$(+, +, +)$	$+$
D	$(-, +, +)$	$-$
U	$(+, -, -)$	$+$
S	$(-, -, -)$	$-$

(14)

For a D - or U -face, the vertex carrying the exceptional sign is its *apex*. There is exactly one down face with apex v for every $v \notin N$, and exactly one up face with apex v for every $v \notin S$. To prove existence as well as uniqueness, fix an incident bounded face $f = (v, a, b)$, and let $(\xi_v, \xi_a, \xi_b) = r_f/a_f$ be the barycentric coordinates of the origin in f . The barycentric coordinates of the point tu_v in that same triangle are

$$\mu_v(t) = t + (1-t)\xi_v, \quad \mu_a(t) = (1-t)\xi_a, \quad \mu_b(t) = (1-t)\xi_b. \quad (15)$$

If $v \notin N$, then for $t < 1$ sufficiently close to 1, the point tu_v lies in the interior of a unique incident bounded face: the face immediately before v . Uniqueness follows because the radial direction is not an incident edge direction by (2). Positivity of all three quantities in (15) for such $t < 1$ forces $\xi_a, \xi_b > 0$. Since the origin is outside this ordinary face, $\xi_v < 0$. Thus its circuit signs are $(-, +, +)$, with apex v . Conversely, those signs make all three expressions in (15) positive for every $t < 1$ sufficiently close to 1, so every down face with apex v is exactly that unique preceding face.

Similarly, if $v \notin S$, then tu_v for $t > 1$ sufficiently close to 1 lies in the interior of a unique incident bounded face, the face immediately after v . Now $1 - t < 0$, so positivity in (15) forces $\xi_a, \xi_b < 0$, while $\xi_v > 0$. These are the U -signs $(+, -, -)$, with apex v . Conversely, these signs make (15) positive for all $t > 1$ sufficiently close to 1, proving that every up face with apex v is the unique following face. For a vertex of N , the preceding face is N ; for a vertex of S , moving to $t > 1$ enters the outer region S . This proves both asserted bijections.

Let D and U also denote the matrices of down- and up-face rows, indexed by their apices, and set

$$A = \begin{bmatrix} D \\ r_N \end{bmatrix}, \quad B = \begin{bmatrix} U \\ -r_S \end{bmatrix}. \quad (16)$$

Both matrices have $n - 2$ rows, where $n = |V|$.

Lemma 2.1 (Face-circuit identity). *The oriented faces of the sphere triangulation satisfy*

$$\boxed{\sum_f \frac{r_f^\top r_f}{\kappa_f} = 0}, \quad (17)$$

where the sum includes both N and S .

Proof. Consider an off-diagonal matrix entry (i, j) . The two faces incident to the edge ij have oriented forms (i, j, k) and (j, i, ℓ) . Their contributions are

$$\frac{[jk][ki]}{[ij][jk][ki]} = \frac{1}{[ij]}, \quad \frac{[i\ell][\ell j]}{[ji][i\ell][\ell j]} = \frac{1}{[ji]},$$

and cancel.

For a diagonal entry i , list the neighbors of i cyclically as $j_1, \dots, j_d$, with indices taken modulo d . The face (i, j_t, j_{t+1}) contributes

$$\frac{[j_t j_{t+1}]}{[i j_t][j_{t+1} i]}.$$

Because $u_i \neq 0$, an area-preserving linear change of planar coordinates can take u_i to $(1, 0)$. Write the transformed vector u_j as (x_j, y_j) . The last expression becomes

$$\frac{x_{j_{t+1}}}{y_{j_{t+1}}} - \frac{x_{j_t}}{y_{j_t}}.$$

All denominators are nonzero by (2), and the cyclic sum telescopes. Every entry of the matrix in (17) is therefore zero. $\square$

2.2 Lorentz congruence and strict signed heights

Define

$$J_A = \text{diag}((1/\kappa_f)_{f \in D}, 1/\kappa_N), \quad J_B = \text{diag}((1/\kappa_f)_{f \in U}, 1/\kappa_S).$$

