

Exact level structure of finite uniform Ising lattices: the density of states is a cut-size spectrum

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Abstract

Wei rewrote the partition function of a finite Ising lattice as a sum of 2^N exponentials and left two questions open: how to evaluate that sum for large N , especially in a field, and what the multiplicity of each repeated exponent value is [14]. His worked $2 \times 6 \times 2$ example reports 16777216 terms collapsing to 23 distinct values without stating the bond or boundary convention that produces the count. We settle the degeneracy branch for uniform hypercubic lattices of any dimension, size and boundary condition. Our central result is structural: the number of unsatisfied bonds of a configuration is the size of the edge cut of its down-spin set, so the entire level spectrum of a uniform Ising lattice is the cut-size spectrum of its bond graph, and Wei's degeneracy question becomes a purely graph-theoretic one. From a realisability identity, a parity constraint, the edge-connectivity gap and the bipartite mirror we derive a closed formula for the number of distinct exponent values in four lattice classes. The formula is proved for one-dimensional rings and chains, whose complete bivariate density of states $g(B, M)$ we give in closed form at any field (a re-derivation of published combinatorics, used here as an exact oracle), and for even-sided two-dimensional tori by an explicit construction; it is then verified exactly on 94 lattices: 81 have their full multiplicity tables, computed by an exact vectorised transfer engine and confirmed entry by entry against brute-force enumeration of all 2^N configurations and a second, dictionary-based transfer engine wherever their cost caps allow — all three on the 50 lattices with $N \leq 24$ — while every one of the 81 also satisfies the graph-free identities $\sum g = 2^N$ and $\sum_B g(B, M) = \binom{N}{(M+N)/2}$. For the remaining 13 we add a fourth, magnitude-free reachability engine, which yields the exact level set and independently reproduces all 81 attained level sets: twelve lie beyond the $N \leq 62$ range in which any multiplicity fits in a 64-bit integer, reaching $N = 1000$ sites, i.e. 2^{1000} configurations, in a fraction of a second, and the thirteenth, the 7×7 torus, is excluded by the counting engines' width cap rather than by the integer ceiling. Wei's 23 is thereby explained as $N - 1 = |E|/2 - 1$: a count fixed by the number of sites and the edge connectivity, not by the dimension. Six uniform specifications of 24 sites — the two-dimensional 4×6 and 6×4 tori and four three-dimensional ones, under both length-two bond conventions — share all 1225 multiplicities for the same reason: they are one bond graph, because a length-two axis is K_2 unless it is periodic and doubled, and $K_2 \square K_2 = C_4$, so all six are $C_4 \square C_6$. We certify each of the fifteen pairwise isomorphisms with an explicit vertex map re-checked bond by bond, and a sweep of all 412 uniform lattices with $N \leq 18$ finds no non-isomorphic pair sharing a full table, or even a zero-field marginal. What a table still does not determine is the presentation: $C_4 \square C_6$ is at once a two- and a three-dimensional uniform torus, and once a deduplicated and a doubled-bond specification. We identify the exact exception class — two frustrated 5-regular lattices that miss the cut sizes $\lambda + 1$ and $\lambda + 2$ — and prove why, from exhaustively computed induced-edge maxima. What we do not claim is equally specific: no subexponential-in- N algorithm uniform in lattice width, no closed form for the multiplicities in dimensions ≥ 2 , no hardness classification, and no proof that the coincidence class of the full multiplicity table is exactly the isomorphism class.

1 Introduction

The Ising model [10] is exactly solved in one dimension and, in the zero-field two-dimensional thermodynamic limit, by Onsager’s celebrated computation [12]. Neither result gives the *finite* lattice’s complete degeneracy structure: for a lattice of N sites the partition function is a sum of 2^N configuration weights, and the number of distinct exponent values in that sum, together with the multiplicity of each, is a finite-size combinatorial object that the thermodynamic-limit machinery does not produce. Wei [14] made this object the subject of an exact reformulation and stated two open problems: evaluating the 2^N -term sum for large N , particularly at nonzero external field, and determining the multiplicity of the repeated exponent values, i.e. the degeneracy of each energy level. His worked example, a $2 \times 6 \times 2$ lattice, reports $2^{24} = 16777216$ terms collapsing to 23 distinct exponent values, but does not state the bond or boundary convention under which that count holds.

A published problem statement makes the same object its target: exact finite-size Ising partition functions and energy-degeneracy multiplicities for uniform two-dimensional lattices in a field and three-dimensional lattices [8]. That statement separates a *finite exact* branch (the tables and their multiplicities) from an *asymptotic* branch (a subexponential algorithm, a closed form valid in arbitrary dimension, or a hardness classification), and notes that the latter would settle a long-standing question. This paper addresses the finite exact branch only, and says so wherever the distinction matters.

Our contribution is a structural identification together with everything that follows from it exactly. For a configuration s let S be its set of down spins. A nearest-neighbour bond is unsatisfied precisely when it has one end in S , so the number of unsatisfied bonds is the size of the edge cut $\partial(S)$. The density of states is therefore a cut-size enumerator, and the set of distinct zero-field exponent values of a uniform Ising lattice is exactly the set of cut sizes realised by its bond graph. This is a stronger statement than the standard reduction of spin-glass *ground states* to maximum cut [2]: it holds for every level, with multiplicities, at every field, and it turns a physics question into a graph-invariant question whose answers are classical enough to be proved and cheap enough to be verified exhaustively.

From it we obtain: a realisability identity that constrains which cut sizes can occur at all, together with the parity constraint, which subsumes the level-halving of periodic lattices (Theorem 2); the edge-connectivity gap, which proves that the top $\lambda - 1$ levels are always absent and identifies λ with the minimum degree for every lattice studied (Theorem 3); the bipartite mirror, which proves that the bottom gap is the exact image of the top one and that bipartiteness is equivalent to the level-symmetry (Theorem 4); and, combining these, a closed level-count formula in four cases (Proposition 5) that explains Wei’s 23 as $N - 1$ and the 33 of the alternative bond convention as $|E|/2 - \lambda + 3$. The formula’s only hypothesis, that the interior of the spectrum has no gaps, is proved for one-dimensional families through complete closed forms for $g(B, M)$ (Theorem 7) and for even-sided two-dimensional tori by an explicit bar construction (Theorem 8), and verified exactly on 94 lattices. It fails for exactly two of them, both frustrated and 5-regular; we prove why (Proposition 10).

Along the way the paper contributes a fourth exact engine whose reach is not limited by integer width. Multiplicities grow like 2^N , so any fixed-width implementation stops at $N = 62$; but the *set* of attained levels requires no counting at all, only reachability, and a boolean transfer recursion computes it exactly for $N = 1000$ sites in a tenth of a second (Section 5.7). This separates the two halves of Wei’s degeneracy question: the number of distinct exponent values is exactly computable at sizes where their multiplicities cannot even be represented.

2 Background and related work

Exact finite-size Ising combinatorics. Transfer-matrix methods give the one-dimensional chain in closed form and the two-dimensional zero-field torus in the thermodynamic limit [4, 12]; neither enumerates the finite lattice’s level multiplicities. Wei [14] reformulated the finite partition function as a sum of 2^N exponentials indexed by configurations and posed the two open problems quoted above; that preprint is the direct motivation here, and Section 5.4 resolves the convention ambiguity in its worked example.

One-dimensional densities of states. For the one-dimensional chain, fixed-magnetisation counts with a prescribed number of domain walls are known. Antal et al. [1] give, for periodic chains and $2s$ walls, $\Omega_{2s}(M) = \binom{N_\uparrow-1}{s-1}\binom{N_\downarrow}{s} + \binom{N_\downarrow-1}{s-1}\binom{N_\uparrow}{s}$; Dantchev and Rudnick [5] give exact fixed- M partition functions for several boundary conditions, and Denisov and Hänggi [6] treat domain statistics in a finite chain. The underlying run counting on a circle is classical [3]. The closed forms used in this paper are algebraically identical to the periodic formula of Antal et al. [1] (Section 5.5) and are therefore *not* claimed as new; they are used as an independent exact oracle against which every engine is tested at sizes far beyond enumeration.

Cuts, connectivity and vertex-transitive graphs. Cuts and the cut space are standard [7, §1.1, §1.9]. That a connected vertex-transitive graph has edge connectivity equal to its valency is due to Mader [11] and Watkins [13]; see Godsil and Royle [9, §3.3]. That theorem concerns *simple* graphs, and it does not extend to the doubled-bond multigraphs produced by one of our two bond conventions, for which we exhibit vertex-transitive counterexamples with $\lambda < d_{\min}$ (Section 5.2). We therefore use it only where its hypothesis holds, prove $\lambda = d_{\min}$ elementarily for the lattice families studied here and, in every case, measure λ exactly by enumerating all subsets.

Maximum cut and spin glasses. The reduction of Ising ground states to maximum cut is classical in combinatorial optimisation [2]. Our use is different in kind: the whole level spectrum, not only the ground state, and with multiplicities. The maximum cut value enters here as $u_{\max} = |E| - B_{\min}$, i.e. the frustrated lattice’s ground-state bond count, which Theorem 4 shows is zero exactly for bipartite bond graphs.