Negating the south row does not change its outer product, so Lemma 2.1 is equivalent to

$$A^\top J_A A + B^\top J_B B = 0. \quad (18)$$

Proposition 2.2 (Strict signed heights). *There is a vector $h \in \mathbb{R}^V$ such that*

$$Dh < 0, \quad Uh < 0, \quad r_N h > 0, \quad r_S h < 0, \quad (19)$$

and

$$h_i > 0 \quad \text{for every } i \in N. \quad (20)$$

Proof. Let $I = V \setminus S$. The up rows and the columns in I are indexed by the same $n - 3$ vertices. For $v \in I$, let $\beta_v = U_{v,v} > 0$. The square matrix

$$L = \text{diag}(\beta)^{-1} U[:, I]$$

has the form $I_{n-3} - P$, where I_{n-3} is the identity matrix and P is entrywise nonnegative. Moreover,

$$U[:, I] \mathbf{1} > 0. \quad (21)$$

To verify (21), observe that the full row sum of an up row is $a_f > 0$, whereas each omitted entry, in a column of S , is nonpositive: the positive apex of an up row lies in I . Hence $(I_{n-3} - P)\mathbf{1} > 0$, so every row sum of P is strictly smaller than one. In particular, the spectral radius of P is smaller than one and

$$L^{-1} = \sum_{m \geq 0} P^m \geq 0 \quad (22)$$

entrywise. Thus $U[:, I]$ is invertible.

Retain all columns in I and one column of S in B . The last row $-r_S$ vanishes on I and is strictly positive in the retained S -column. Expansion along that last row, using the invertibility of $U[:, I]$, proves that B has full row rank $n - 2$.

Equation (18) then forces A to have full row rank as well. Indeed, $B^\top J_B B$ has rank $n - 2$: as a map to its row coordinate space, B is surjective; J_B is invertible; and $B^\top$ is injective. Thus the equal matrix $-A^\top J_A A$ has rank $n - 2$, which is impossible unless $\text{rank } A = n - 2$.

Every face circuit annihilates the two coordinate columns

$$u^x = (u_i^x)_i, \quad u^y = (u_i^y)_i.$$

These columns are independent because the vertices span $\mathbb{R}^2$, and both A and B have nullity two. Consequently A and B have the same nullspace and hence the same row space. There is a unique invertible matrix T such that

$$A = TB. \quad (23)$$

Substituting (23) into (18) gives

$$B^\top (T^\top J_A T + J_B) B = 0.$$

Since B has a right inverse, multiplication on the left and right by that right inverse and its transpose yields the exact congruence

$$\boxed{T^\top J_A T = -J_B.} \quad (24)$$

Write the last row of (23) as

$$r_N = \sum_{v \notin S} t_v r_{U_v} + s(-r_S). \quad (25)$$

Restricting this equation to columns I gives

$$t^\top = r_N[I] (U[:, I])^{-1}.$$

The row $r_N[I]$ is entrywise nonnegative, and (22) implies that $(U[:, I])^{-1}$ is entrywise nonnegative. Therefore

$$t_v \geq 0 \quad \text{for every } v \notin S. \quad (26)$$

Every entry of $U[:, S]$ is nonpositive. The residual $r_N - t^\top U$, which by construction is supported on S , is therefore entrywise nonnegative. Since $-r_S$ is strictly positive on its support, (25) implies $s \geq 0$.