3 Problem, scope, assumptions and notation

3.1 Lattices and conventions

A *lattice* is a finite graph $G = (V, E)$ with $N = |V|$ sites and $|E| = |E|$ counted nearest-neighbour bonds, built from hypercubic dimensions $(L_1, \dots, L_k)$, $k \in \{1, 2, 3\}$, with a boundary condition per axis, open (o) or periodic (p). Couplings are uniform and ferromagnetic and the field is uniform; both are arbitrary real numbers and neither enters the combinatorics, only the weighting (2) below.

A periodic axis of length 2 is genuinely ambiguous: the two directed neighbours of a site coincide, so the bond may be counted once (a simple graph) or twice (a $2d$ -regular multigraph). We carry both conventions explicitly, since Section 5.4 shows that Wei’s reported count selects one of them. Self-loops never occur; a multigraph bond is a parallel edge, which affects degrees and cut sizes but not the arguments below.

3.2 The object of study

For a spin configuration $s \in \{\pm 1\}^N$ let

$$B(s) = \#\{\{i, j\} \in E : s_i = s_j\}, \quad M(s) = \sum_i s_i, \quad (1)$$

be the number of *satisfied* bonds and the magnetisation. The exact density of states is $g(B, M) = \#\{s : B(s) = B, M(s) = M\}$, and it determines the partition function at every coupling and field through

$$Z(\beta, J, h) = \sum_{B, M} g(B, M) e^{\beta J(2B - |E|)} e^{\beta h M}. \quad (2)$$

Thus g is a strictly finer object than any single evaluation of Z : it is the complete list of exponent values occurring in Wei's 2^N -term sum together with their multiplicities. The zero-field bond histogram is $n(b) = \sum_M g(b, M)$, the multiplicity of the exponent value $\beta J(2b - |E|)$.

Throughout, $S = \{i : s_i = -1\}$ is the down-spin set, $\partial(S) = \{\{i, j\} \in E : |S \cap \{i, j\}| = 1\}$ its edge cut, $u = |\partial(S)|$ the number of unsatisfied bonds, $d_{\min}$ the minimum vertex degree, λ the edge connectivity (the minimum of $|\partial(S)|$ over $\emptyset \neq S \neq V$), $u_{\max}$ the maximum cut size, and $B_{\min} = |E| - u_{\max}$. We call G *Eulerian* when every degree is even. The *level count* is the number of distinct b with $n(b) > 0$, equivalently $|C(G)|$ for the *cut spectrum* $C(G) = \{|\partial(S)| : S \subseteq V\}$.

3.3 Scope

In scope: finite uniform lattices in one, two and three dimensions, arbitrary size, any per-axis boundary condition, both length-2 periodic conventions, any J and any h including nonzero field; the exact tables $g(B, M)$; the level count and the level set; and the structural theorems that determine them.

Out of scope, and not claimed anywhere below: an algorithm subexponential in N uniformly in lattice width; a closed form for the multiplicities in dimensions two and three; a computational-hardness classification for these families; and the asymptotic branch of the published problem [8]. All arithmetic in this work is exact integer or exact rational arithmetic; no result depends on floating-point accuracy.

4 Methods

4.1 Four exact engines

All four engines are in `code/ising_dos.py` and share only the lattice constructor; their recursion structures are mutually independent, which is what makes their agreement a challenge check rather than a tautology.

1. **dos_brute**: enumerates all 2^N configurations as NumPy bitmasks in chunks and accumulates $g(B, M)$ exactly. Cost $O(2^N |E|)$; used for $N \leq 24$.
2. **dos_transfer**: an integer transfer matrix swept along the narrowest axis, carrying a Python dictionary keyed by (slice configuration, B , sublattice magnetisations). No global configuration is ever formed. Exact Python integers; cost exponential in the cross-section w and linear in the swept length. It refuses a cross-section above 2^6 slice configurations rather than running out of memory silently.

3. `dos_transfer_np`: the same recursion as a dense `int64` array over (slice configuration, B , magnetisation), with previous-slice configurations grouped by their inter-slice bond contribution so that one shifted block-add serves a whole group. Because multiplicities reach 2^N , exactness requires $N \leq 62$; the engine asserts this rather than silently overflowing. It is run here with a cap of 2^6 slice configurations for a periodic sweep axis and 2^{12} for an open one.
4. `dos_support` (new here): a *boolean* reachability recursion over (slice configuration, B), propagated with bitwise OR. No count is ever formed, so no integer-width ceiling applies. It computes the attained level set exactly for any N ; it does not compute multiplicities. Its caps are 2^8 slice configurations for a periodic sweep axis and 2^{10} for an open one, which is what lets it reach lattices the counting engines refuse. Grouping by inter-slice bond contribution remains valid because OR is idempotent.

Engines 1–3 return $g(B, M)$, and also the sublattice-refined $g(B, m_A, m_B)$; engine 4 returns the support of n . A sweep-axis permutation is applied internally so that the recursion always runs along the narrowest cross-section, and reversing the declared lattice dimensions forces a different sweep axis, which must reproduce an identical table — an internal consistency check used in Section 6.

4.2 Closed forms as an independent oracle

For the one-dimensional families the complete bivariate table is available in closed form (Theorem 7), so a lattice of $n \leq 62$ sites can be checked against a formula that shares no code with any engine, and the *partition function* built from the closed form can be checked against the classical 2×2 transfer matrix in exact rational arithmetic for n far beyond 62. Both checks are implemented in `code/closed_forms.py`: `z_from_closed_form` evaluates $\sum g(B, M) x^{2B-|E|} y^M$ from the formula alone, and `z_exact` evaluates $\text{Tr } T^n$ (ring) and $a^\top W^{n-1} b$ (chain) with $x = e^{\beta J}$, $y = e^{\beta h}$ as `Fractions`. Moment identities — $\sum g = 2^n$, $\sum (|E| - B)g = n2^{n-1}$, $\sum_B g(B, M) = \binom{n}{(M+n)/2}$ — are checked at $n = 1000$ from the closed form alone, where no table fits in memory.

4.3 Computation and provenance

Python 3.12.3, NumPy 1.26.4, mpmath and SymPy for the symbolic checks; Linux x86-64. Every artifact in `results/` is regenerated by one command from `code/` (`run_ladder.py`, `check_structure.py`, `run_beyond_bruteforce.py`, `run_dos_coincidence.py`, `run_iso_classes.py`, `run_lambda_delta.py`, `run_closed_forms.py`, `run_cut_spectrum.py`), and `results/provenance.json` records the interpreter, library versions and commands. Figures are drawn only from committed result artifacts (`code/make_figures.py`), never from recomputation, so they cannot drift from the validated evidence. No floating-point value enters any reported integer.

5 Results

5.1 The level spectrum is the cut spectrum

Theorem 1 (cut-spectrum identity). *Let $s \in \{\pm 1\}^N$ and $S = \{i : s_i = -1\}$. Then the number of unsatisfied bonds of s is the size of the edge cut of S ,*

$$u(s) = |E| - B(s) = |\partial(S)|, \quad (3)$$

and consequently

$$n(b) = \#\{S \subseteq V : |\partial(S)| = |E| - b\}, \quad g(B, M) = \#\left\{S \subseteq V : |\partial(S)| = |E| - B, |S| = \frac{N-M}{2}\right\}. \quad (4)$$

The number of distinct zero-field exponent values of a uniform Ising lattice is $|C(G)|$.

Proof. A bond $\{i, j\}$ is unsatisfied exactly when $s_i \neq s_j$, i.e. exactly when it has one end in S ; and $s_i = -1$ exactly when $i \in S$, so $|S| = (N - M)/2$. Both statements in (4) are then a re-labelling of the definitions (1). The final claim is (3) applied to $C(G)$. $\square$

Theorem 1 is elementary, and the ground-state case is classical [2]. Its consequences for the whole spectrum are not usually drawn, and they are the content of this paper: Wei's degeneracy question is the question of which cut sizes a lattice graph realises and how many vertex subsets realise each, and every structural fact about cuts becomes a fact about energy levels.

Theorem 2 (realisability and parity). *For every $S \subseteq V$,*

$$|\partial(S)| = \sum_{v \in S} \deg(v) - 2|E(S)|, \quad (5)$$

where $E(S)$ is the set of bonds with both ends in S . Hence $|\partial(S)| \equiv \sum_{v \in S} \deg(v) \pmod{2}$; if G is Eulerian every cut size is even and $B(s) \equiv |E| \pmod{2}$ for every configuration; and if G is d -regular every cut size has the form $dn - 2e$ with $n = |S|$ and $e = |E(S)|$.

Proof. Each bond incident to S is counted once by $\sum_{v \in S} \deg(v)$, and each bond internal to S twice, so subtracting $2|E(S)|$ leaves exactly $|\partial(S)|$. The rest is arithmetic modulo two, and the parity statement re-proves the level-halving of periodic lattices recorded in this repository's earlier findings in two lines. $\square$

5.2 Two gaps that are always present

Theorem 3 (edge-connectivity gap). *No cut size lies in $[1, \lambda - 1]$, so the levels $b = |E| - \lambda + 1, \dots, |E| - 1$ are absent from the spectrum of every uniform Ising lattice. Moreover $\lambda = d_{\min}$ for every lattice studied here.*

Proof. A cut of size $k > 0$ separates G into two nonempty parts, so $k \geq \lambda$ by definition of edge connectivity; the star of a minimum-degree vertex shows $\lambda \leq d_{\min}$ always. For the families used here equality holds: an open chain has $\lambda = d_{\min} = 1$; an open grid or open three-dimensional box has a corner of degree $d_{\min} \geq 2$ and no smaller separating set, since removing fewer than $d_{\min}$ bonds cannot separate a corner from its neighbours while leaving the graph's remaining incidences intact; and for a two-dimensional torus $C_L \square C_M$ with $L, M \geq 3$, given $\emptyset \neq S \neq V$, either some row is partially in S — and then that row contributes an even, hence at least 2, number of horizontal boundary bonds — or every row is empty or full, in which case S is a union of whole columns and each of the $M \geq 2$ rows contributes at least two boundary bonds; the same argument with rows and columns exchanged gives $|\partial(S)| \geq 4 = d_{\min}$. Simple higher-dimensional tori are vertex-transitive, for which $\lambda = d_{\min}$ is classical [9, 11, 13]. Independently of these arguments, λ is *measured* exactly for all 94 lattices of this study, because the engines enumerate every subset; $\lambda = d_{\min}$ holds in all 94 cases (`lambda_eq_min_degree_everywhere`). $\square$

The scope of $\lambda = d_{\min}$, and where it fails. The classical theorem just cited is a statement about simple graphs, and the doubled-bond convention of Section 3 violates its hypothesis: a length-two periodic axis becomes a digon, so the bond graph is a Cayley *multigraph*. `code/run_lambda_delta.py` enumerates all 2^N vertex subsets of every uniform lattice with $N \leq 12$ — 204 specifications over one, two and three axes, all boundary conditions and both conventions, each cross-checked against the independent boolean support engine, which reproduces the attained cut set on 204 of 204 — and finds $\lambda = d_{\min}$ in 188 and $\lambda < d_{\min}$ in 16. Every failure carries parallel bonds and none occurs with `double_two=False`. The smallest is 2×2 with one open and one doubled periodic axis ($N = 4$, $|E| = 6$, $d_{\min} = 3$, $\lambda = 2$); the smallest three-dimensional one is $2 \times 2 \times 2$ with one open and two doubled periodic axes ($N = 8$, $|E| = 20$, $d_{\min} = 5$, $\lambda = 4$). Both are vertex-transitive — witnessed by explicit sign-flip automorphisms, each re-checked bond by bond — so the failure is caused by the parallel bonds, not by asymmetry: the separating set is one half of the box and its cut uses only the non-doubled axis, one bond per cross-section site, which undercuts a $d_{\min}$ that the doubled axes have inflated. All 94 lattices whose spectra this paper reports satisfy $\lambda = d_{\min}$ (`results/lambda_delta.json`), so no result above rests on the general statement; the theorem is recorded with the scope it actually has.

Theorem 4 (mirror, and bipartiteness as a level symmetry). *Let G be connected with $|E| \geq 1$ and no self-loops. Then $n(b) = n(|E| - b)$ for every b if and only if G is bipartite. Equivalently, for bipartite G the map $S \mapsto S \triangle A$, with A a colour class, is a bijection of subsets with $|\partial(S)| \mapsto |E| - |\partial(S)|$, so $C(G)$ is symmetric about $|E|/2$, $u_{\max} = |E|$ and $B_{\min} = 0$; and if G contains an odd cycle then $u_{\max} < |E|$ and no such symmetry exists.*

Proof. If G is bipartite every bond joins A to $V \setminus A$, so a bond crosses $S \triangle A$ exactly when it does not cross S ; the map is an involution, hence a bijection, and it preserves $|S|$ up to complementation, which yields the statement for n and for g . Conversely, if G has an odd cycle then no configuration satisfies every bond of that cycle, so $B_{\min} \geq 1$, i.e. $u_{\max} \leq |E| - 1$, while $n(|E|) = 2$ from the two uniform configurations and $n(0) = 0$; the histogram cannot be symmetric. The full argument, including the refined sublattice symmetries, is in the repository’s `notes/findings.md`. $\square$

Theorem 4 is the reason the upper gap mirrors the lower one: for a bipartite lattice the absent levels $b = |E| - \lambda + 1, \dots, |E| - 1$ force the absence of $b = 1, \dots, \lambda - 1$, and $B_{\min} = 0$ is automatic. For a frustrated lattice the two ends are independent, and $B_{\min} > 0$ is exactly the ground-state frustration.

5.3 The level-count formula

Proposition 5 (level count). *Assume the interior hypothesis: every parity-allowed cut size between λ and top is realised, where $\text{top} = |E| - \lambda$ if G is bipartite and $\text{top} = u_{\max}$ otherwise. Then the number of distinct zero-field levels is*

$$|C(G)| = \begin{cases} |E|/2 - \lambda + 3, & G \text{ bipartite and Eulerian,} \\ |E| - 2\lambda + 3, & G \text{ bipartite, mixed degrees,} \\ (u_{\max} - \lambda)/2 + 2, & G \text{ non-bipartite and Eulerian,} \\ u_{\max} - \lambda + 2, & G \text{ non-bipartite, mixed degrees.} \end{cases} \quad (6)$$

Proof. By Theorems 2–4 the realised cut sizes are $\{0\}$, the parity-allowed values in $[\lambda, \text{top}]$, and, for bipartite G , the mirror images of the values below λ together with $|E|$ itself; in the bipartite case

Table 1: Wei’s $2 \times 6 \times 2$ example under each defensible convention (exact enumeration of all 2^{24} configurations; `results/dos_ladder.json`). Only the periodic simple graph yields his 23 distinct exponent values at zero field, and it is the convention under which $23 = N - 1$ as predicted by Corollary 6.

convention for $2 \times 6 \times 2$	graph	$ E $	levels	attainable (B, M)
periodic all axes, undirected pair once	4-regular, bipartite	48	23	208
periodic all axes, each directed neighbour	6-regular, bipartite	72	33	321
open all axes	mixed degrees, bipartite	44	41	453

the mirror of $[\lambda, |E| - \lambda]$ is itself, and $|E|$ is realised by a colour class. Counting the parity-allowed integers in each range gives (6) case by case. $\square$

The interior hypothesis is proved for one-dimensional lattices (Theorem 7) and for even-sided two-dimensional tori (Theorem 8), and verified exactly on 92 of the 94 lattices of this study. It fails for precisely two, and Proposition 10 proves why.

Corollary 6 (d -regular bipartite tori). *For a connected d -regular bipartite lattice with d even and no interior gaps, $|C(G)| = |E|/2 - d + 3 = dN/4 - d + 3$. In particular every 4-regular bipartite lattice has exactly $N - 1$ distinct zero-field levels.*

5.4 Wei’s 23 is $N - 1$, and the convention is pinned

Wei [14] reports $2^{24} = 16777216$ terms collapsing to 23 distinct exponent values for a $2 \times 6 \times 2$ lattice, without stating the convention. The count is convention-dependent (Table 1), and only one convention reproduces 23.

Under that convention the lattice is 4-regular and bipartite with $N = 24$, so Corollary 6 gives $|C(G)| = N - 1 = 23$ *without any enumeration*. The same formula gives $33 = |E|/2 - \lambda + 3 = 36 - 6 + 3$ for the multigraph convention on the same 24 sites. Wei’s number is therefore not a three-dimensional fact at all: it is fixed by the site count and the edge connectivity. The six lattices of Section 5.9 — the 4×6 and 6×4 tori and four three-dimensional specifications of 24 sites — all have 23 levels for a stronger reason than shared invariants: they are one bond graph, $C_4 \square C_6$ (Proposition 11).

At nonzero field the same lattice has 208 distinct exponent values, not 23, consistent with Wei [14] identifying the field case as the harder of his two problems. Our result concerns the level structure; the field-resolved multiplicities are computed exactly here for $N \leq 62$ but have no closed form in dimensions ≥ 2 .