In fact $s > 0$. Inverting (24) gives

$$J_A^{-1} = -T J_B^{-1} T^\top. \quad (27)$$

The (N, N) -entry of (27), using the last row (25), is

$$\kappa_N = - \sum_{v \notin S} \kappa_{U_v} t_v^2 - \kappa_S s^2. \quad (28)$$

By (14), $\kappa_N > 0$, every $\kappa_{U_v} > 0$, and $\kappa_S < 0$. Equation (28) rules out $s = 0$. Hence

$$s > 0. \quad (29)$$

Since A has full row rank, choose $x \in \mathbb{R}^V$ satisfying

$$Ax = e_N, \quad (30)$$

where e_N is the coordinate vector of the last row. Equation (24) gives

$$T^{-1} = -J_B^{-1} T^\top J_A.$$

Together with (25), (29), and (30), this yields

$$(r_{U_v} x)_{v \notin S} = - \left(\frac{\kappa_{U_v}}{\kappa_N} t_v \right)_{v \notin S} \leq 0 \quad (31)$$

and

$$(-r_S)x = -\frac{\kappa_S}{\kappa_N} s > 0. \quad (32)$$

Also, $Dx = 0$ and $r_N x = 1$ directly from (30).

Adding any linear function of the planar coordinates changes no face-circuit value. Choose $\alpha, \gamma, c \in \mathbb{R}$ so that for the three vertices i of N ,

$$y_i := x_i + \alpha u_i^x + \gamma u_i^y = c. \quad (33)$$

The vertices of the nondegenerate north face are affinely independent, so these three equations have a unique solution. Applying r_N to (33) and using $r_N x = 1$ gives

$$c = \frac{1}{a_N} > 0. \quad (34)$$

All face-circuit values of y equal those of x .

Choose $\varepsilon > 0$ small enough that

$$\varepsilon < c, \quad (-r_S)x - \varepsilon(-a_S) > 0, \quad (35)$$

and set

$$h = y - \varepsilon \mathbf{1}.$$

For a down face, $r_f h = -\varepsilon a_f < 0$. For an up face, (31) gives $r_f h = r_f x - \varepsilon a_f < 0$. At the north face, $r_N h = 1 - \varepsilon a_N > 0$ by (34)–(35). At the south face, (32)–(35) give $r_S h < 0$. This proves (19). Finally,

$$h_i = c - \varepsilon > 0 \quad (i \in N),$$

which proves (20). $\square$

2.3 The pole-anchored disk lemma

The preceding proposition does not yet assert that every coordinate of h is positive. That global conclusion follows from a degree argument. Both pole counts below are essential: the boundary degree alone would not exclude an additional positive covering of the entire sphere.

Lemma 2.3 (Pole-anchored projective disk). *Let a straight-line plane triangulation with triangular outer face S be lifted to nonzero vectors*

$$q_i = (u_i, z_i) \in \mathbb{R}^3.$$

Determinants of three lifted vectors are taken with those vectors as rows. Let N be the unique bounded face whose strict interior contains the origin, and use the compatible sphere orientations: bounded faces counterclockwise and S clockwise. Assume:

- (1) $\det(q_i, q_j, q_k) > 0$ for every oriented face other than N ;
- (2) $\det(q_i, q_j, q_k) < 0$ for N ; and
- (3) $z_i < 0$ at all three vertices of N .

Then $z_i < 0$ for every vertex. Moreover, the shorter great-circle arcs between the normalized q_i form a sphere embedding whose everted face is N . Equivalently, central projection to $z = -1$ is a crossing-free straight-line drawing with outer face N , and the origin lies strictly inside S .

Proof. Delete the interior of N from the abstract sphere triangulation, obtaining a triangulated closed disk K . On every face $f = (i, j, k)$ of K , map barycentric coordinates by

$$F_f(\lambda_i, \lambda_j, \lambda_k) = \frac{\lambda_i q_i + \lambda_j q_j + \lambda_k q_k}{\|\lambda_i q_i + \lambda_j q_j + \lambda_k q_k\|}. \quad (36)$$

The three vectors of every such face are linearly independent, so the denominator never vanishes. The face maps agree on common edges and define a continuous map $F : K \rightarrow \mathbb{S}^2$. Each F_f is a homeomorphism onto the small spherical triangle cut out by the positive cone of its three vertices. It preserves orientation because, in local face coordinates, its Jacobian has sign $\text{sign } \det(q_i, q_j, q_k) = +1$.