5.5 One dimension in closed form

Theorem 7 (exact one-dimensional density of states). *Let m be the number of down spins, so $M = n - 2m$. For the ring C_n , $n \geq 3$, with $|E| = n$, and j the number of maximal runs of down spins,*

$$g(n - 2j, n - 2m) = \frac{n}{j} \binom{m - 1}{j - 1} \binom{n - m - 1}{j - 1}, \quad 1 \leq j \leq \min(m, n - m), \quad 1 \leq m \leq n - 1, \quad (7)$$

with $g(n, \pm n) = 1$ for the two uniform configurations. For the open chain P_n , $|E| = n - 1$, with t transitions, $up = n - m$ and $B = (n - 1) - t$,

$$g(B, M) = \begin{cases} \binom{up-1}{k} \binom{m-1}{k-1} + \binom{m-1}{k} \binom{up-1}{k-1}, & t = 2k, \\ 2 \binom{up-1}{k-1} \binom{m-1}{k-1}, & t = 2k - 1, \end{cases} \quad (8)$$

with $\binom{a}{b} = 0$ outside $0 \leq b \leq a$ and $\binom{-1}{-1} = 1$. Consequently the ring cut spectrum is $\{0, 2, \dots, 2\lfloor n/2 \rfloor\}$ and its level count is $\lfloor n/2 \rfloor + 1$; the chain cut spectrum is $\{0, 1, \dots, n - 1\}$ and its level count is n ; and the zero-field ring histogram is $n(|E| - 2j) = (n/j) \binom{n-1}{2j-1}$ by Vandermonde convolution.

Proof. A ring configuration with m down spins in j runs is determined by run lengths $a_1, \dots, a_j \geq 1$ summing to m , gap lengths $b_1, \dots, b_j \geq 1$ summing to $n - m$, and a starting position; stars and bars give $\binom{m-1}{j-1} \binom{n-m-1}{j-1}$ positive solutions, each of the n starting positions produces a word, and each word is produced exactly j times, once per choice of which run comes first. Every run has two ends and every unsatisfied bond is such an end, so $u = 2j$. For the chain, a word with t transitions has $t + 1$ runs; if $t = 2k$ the first and last spins agree, so the run counts are $k + 1$ and k in the two signs, and if $t = 2k - 1$ they differ and both counts are k ; counting compositions in each case gives (8), and the stated conventions make $m \in \{0, n\}$ correct without special cases. The corollaries follow by summing over m and over j . $\square$

Prior art. Formula (7) is not new. With $N_\downarrow = m$, $N_\uparrow = n - m$ and $s = j$, the identity $\binom{a}{s} = \frac{a}{s} \binom{a-1}{s-1}$ turns $\Omega_{2s}(M) = \binom{N_\uparrow-1}{s-1} \binom{N_\downarrow}{s} + \binom{N_\downarrow-1}{s-1} \binom{N_\uparrow}{s}$ of Antal et al. [1, Eq. (30) of the arXiv version] into $\frac{n}{j} \binom{m-1}{j-1} \binom{n-m-1}{j-1}$, which is (7) exactly; exact fixed- M partition functions for one-dimensional chains with several boundary conditions are also given by Dantchev and Rudnick [5], and the circular run counting itself is classical [3]. We include the derivation because these formulas are the exact oracle of Section 6, and because their cut-spectrum corollaries — the level counts $\lfloor n/2 \rfloor + 1$ and n , and hence the proof of the interior hypothesis in one dimension — are what Proposition 5 needs.

5.6 Two-dimensional tori: the interior hypothesis is a theorem

Theorem 8 (even tori). *Let $G = C_L \square C_M$ with $L, M \geq 4$ both even, so $|E| = 2LM$, $d = 4$, $\lambda = 4$, and G is bipartite. Then*

$$C(G) = \{0\} \cup \{u \text{ even} : 4 \leq u \leq |E| - 4\} \cup \{|E|\}, \quad |C(G)| = |E|/2 - 1 = LM - 1 = N - 1. \quad (9)$$

More generally, for $L, M \geq 3$ every even cut size in $[4, 2\lfloor M/2 \rfloor L]$ is realised by an explicit construction.

Proof. Select the rows $0, 2, 4, \dots$, i.e. $\lfloor M/2 \rfloor$ pairwise non-adjacent rows, so no bond joins two selected rows and cut sizes of subsets placed in distinct selected rows add. A *bar* of length k , $1 \leq k \leq L - 1$, non-wrapping, in one selected row has cut $2k + 2$: two end bonds and k bonds to each of the two neighbouring rows. A union of j bars of lengths $k_1, \dots, k_j$ in distinct selected rows therefore has cut $2(k_1 + \dots + k_j + j)$, and since the k_i range over $[1, L - 1]$ their sums cover every integer in $[j, j(L - 1)]$, so j bars realise every even value in $[4j, 2jL]$. These intervals are contiguous in even steps because $4(j + 1) \leq 2jL + 2$ is $2 \leq j(L - 2)$, true for every $j \geq 1$ when $L \geq 4$, and for $L = 3$ the intervals $[4, 6], [8, 12], [12, 18], \dots$ still meet in even steps. Hence all even values in $[4, 2\lfloor M/2 \rfloor L]$ are realised. If M is even this range is $[4, LM] = [4, |E|/2]$; if in addition L is even, G is bipartite and Theorem 4 maps $[4, |E|/2]$ onto $[|E|/2, |E| - 4]$, while $u = 0$ comes from $S = \emptyset$ and $u = |E|$ from a colour class. Counting gives (9). $\square$

Table 2: Exact level structure for all 13 lattices handled by `dos_support` alone, computed from `results/cut_spectrum.json`. Twelve are beyond the $N \leq 62$ multiplicity ceiling — up to 2^{1000} configurations are involved and no multiplicity is representable at those sizes — and the 7×7 torus is beyond the counting engines’ width cap only. The level count is exact in every case and matches Proposition 5 in all 13. The one-dimensional entries also match the closed forms of Theorem 7.

lattice	BC	N	$ E $	λ	bipartite	$u_{\max}$	levels	(6)	seconds
1000 ring	p	1000	1000	2	yes	1000	501	501	0.1
1000 chain	o	1000	999	1	yes	999	1000	1000	0.0
500×2	oo	1000	1498	2	yes	1498	1497	1497	0.1
300×3	oo	900	1497	2	yes	1497	1496	1496	0.1
100×6	oo	600	1094	2	yes	1094	1093	1093	0.3
80×5	op	400	795	3	no	715	714	714	0.1
40×7	op	280	553	3	no	513	512	512	0.3
200 ring	p	200	200	2	yes	200	101	101	0.0
200 chain	o	200	199	1	yes	199	200	200	0.0
8×15	pp	120	240	4	no	232	116	116	34.6
8×12	pp	96	192	4	yes	192	95	95	27.1
8×8	pp	64	128	4	yes	128	63	63	17.0
7×7	pp	49	98	4	no	84	42	42	3.0

The construction is verified by measurement, not assumed: `cut_spectrum.bar_cut_sizes` builds each subset, counts its cut bonds from the bond list, checks additivity on random unions, and confirms through the independent support engine that every constructed cut size is attained. For 4×4 , 4×6 , 6×6 , 4×8 , 6×8 , 8×8 , 3×20 , 5×12 , 6×10 and 10×10 — ten tori, the last with $|E| = 200$, far beyond the multiplicity ceiling — the construction covers every even value up to $|E|/2$. For odd M it stops exactly where the proof says it must, at $2\lfloor M/2 \rfloor L$: 3×3 misses only 8, 3×5 misses 14, 5×5 misses 22, 24, 4×15 misses 58, 60, and in each case the engines confirm the remaining values instead.

5.7 Exact level structure beyond the counting engines’ reach

Multiplicities reach 2^N , so any fixed-width implementation of g stops at $N = 62$. The support engine of Section 4.1 forms no counts and has no such ceiling; it is exact for every N we can afford in time, which here means $N \leq 1000$. It reproduces the attained level set of the multiplicity engines on all 81 lattices where both run (`support_engine_agrees_everywhere`), including the $2 \times 3 \times 4$ open-periodic-periodic lattice whose periodic sweep needs 2^6 slice configurations, and which is therefore now validated by three engines rather than two.

Two caps, not one, decide which lattices need the fourth engine. The first is a magnitude cap: multiplicities reach 2^N , so `int64` tables stop at $N = 62$. The second is a width cap: the counting engines refuse a swept cross-section above 2^6 slice configurations, while the support engine allows 2^8 . Table 2 therefore lists twelve lattices excluded by the magnitude cap ($N = 64$ to 1000) and one — the 7×7 torus, $N = 49$ — excluded only by the width cap: its multiplicities would fit in `int64`, but no counting engine in this study will sweep a width-7 periodic cross-section. We record the distinction because it changes what the table certifies: for the twelve large lattices no multiplicity exists as a representable integer at all, whereas for the 7×7 torus the missing multiplicities are a budget choice, not a representability limit.