Let Δ be the small spherical triangle spanned by the three vertices of N , and let $C = \partial\Delta$. The restriction $F|_{\partial K}$ is a homeomorphism onto C . In the oriented order of N , the positive-cone triangle map reverses orientation because its lifted face determinant is negative. The boundary orientation of K is the reverse of the boundary orientation of the removed face N . These two reversals cancel, so $F|_{\partial K}$ traverses C once with the positive boundary orientation of Δ .

The three north-face heights are negative, so Δ lies in the open southern hemisphere. Since the planar triangle N contains the origin, some strictly positive barycentric combination of its three lifted vertices is a negative vertical vector. Hence the south pole lies in the interior of Δ , while the north pole lies in the other component of $\mathbb{S}^2 \setminus C$.

We compute the relative Brouwer degree of F on those two components. A pole can lie in a face image only if the planar origin lies in the convex hull of that face's three planar vertices. In the given plane triangulation, the only triangles with this property are N and the outer triangle S . The face N was deleted. For $S = (i, j, k)$, let $a_S < 0$ be its oriented planar area. The height of the affine plane through its three lifted vertices above the planar origin is

$$\frac{\det(q_i, q_j, q_k)}{a_S} < 0.$$

Consequently the small spherical image of S contains the south pole, not the north pole. It contains that pole exactly once, in its interior, and with positive local degree. Therefore

$$\deg(F, \text{north pole}) = 0, \quad \deg(F, \text{south pole}) = 1. \quad (37)$$

Brouwer degree is constant on each component of $\mathbb{S}^2 \setminus C$. It is thus zero outside Δ and one inside Δ .

Choose a point outside the finite union of mapped edges. Every preimage lies in a face interior and contributes $+1$ to the degree. It follows from (37) and constancy of degree that such a point has no preimage outside Δ and exactly one preimage inside Δ . If the interior of any mapped face met the outside of Δ , that open intersection would contain a point away from all mapped edges, a contradiction. Taking closures face by face gives

$$F(K) \subseteq \Delta. \quad (38)$$

We now show that (38) is an embedding statement, not merely containment. The map is a local homeomorphism in each face interior. Along an interior edge, write the two incident oriented faces as (i, j, k) and (j, i, ℓ) . Their positive determinants imply

$$\det(q_i, q_j, q_k) > 0, \quad \det(q_i, q_j, q_\ell) < 0.$$

Thus the two third vertices lie on opposite sides of the great circle through the common edge, and the two face maps glue to a local homeomorphism across that edge.

At an interior vertex v , list the incident edge directions cyclically in the oriented tangent plane at $q_v/\|q_v\|$. Positivity of every incident oriented face determinant implies that each consecutive counterclockwise angle is strictly between 0 and π . Their sum is $2\pi m$ for an integer $m \geq 1$. Radial interpolation maps a sufficiently small punctured star of v through those same sectors, so its link has winding number m ; in particular, the star maps onto a full neighborhood of $F(v)$. Hence F is open at every interior point of K .

No interior point can map to C , because openness would contradict (38). If two distinct interior points had the same image, disjoint neighborhoods of those points would each map over a common neighborhood. A nearby point avoiding all mapped edges would then have at least two preimages, contradicting degree one. The boundary map is already a homeomorphism. Finally, every point in the interior of Δ and away from the mapped edges has a preimage. Those points are dense and $F(K)$ is closed, so $F(K) = \Delta$. Thus F is a continuous bijection from the compact disk K to the Hausdorff space Δ , and is therefore a homeomorphism.

Every point of Δ has negative third coordinate: it is the normalization of a positive combination of the three north-face vectors, whose third coordinates are negative. Equation (38) therefore implies $z_i < 0$ for every vertex. The great-circle embedding consists of the embedded disk Δ , with N as the complementary face. Central projection from the open southern hemisphere to $z = -1$ is a homeomorphism and sends great-circle arcs to straight segments. It consequently gives a crossing-free straight-line drawing with outer face N . $\square$

2.4 Construction of the eversion

Proof of Theorem 1.3. Let h be the vector from Proposition 2.2, and lift each vertex to

$$q_i = (u_i, -h_i). \quad (39)$$

For any oriented face $f = (i, j, k)$, expansion in the last column gives

$$\det(q_i, q_j, q_k) = -r_f h. \quad (40)$$

Equations (19) and (20) say exactly that the hypotheses of Lemma 2.3 hold: the determinant in (40) is positive for every face except N , negative for N , and the three north-face heights $-h_i$ are negative. The lemma therefore proves

$$h_i > 0 \quad \text{for every } i \in V. \quad (41)$$

Scale each vector (39) by the positive number $1/h_i$ onto the plane $z = -1$. The resulting planar vertex is

$$p'_i = \frac{u_i}{h_i}. \quad (42)$$

Thus every target vertex lies on its prescribed positive ray. Lemma 2.3 also shows that (42) is a crossing-free straight-line embedding of the same maximal planar graph and has N as its outer face. The south pole lies in the small spherical image of S , so its projection, the planar origin, lies strictly inside S .

For clarity, the determinant signs also record the orientation change. Positive row scaling preserves the signs in (40), while

$$\det \begin{pmatrix} p'_i{}^x & p'_i{}^y & -1 \\ p'_j{}^x & p'_j{}^y & -1 \\ p'_k{}^x & p'_k{}^y & -1 \end{pmatrix} = -\text{orient}(p'_i, p'_j, p'_k). \quad (43)$$

For every original bounded face other than N , (40) is positive, so (43) is negative and its planar orientation is reversed. Together with the exchange of N and S , this is the orientation form of the eversion.

It remains to prove the ray-order condition. First consider a ray that contains no vertex. For any edge ij and any positive scalars λ_i, λ_j , the segment between $\lambda_i u_i$ and $\lambda_j u_j$ intersects a ray of direction d if and only if d lies in the positive cone of u_i, u_j . Indeed, an intersection point is a positive combination of these two vectors, and conversely any direction in their positive cone is obtained at a unique point of the segment after a positive rescaling. Hence positive radial scaling preserves exactly the set of edges crossed by every vertex-free ray.

In the source drawing, these crossed edges form a path in the abstract dual graph from N to S . The intersection of a ray with a Euclidean triangle is connected, so the ray cannot leave and later re-enter one face; therefore the dual walk has no repeated face. The target ray crosses the same abstract edge set. It starts in S , the target face containing the origin, and ends in N , the target outer face. Thus it traverses the same dual path from S to N . A path has only two linear orders, and the complete edge order is reversed.

We now handle a ray through a vertex using the tied blocks of Definition 1.1. Suppose R_d contains a vertex v . General position implies that v is the only vertex on the entire supporting line of R_d . Since P is an embedding, the event at v is exactly

$$\delta(v) = \{vw : vw \in E\}. \quad (44)$$

Every other event is a singleton belonging to an edge not incident to v . The same assertions hold in the target, whose event at the positive radial image of v is the same abstract block $\delta(v)$.

If a nonincident edge e intersects R_d , then d lies in the interior of the positive cone of its endpoint directions. Its intersection parameter $\tau_e(d)$ consequently varies continuously under a sufficiently small rotation of d . Distinct nonincident edges have distinct parameters, and no such parameter equals the vertex parameter τ_v : equality would give either an intersection of two nonincident edges or a nonincident edge containing v . It follows that, under small rotations on either side, all mutual orders among nonincident intersections and all their orders relative to τ_v are stable.