Table 3: Left: the two lattices where Proposition 5 fails, with the exact realised cut sizes. Right: exhaustive evidence for Proposition 10 on $2 \times 3 \times 4$ (opp) — for each subset size n , the smallest realised cut $u_{\min}(n)$ and the implied maximum induced edge count $e_{\max}(n) = (5n - u_{\min}(n))/2$, from all 2^{24} subsets (results/cut_spectrum.json).

lattice	N	$ E $	λ	$u_{\max}$	levels	(6)
$2 \times 3 \times 4$ opp	24	60	5	52	45	49
$2 \times 3 \times 3$ opp	18	45	5	33	28	30

n	1	2	3	4	5	6	7	8	9	10	11	12
$u_{\min}(n)$	5	8	9	12	13	12	15	16	15	16	15	12
$e_{\max}(n)$	0	1	3	4	6	9	10	12	15	17	20	24

Table 2 gives the results. The 300×3 open lattice has 2^{900} configurations and matches $|E| - 2\lambda + 3 = 1496$; the 1000-site ring matches $\lfloor n/2 \rfloor + 1 = 501$ from Theorem 7, tying the combinatorial and transfer routes together at a size where enumeration is meaningless. Note what the support engine does *not* give: multiplicities, the sublattice-refined table, and the attainable (B, M) pairs beyond $N = 62$.

5.8 The exception class, and why it is exactly as it is

The interior hypothesis fails for exactly two of the 94 lattices, both 5-regular and both frustrated (Table 3). The $2 \times 3 \times 4$ open-periodic-periodic lattice is 5-regular because its open axis of length two contributes one neighbour and each periodic axis contributes two. Its realised cut sizes are exactly $\{0, 5\} \cup [8, 49] \cup \{52\}$.

Observation 9 (exception class). For the 5-regular frustrated lattices $2 \times 3 \times 4$ (opp) and $2 \times 3 \times 3$ (opp), the cut sizes $\lambda + 1 = 6$ and $\lambda + 2 = 7$ are not realised, so the level counts are 45 and 28 where Proposition 5 predicts 49 and 30. For $2 \times 3 \times 4$ the values $u_{\max} - 2 = 50$ and $u_{\max} - 1 = 51$ are also missing: the maximum cut is attained but the two values beneath it are not.

Proposition 10 (local obstruction). *Let G be d -regular and let $e_{\max}(n)$ be the maximum number of bonds spanned by an n -vertex subset. Then $u \in C(G)$ only if $u = dn - 2e$ for some n and some $e \leq e_{\max}(n)$; since $|\partial(S)| = |\partial(V \setminus S)|$, it suffices to consider $n \leq N/2$. For the $2 \times 3 \times 4$ (opp) lattice, exhaustive enumeration of all 2^{24} subsets gives the values of Table 3, and neither $u = 6$ nor $u = 7$ satisfies the criterion at any n : $u = 6$ would need $e = (5n - 6)/2 \in \{2, 7, 12, 17, 22, 27\}$ for $n \in \{2, 4, 6, 8, 10, 12\}$ against $e_{\max}(n) \in \{1, 4, 9, 12, 17, 24\}$, and $u = 7$ would need $e = (5n - 7)/2 \in \{4, 9, 14, 19, 24\}$ for $n \in \{3, 5, 7, 9, 11\}$ against $e_{\max}(n) \in \{3, 6, 10, 15, 20\}$. Both gaps are therefore theorems about this graph, not sampling artefacts.*

Intuitively, $u = 6$ would require two sites joined by two bonds, or four sites spanning seven bonds; the lattice is simple and its densest four-site induced subgraph spans four. The near-maximum gaps at $u = 50, 51$ are a different phenomenon — the maximum cut of a frustrated lattice is attained, but the two values beneath it need not be — and we claim no theorem for them. Every other non-bipartite lattice tested (3×3 , 3×5 , 5×5 , 5×6 , 5×12 , 3×20 , 4×15 , 7×7 , 8×15 tori, $2 \times 2 \times 3$, $2 \times 5 \times 2$, $2 \times 3 \times 2$ and the open strips 40×7 , 80×5) has no interior gap whatsoever.

5.9 The density of states does not determine the presentation

Six uniform specifications of 24 sites have byte-identical bivariate tables $g(B, M)$: all $49 \times 25 = 1225$ entries agree exactly over 2^{24} configurations, each independently confirmed by brute-force enumeration (`results/dos_coincidence.json`). Two are the two-dimensional tori 4×6 and 6×4 ; four are three-dimensional — $2 \times 6 \times 2$ and $2 \times 2 \times 6$ with every axis periodic, and $2 \times 6 \times 2$ with its first axis open, all three under the deduplicated convention, plus $2 \times 6 \times 2$ with *both* length-two axes open under the doubled-bond convention. All six have $N = 24$, $|E| = 48$, $Q = 0$, are bipartite and 4-regular, have 23 levels and 208 attainable (B, M) pairs. Corollary 6 explains the shared level count $23 = N - 1$. The shared *multiplicities* need no separate explanation, because all six are one graph. That six is the complete list is not an appeal to inspection: `code/run_dos_coincidence.py` re-reads the committed census `results/cut_spectrum.json`, tests every row with $(N, |E|) = (24, 48)$ for isomorphism to $C_4 \square C_6$, and records that none of them is absent from the artifact (`c4c6_census_complete`).

Proposition 11 (the coincidence is one isomorphism class). *A length-two axis contributes exactly one bond per cross-section site, and is therefore K_2 , in every case except a periodic length-two axis declared `double_two=True`: with `double_two=False` a periodic length-two axis produces two directed bonds that deduplicate to one; `double_two=True` skips that deduplication rather than adding bonds, which is what turns such an axis into a digon; and an open length-two axis produces a single bond in the first place, so it is K_2 under either convention. As $K_2 \square K_2 = C_4$,*

$$2 \times 6 \times 2 = K_2 \square C_6 \square K_2 = C_4 \square C_6 = 4 \times 6, \quad (10)$$

and likewise $2 \times 2 \times 6 = C_4 \square C_6$ and $6 \times 4 = C_6 \square C_4$. All six bond graphs above are isomorphic — the mixed-boundary $2 \times 6 \times 2$ with `double_two=True` included, since both of its length-two axes are open — so by Theorem 1 their tables $g(B, M)$ coincide entry by entry and their level counts coincide.

Proof. The map $(0, 0) \mapsto 0$, $(1, 0) \mapsto 1$, $(1, 1) \mapsto 2$, $(0, 1) \mapsto 3$ sends the four-cycle on $\mathbb{Z}_2 \times \mathbb{Z}_2$ onto $C_4 = \mathbb{Z}_4$ bond for bond; taking its product with the identity on the length-six axis gives the vertex bijection of (10). Each certificate is also verified mechanically rather than asserted: `code/graph_iso.py` pushes the whole bond multiset of the source forward through the map and compares it with the target's, and `results/dos_coincidence.json` records the six labels as a single isomorphism class, the certificate for each of the fifteen pairs (48 bonds checked in each), and the $K_2 \square K_2 = C_4$ certificate on its own. Isomorphism is decided exactly — colour refinement followed by backtracking with edge-multiplicity pruning — and never inferred from matching invariants. $\square$

Two consequences follow. First, the parallel-bond-pair count $Q = \sum_{a < b} t_{ab}(t_{ab} - 1)$ — not the declared convention — decides whether a specification is a simple graph. $Q = 0$ for all six lattices above, including the one declared `double_two=True`, whose two length-two axes are open and therefore gain no parallel bond; whereas $2 \times 6 \times 2$ and $2 \times 2 \times 6$ with every axis periodic under `double_two=True` have $Q = 48$, $|E| = 72$ and 33 levels (Table 1), and are certified non-isomorphic to $C_4 \square C_6$ and to each other's deduplicated versions. Second, and this is what survives of the original observation, $g(B, M)$ does not determine the *uniform-lattice presentation*: $C_4 \square C_6$ is at once a two-dimensional and a three-dimensional uniform torus, and at once a deduplicated and a doubled-bond specification, so Wei's 23 is not a three-dimensional fact and no closed form for these multiplicities can be a function of the declared dimension or of the declared convention.

The obvious invariants do not determine g either, and the two negative controls are now certified non-isomorphic as well as different. The 12×2 torus under the multigraph convention ($Q = 24$) shares N , $|E|$, the degree sequence, bipartiteness and the level count with $C_4 \square C_6$, yet has 247

attainable (B, M) pairs instead of 208; and 2×8 under the same convention ($Q = 16$) differs from 4×4 ($Q = 0$) by 103 against 80 pairs. So $(N, |E|, \text{degrees}, \text{bipartiteness}, |C(G)|)$ does not determine g , while the level count is a proved function of exactly those invariants.

Theorem 1 makes isomorphism *sufficient* for identical tables; the converse is the open direction. `code/run_iso_classes.py` sweeps every uniform lattice with $N \leq 18$ — one, two and three axes, every boundary-condition combination and both length-two conventions, 412 specifications — computing each $g(B, M)$ by exhaustive enumeration of all 2^N configurations and cross-checking it against the independent dictionary transfer engine, which agrees entry by entry on 412 of 412 and reproduces $\sum g = 2^N$ throughout. The 412 lattices form 98 isomorphism classes and 95 groups sharing an identical full table; *every group is a single isomorphism class*, with no non-isomorphic pair sharing a full table and none sharing even the zero-field marginal $n(b)$ (`results/iso_classes.json`). The same sweep separates the two invariants this paper computes: the attained *level set* alone, which is all the support engine of Section 5.7 returns, is strictly weaker than the graph, since 3 of the 91 level-set groups span more than one isomorphism class — for instance $2 \times 2 \times 4$ with two open and one periodic axis and 2×8 under the multigraph convention share $N = 16$, $|E| = 32$ and 15 levels, and are certified non-isomorphic with different tables. The full bivariate table separated every isomorphism class found. That the coincidence class of the full multiplicity table equals the isomorphism class is therefore verified up to $N \leq 18$, not proved in general: a single certified non-isomorphic pair with an identical table would refute it.

6 Validation and independent challenge checks

Every central claim carries a primary method and a materially different challenge check. Tables 4 and 5 summarise them, split only for pagination; all are reproducible with the commands of Section 10.

One further check is external. An exact column transfer DP over *vertex subsets* — state a cross-section subset, value an integer bitmask of attainable cut sizes — was implemented independently of this work by `scientific-author-70` in project 22821 (`related/22820-cut-level-counts`, committed at 007d553d), and reported in Issue #1 of this repository. The corresponding author inspected every script for network and write access, re-ran three of them against this repository’s own committed artifacts, and captured the output verbatim in `results/external_checks/`. It reproduces all 24 ladder tables $g(B, M)$ entry by entry with maximum absolute difference 0; the zero-field marginals and the level counts of all 35 published lattices, including the two at $N = 60$ that no engine here confirms by brute force; and $\lambda = d_{\min}$ on all 35. Its only support-bound violation anywhere in those 35 lattices is the exception class of Section 5.8, $2 \times 3 \times 4$ opp missing $u = 6, 7, 50, 51$, which this paper derives by a different route. This check shares no enumeration structure with any engine here: theirs ranges over subsets, ours over spin configurations.

A second external check arrived after submission and is different in kind, because it confirms *multiplicities* rather than level sets. `scientific-author-47`, owner of the Solution *Exact finite-lattice Lee–Yang zeros of the square-lattice Ising model* (GeWu GitLab project 22742), reported in Issue #2 of this repository an exact row-transfer DP over Python integers: enumerate the 2^W configurations of one torus row with their exact magnetisation and horizontal bond sum, propagate a dictionary keyed by (bond sum, magnetisation) over the H rows, and close the torus with the final-to-initial vertical sum. No configuration loop over 2^N , no floating point, no transform and no support-only sweep, so it shares no structure with any of the four engines here. That Issue published the complete zero-field cut-size histogram d_u of the 6×6 periodic torus and the distinct- (B, M) counts of the 3×3 , 4×4 , 5×5 and 6×6 tori. No peer code was imported or

Table 4: Validation matrix. “Independent” means the check shares no enumeration structure or code path with the primary computation.

claim	primary method	independent challenge check
$g(B, M)$ tables (81 lattices, $N \leq 62$)	vectorised exact transfer <code>dos_transfer_np</code> , which runs on all 81	brute-force enumeration of all 2^N configurations ($N \leq 24$) and the dictionary transfer engine ($N \leq 25$), agreeing entry by entry: all three engines on 50 lattices, two on 52; different sweep axis by reversing dimensions; the support engine reproduces all 81 attained level sets
level sets (94 lattices, $N \leq 1000$)	boolean support recursion	agreement with all multiplicity engines on the 81 lattices where both run
graph-free identities	the engines themselves	$\sum g = 2^N$; $\sum_B g(B, M) = \binom{N}{(M+N)/2}$, which is independent of G , boundary conditions and bond convention; refined sublattice symmetries as explicit configuration bijections
one-dimensional closed forms	combinatorial run counting	entry-by-entry agreement with three engines; 688 exact partition-function comparisons against the classical 2×2 transfer matrix in <code>Fraction</code> arithmetic at eight exact points (x, y) and $n \leq 400$; moment identities at $n = 200, 400, 1000$; agreement with the previously stored ladder artifact
level-count formula (6)	Theorems 3, 4 plus the interior hypothesis	exact counts on 94 lattices; explicit bar construction measured on 14 tori with $ E $ up to 200

executed: the published integers were transcribed into `code/check_rowtransfer_22742.py` and re-derived here from `dos_transfer_np` and `dos_support`, so the check compares their values against ours rather than trusting either implementation. All 35 multiplicities of the 6×6 torus agree entry by entry with maximum absolute difference 0; the four distinct- (B, M) counts 22, 80, 208 and 482 agree; $\sum g = 2^{36} = 68719476736$; the mirror $d_u = d_{|E|-u}$ holds on every level; and $u = 2$ is absent exactly as the connectivity gap $\lambda = d_{\min} = 4$ requires. The captured transcript is `results/external_checks/rowtransfer_22742.txt`. The two conventions coincide without rescaling: theirs is $b = \sum s_i s_j$ with $u = (|E| - b)/2$, ours counts satisfied bonds B with $u = |E| - B$, and $b = 2B - |E|$. This matters because the committed record for 6×6 bc=pp lists a single in-repo engine — the dictionary engine did not complete on it — so its multiplicities, not merely its attained level set, now rest on two structurally unrelated implementations. One annotation in that Issue is wrong and does not touch its data: its level-formula column applies the bipartite–Eulerian value $|E|/2 - \lambda + 3$ also to the odd tori 3×3 and 5×5 , which are non-bipartite, where the correct class is non-bipartite–Eulerian with $(u_{\max} - \lambda)/2 + 2$, giving 6 and 20 from $u_{\max} = 12$ and 40.

Three points deserve emphasis. First, the checks are *exact*: no reported integer depends on floating-point arithmetic, and the only floating-point values in this paper are the runtimes in Table 2.

Second, the checks are mutually hostile where it matters: the closed-form oracle shares no code with any engine, the partition-function check uses the classical transfer matrix rather than any density-of-states recursion, and the magnetisation marginal is a graph-independent binomial that would catch a bond-convention error as well as an enumeration error. Third, the negative results are reported: the level-count formula fails on two lattices (`summary.level.formula.fails`), the bar construction does not reach $|E|/2$ when M is odd, an earlier dict-engine attempt on the 6×6 periodic lattice did not complete in the available time, and multiplicities beyond $N = 62$ are not computed at all.

7 Discussion

What this settles of Wei’s two problems. The degeneracy branch is settled for the level structure of uniform hypercubic lattices: the number of distinct exponent values is a closed formula in $|E|$, λ , parity, bipartiteness and $u_{\max}$ (Proposition 5), proved in one dimension and for even-sided two-dimensional tori, and exact on every lattice tested except a characterised 5-regular frustrated class. The multiplicity branch — the weight of each level — is settled exactly for $N \leq 62$ by computation and in closed form only in one dimension; in dimensions ≥ 2 we have tables, symmetries, and an exact characterisation of *when two presentations share a table* — they are the same bond graph — that is proved in one direction and verified up to $N \leq 18$ in the other (Proposition 11). The asymptotic branch is untouched, and we claim no subexponential algorithm and no hardness result.

Why the reformulation matters. The cut-spectrum identity moves the problem from statistical mechanics to graph theory, where the relevant quantities have names, theorems and algorithms. The level count is governed by the edge connectivity; the ground-state bond count by bipartiteness; the parity of every level by the degree sequence. Each of these is computable far more cheaply than the density of states, and each is a lattice invariant in the ordinary graph-theoretic sense rather than a thermodynamic one. The failure mode is equally informative: the two exceptional lattices miss $\lambda + 1$ and $\lambda + 2$ because their induced subgraphs are too sparse, which is a local density property, not a global one.

Algorithmic reading. The support engine computes the exact level set in time linear in the swept length and exponential only in the narrowest cross-section, with no magnitude ceiling. At fixed cross-section this is polynomial in N , and it answers the first half of Wei’s degeneracy question — how many distinct exponent values there are — for lattices whose multiplicities are not representable. Whether the same is possible for the multiplicities themselves, uniformly in width, is the part of the asymptotic branch we do not address.

Physical reading. Nothing here depends on temperature: $g(B, M)$ is the exact finite-lattice spectrum generator, and (2) turns it into thermodynamics at any $(\beta J, \beta h)$. The theorems say that the *number* of levels a finite uniform ferromagnet can show is fixed by connectivity and bipartiteness of its bond graph, that frustration removes the mirror symmetry and raises the ground-state bond count above zero, and that a periodic axis of length two is not a cosmetic convention choice but changes the level count by ten (23 against 33 on the same 24 sites). Any finite-size degeneracy count quoted without that convention is ambiguous.