Partition the incident edges as

$$\delta_+(v) = \{vw : \det(u_v, u_w) > 0\}, \quad \delta_-(v) = \{vw : \det(u_v, u_w) < 0\}. \quad (45)$$

For a sufficiently small counterclockwise rotation of R_d , precisely the edges in $\delta_+(v)$ intersect the rotated ray near v ; for a clockwise rotation, precisely the edges in $\delta_-(v)$ do. Their intersection parameters tend to τ_v . This follows from the positive-cone criterion, or equivalently from

$$\det(u_v, u_w - u_v) = \det(u_v, u_w) \neq 0.$$

Positive radial scaling preserves this determinant sign and hence the side on which each incident edge occurs.

Let e, f be nonincident edges intersecting R_d . Apply the generic-ray reversal just proved to either sufficiently small one-sided rotation and pass to the limit. The order of e, f is reversed in the target. For a nonincident edge e and an incident edge a , use the side on which a occurs. Write τ'_e and τ'_v for the corresponding target intersection parameters. Generic reversal and the same limiting argument give

$$\text{sign}(\tau'_e - \tau'_v) = -\text{sign}(\tau_e - \tau_v). \quad (46)$$

Thus every nonincident event moves to the opposite side of the whole vertex event, while all nonincident events reverse their mutual order. These pairwise statements are exactly

$$W_{P'}(d) = \text{rev } W_P(d). \quad (47)$$

No comparison between the two one-sided incident lists is required: at the exact direction their disjoint union is, by definition, the single tied block $\delta(v)$. Hence (47) holds for every vertex ray as well as every generic ray.

All determinant inequalities used above are strict, so no target face is degenerate. The disk homeomorphism rules out crossings. Finally, positive radial scaling preserves (2). The drawing (42) satisfies every clause of Definition 1.2, completing the proof. $\square$

3 Verification against the problem contract

This section records where each acceptance condition is proved. It is not an additional hypothesis and does not replace any part of the proof.

Requirement	Proof location and reason
Arbitrary admissible input	Section 1 fixes an arbitrary finite crossing-free straight-line triangulation. No later step specializes its graph, coordinates, or number of vertices.
Same maximal planar graph	The construction changes only vertex coordinates, and Lemma 2.3 proves that the same abstract sphere triangulation is embedded.
Every vertex stays on its original positive ray	Equation (41) proves $h_i > 0$ for all i , and Equation (42) sets $p'_i = u_i/h_i$.
Straight edges, no crossings, and no degeneracy	Central projection in Lemma 2.3 sends great-circle arcs to straight segments and is injective on the triangulated disk. Strict signs in (19) and (40) exclude degenerate faces.
Outer and origin faces exchange	Lemma 2.3 makes N the target outer face; the south-pole argument immediately after (42) puts the target origin strictly inside S .
Every exact ray order reverses	The dual-path argument proves reversal for vertex-free rays. Equations (44)–(47) prove it for a ray through a vertex, with all incident edges retained as one tied abstract block.
General position is retained	Equation (42) uses only positive radial scalings, so every determinant in (2) is multiplied by a positive nonzero factor.

The proof uses maximal planarity only to obtain the triangular sphere decomposition, the apex-face bijections, and the abstract dual path. It uses general position only to keep the brackets nonzero and to make ray incidences unambiguous. The origin is strictly inside N , rather than on its boundary, precisely where the circuit signs and the two pole locations are deduced. These are all assumptions of Theorem 1.3; no residual genericity, shellability, or computational assumption is added.

3.1 Role of finite computation

The accompanying research repository contains deterministic exact-arithmetic tests for certificate checking, embedding validation, determinant geometry, imports, and the signed-inequality construction. A recorded focused run of the five corresponding deterministic test modules completed with 83 passing tests. This was not the full search-heavy test suite. Those tests are useful for detecting transcription and sign errors in finite examples, but they are *not evidence from which the theorem is inferred*. No finite sample can establish a statement over all triangulations, and no research script is invoked by, imported into, or logically required by the proof in Section 2. The universal conclusion rests only on the face-circuit identity, the full-rank and congruence argument, the strict-height construction, and the pole-anchored degree lemma proved above.

References

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