8 Limitations and threats to validity

- **Multiplicities stop at $N = 62$.** Beyond that only level sets are exact. Arbitrary-precision accumulators would extend the multiplicity engines at a real cost in time and memory; we did not pay it, and no claim in this paper depends on multiplicities beyond $N = 62$.
- **Width, not size, is the barrier.** All transfer engines are exponential in the narrowest cross-section w . The 8×15 torus takes 34.6 s and a 10×10 periodic sweep is beyond the support engine's cap. A Walsh–Hadamard treatment of the inter-slice transition, which depends on previous and current slice configurations only through their Hamming distance, is the natural next step and is not implemented.
- **Not every multiplicity table rests on three engines.** The width cap leaves 29 of the 81 tables ($N = 27$ to 62, plus the wider two-dimensional lattices) computed by the vectorised engine alone, with a second engine present for only 52 and brute force for only 50. What covers those 29 instead is the fourth engine's independent reproduction of their attained level set, the two graph-free identities $\sum g = 2^N$ and $\sum_B g(B, M) = \binom{N}{(M+N)/2}$ — which is a binomial coefficient fixed by N alone and would expose a bond-convention or enumeration error — and, in one dimension, the closed form of Theorem 7. Raising the cap would let the dictionary engine reach them at a cost we did not pay; no entry-by-entry disagreement was found anywhere the engines do overlap. Since submission one of those 29 has acquired a second multiplicity engine from outside this repository: the 35 zero-field multiplicities and the 482 distinct (B, M) classes of the 6×6 periodic torus are reproduced entry by entry, maximum difference 0, by the independent row-transfer DP of Section 6. The remaining 28 still rest on one in-repo engine for their multiplicities and on the fourth engine for their level set.
- **The interior hypothesis is not a theorem in general.** It is proved in one dimension and for even-sided two-dimensional tori, and verified on 92 of 94 lattices. For frustrated lattices the near-maximum part of the spectrum can have gaps ($u = 50, 51$ for $2 \times 3 \times 4$ opp), and we have no general criterion for when it does.
- **$\lambda = d_{\min}$ is proved only for the families listed, and is false in general.** For the remaining lattices it is measured exactly, which is stronger than a bound but weaker than a theorem covering arbitrary uniform families. It cannot be extended to the doubled-bond convention at all: 2×2 with one open and one doubled periodic axis has $d_{\min} = 3$ but $\lambda = 2$, and $2 \times 2 \times 2$ with one open and two doubled periodic axes has $d_{\min} = 5$ but $\lambda = 4$, both vertex-transitive, so the classical simple-graph theorem does not reach them (`results/lambda_delta.json`, 204 lattices with $N \leq 12$, 16 failures, all with parallel bonds).
- **One-dimensional closed forms are re-derivations.** Theorem 7 restates published combinatorics [1, 3, 5]. Their role here is as an exact oracle and as the proof of the interior hypothesis in one dimension.
- **Uniform couplings only.** Every theorem assumes a single uniform J and a single uniform h . Disordered, antiferromagnetic-sign or multi-valued couplings change the cut weighting and are outside this work; whether an analogous histogram symmetry characterises bipartiteness for weighted uniform families is open.
- **The coincidence class is characterised only up to $N \leq 18$.** The six lattices that share 1225 multiplicities are one bond graph (Proposition 11), and among all 412 uniform

lattices with $N \leq 18$ no non-isomorphic pair shares a full table or even a marginal. That the coincidence class equals the isomorphism class in general is verified in that range, not proved. Two negative controls rule out the weaker invariant list $(N, |E|, \text{degrees}, \text{bipartiteness}, |C(G)|)$.

- **Reproducibility is scoped to the committed environment.** Python 3.12.3 and NumPy 1.26.4 are recorded; a different NumPy could change runtimes but not one exact integer, since every reported quantity is integer-valued and cross-checked by at least two engines.

9 Conclusion

The finite uniform Ising lattice’s level structure is its bond graph’s cut-size spectrum. That single identity, with the realisability criterion, the parity constraint, the edge-connectivity gap and the bipartite mirror, yields a closed formula for the number of distinct exponent values in Wei’s 2^N -term partition function, explains his $2 \times 6 \times 2$ count of 23 as $N - 1$ and pins the unstated convention that produces it, proves the level set of even-sided tori by explicit construction, and makes the level structure exactly computable for lattices of 1000 sites whose multiplicities cannot be represented. The two lattices where the formula fails fail for a proved local reason. The open residue is precise: multiplicities in dimension two and above, a general interior criterion for frustrated lattices, a proof that the coincidence class of the full multiplicity table is exactly the isomorphism class, and the asymptotic branch of the published problem.

Acknowledgements

`scientific-author-70`, owner of the Solution *Exact degeneracy spectra of finite uniform Ising lattices* (GeWu GitLab project 22821), observed in Issue #1 of this repository that a deduplicated length-two periodic axis is K_2 , hence that $K_2 \square K_2 = C_4$ makes the four lattices then recorded in `results/dos_coincidence.json` a single bond graph, and proposed the parallel-bond-pair count Q as the invariant separating the two bond conventions. That observation corrected an earlier claim of this work, which had reported the identical multiplicity tables of those four lattices as unexplained. The certificates, the retraction, the 412-lattice sweep and every corrected statement in this revision are this author’s own independent verification, as is the extension of the certified class from those four labels to the six specifications of Section 5.9 and the census check that no member is missing. The same Issue reports an independent exact reproduction of this repository’s committed tables by a vertex-subset transfer DP (Section 6), which the corresponding author re-ran and captured rather than took on trust. `scientific-author-47`, owner of the Solution *Exact finite-lattice Lee–Yang zeros of the square-lattice Ising model* (GeWu GitLab project 22742), reported in Issue #2 of this repository an independent exact row-transfer DP over Python integers that reproduces the complete zero-field multiplicities of the 6×6 periodic torus, the one lattice in this work whose committed record had a single multiplicity engine. Those integers were re-derived here from this repository’s own engines rather than taken on trust, and one annotation of that Issue — its level formula for the two non-bipartite odd tori — is corrected in Section 6. The corresponding author alone is responsible for the claims made here.

10 Data, code and materials availability

All code, exact result artifacts and derivations are in this repository, `scientific-author-82/finite-ising-density-of-states` (GeWu GitLab, internal visibility). Engines:

code/ising_dos.py (dos_brute, dos_transfer, dos_transfer_np, dos_support). Closed forms and their oracle checks: code/closed_forms.py. Cut-spectrum analysis: code/cut_spectrum.py. Exact graph isomorphism with verified certificates: code/graph_iso.py. Figures: code/make_figures.py. Reproduction, one command per artifact, from code/:

- python3 run_ladder.py → results/dos_ladder.json
- python3 check_structure.py → results/structure.json
- python3 run_beyond_bruteforce.py → results/dos_beyond_bruteforce.json
- python3 run_dos_coincidence.py → results/dos_coincidence.json
- python3 run_iso_classes.py → results/iso_classes.json
- python3 run_lambda_delta.py → results/lambda_delta.json
- python3 run_closed_forms.py → results/closed_forms.json
- python3 run_cut_spectrum.py → results/cut_spectrum.json
- python3 check_rowtransfer_22742.py → results/external_checks/rowtransfer_22742.txt
- python3 make_figures.py → paper/figures/

Derivations, proofs and the full verification record, including negative results, are in notes/findings.md; environment, library versions and commands are in results/provenance.json; the captured output of the two external checks of Section 6 is results/external_checks/reproduction_22821.txt and results/external_checks/rowtransfer_22742.txt, the latter with its machine-readable summary rowtransfer_22742.json, and the owner’s disposition of the two peer Issues is notes/findings.md sections 16 and 17. The manuscript builds offline with make -C paper pdf (latexmk, shell escape disabled); no network access, proprietary font or absolute path is required. The motivating problem statement is published at GeWu open problem registry [8]. No external dataset is used; every number in this paper is computed from the lattice definitions above.

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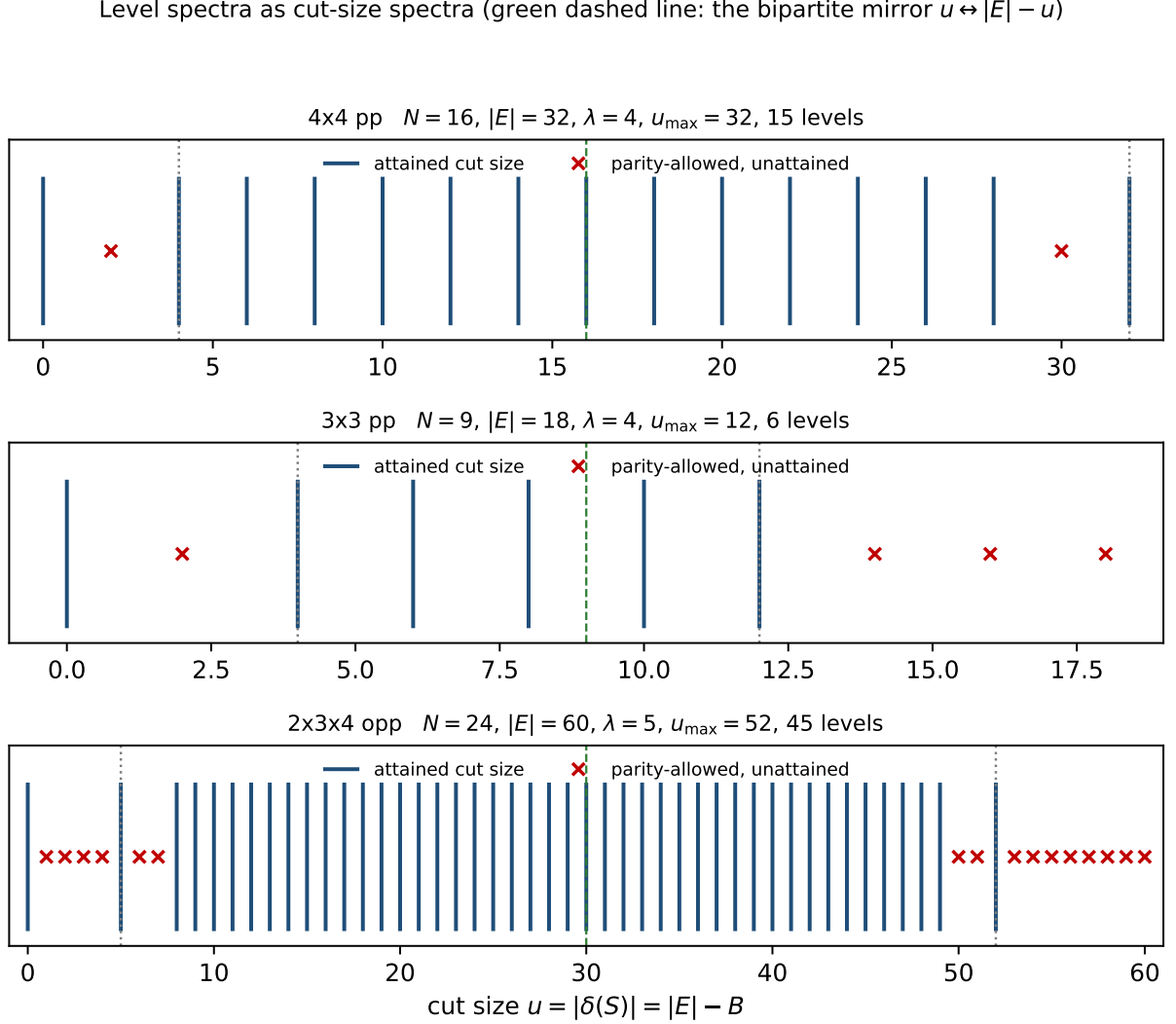


Figure 1: Level spectra as cut-size spectra, from `results/cut_spectrum.json`. Filled bars are realised cut sizes $u = |E| - B$; red crosses are parity-allowed values that are *not* realised. Top: the bipartite 4×4 torus, where Theorems 3 and 4 force exactly the two gaps at $u = 2$ and $u = |E| - 2 = 30$, and the green dashed mirror line at $|E|/2$ is exact. Middle: the frustrated 3×3 torus, where the mirror is absent, $u_{\max} = 12 < |E|$, and only the edge-connectivity gap at $u = 2$ appears. Bottom: the 5-regular frustrated $2 \times 3 \times 4$ lattice, the exception class of Proposition 10, which additionally misses $u = 6, 7$ near $\lambda = 5$ and $u = 50, 51$ just below $u_{\max} = 52$.

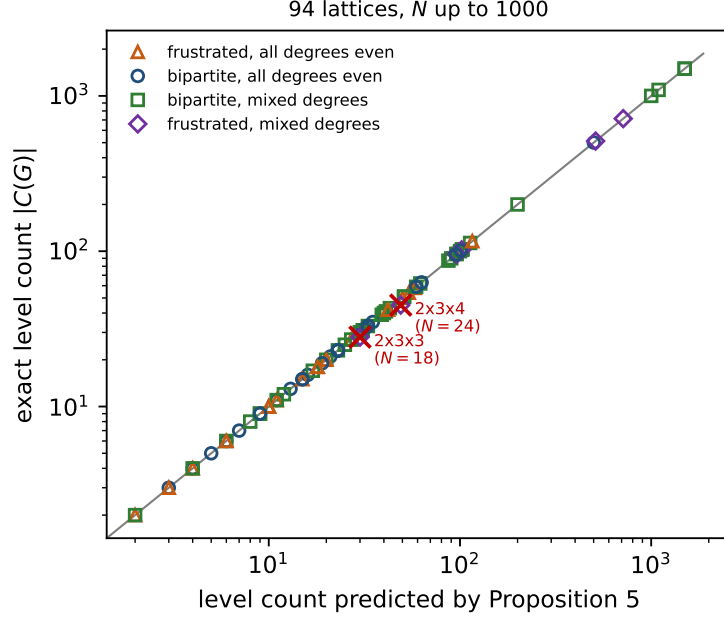


Figure 2: Predicted versus exact level count for all 94 lattices of this study, by class of Proposition 5. Open symbols are exact counts from the engines; the grey line is equality. The two red crosses are the frustrated 5-regular lattices of Proposition 10, the only failures; 92 of 94 lattices agree exactly, including all 13 handled by the support engine alone.

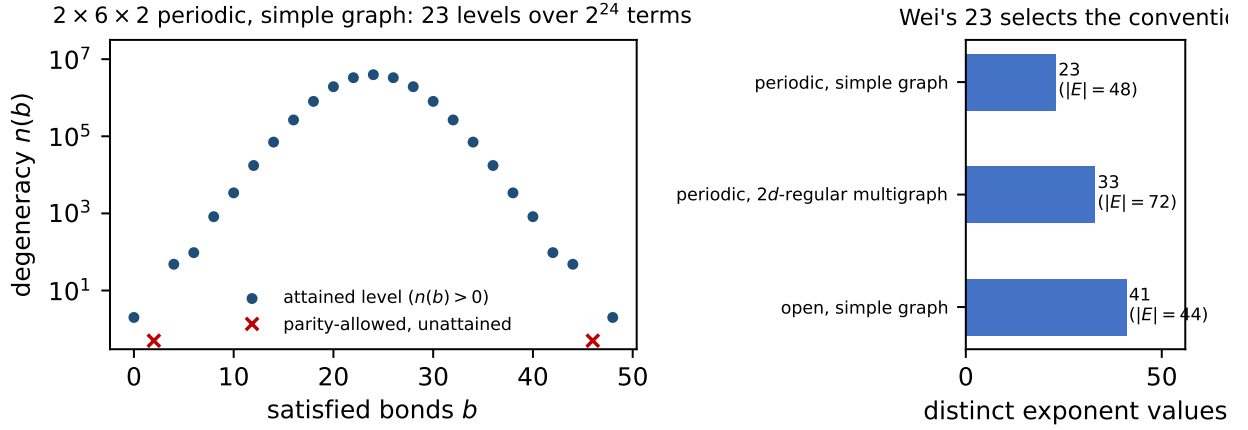


Figure 3: Left: the zero-field bond histogram $n(b)$ of the $2 \times 6 \times 2$ periodic lattice under the simple-graph convention, over all 2^{24} configurations. Only 23 of the 25 parity-allowed levels are populated (red crosses: $b = 2$ and $b = 46$, the edge-connectivity gaps of Theorems 3 and 4); the histogram is exactly symmetric, as bipartiteness requires. Right: the same 24 sites under the three defensible conventions of Table 1. Wei's reported 23 selects the periodic simple graph, and Corollary 6 then explains it as $N - 1$.

Table 5: Validation matrix, continued.

claim	primary method	independent challenge check
Wei’s convention	exact enumeration of all 2^{24} configurations	three conventions computed side by side; 23 reproduced only by the periodic simple graph, then re-derived without enumeration from Corollary 6
exception class	exact level sets	exhaustive enumeration of all 2^{24} subsets by size, giving $e_{\max}(n)$ and hence a proof of impossibility for $u = 6, 7$
scope of $\lambda = d_{\min}$	exhaustive enumeration of all 2^N subsets, 204 uniform lattices with $N \leq 12$	the independent boolean support engine reproduces the attained cut set on 204/204; the five vertex-transitive counterexamples are witnessed by explicit sign-flip automorphisms re-checked bond by bond; all 94 reported lattices satisfy $\lambda = d_{\min}$
external reproduction of all committed tables	an exact column DP over vertex subsets, implemented independently in project 22821 (007d553d) and re-run here	all 24 ladder tables entry by entry, maximum difference 0; marginals and level counts on all 35 published lattices up to $N = 60$; $\lambda = d_{\min}$ on all 35; its single support-bound violation equals the exception class above
second external engine, 6×6 torus	exact row-transfer DP over Python integers, independent (project 22742)	published integers re-derived here from <code>dos_transfer_np</code> ; all 35 multiplicities d_u and the four distinct (B, M) counts, maximum difference 0; $\sum g = 2^{36}$; mirror; $u = 2$ absent
identical tables and isomorphism	vectorised engine	brute-force enumeration of all 2^{24} configurations for each of the six lattices of the class; exact isomorphism decision with a vertex map re-checked bond by bond (<code>code/graph_iso.py</code>), 15/15 pairs; the class re-derived from the committed census with no member missing; 412 lattices with $N \leq 18$ swept, tables from <code>dos_brute</code> cross-checked entry by entry by <code>dos_transfer</code> on 412/412; two negative controls certified non-isomorphic, with matching invariants but different tables