

The affinity–topology bound on current fluctuations in finite Markov networks: convention, exhaustive certification, sharpness and an exact minimiser law

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`finite-markov-affinity-current-bounds`

(internal GitLab project id 22780)

8 September 2026

Abstract

For a finite, irreducible continuous-time Markov jump network in a nonequilibrium steady state, Pietzonka, Barato and Seifert bounded the scaled cumulant generating function $\lambda_s(z)$ of the medium entropy current below by a hyperbolic cosine whose only network-dependent input, besides the entropy production rate σ , is the ratio $r = \mathcal{A}^*/n^*$ of the smallest cycle affinity to its length [Phys. Rev. E **93**, 052145 (2016), Eqs. (55)–(56)]; the bound was subsequently proved via uniform-cycle decompositions [J. Phys. A **49**, 34LT01 (2016)]. Validity is therefore settled, but the content of the bound is not: the definition of r is stated differently in the two papers, no exact certification on a complete finite network class existed, and the equality set was unknown. We resolve all three. (i) The operative convention is the *minimal positive* affinity per edge over simple cycles; the literal “smallest ratio among all cycles” reading of the earlier paper is strictly weaker, and a signed/maximal reading is falsified by an exact four-state counterexample on which the Fano factor falls *below* its purported bound. (ii) We certify the bounds in exact and 80-digit arithmetic on the complete class of 877 626 nonequilibrium four-state multicyclic networks over the integer rate grid 1..3 (seven topologies) and on all 4 917 816 of their edge currents, under both readings: the entropy bounds on a seven-point z -grid and the *full* z -dependent per-current bounds, not merely their local consequences, on the symmetric ten-point grid that covers both orientations of every edge. Every margin is strictly positive, the smallest being 1.96×10^{-9} for the hyperbolic-cosine bound and 4.24×10^{-8} for its per-current analogue under the theorem’s reading, with Fano minima 2.858×10^{-4} (absolute) and 2.888×10^{-5} (positive); each reported extremum is computed twice by algebraically independent routes. (iii) Uniform unicyclic rings saturate the bound for *every* real z as a closed-form symbolic identity, and on exhaustive grids they are the only saturators; the companion equilibrium bound is saturated by the same family at $k^+ = k^-$. (iv) We prove a chord-invariance lemma: adding symmetric chords leaves $\lambda_s(z)$ exactly fixed while strictly lowering r , so the margin is pinned at a positive constant and the equality set is *topologically discontinuous*. (v) For the uniform-field family on the complete graph K_N we prove and exhaustively verify the exact minimiser law $r = |\ln \rho|/L_{\text{odd}}(N)$, where $L_{\text{odd}}(N)$ is the largest odd integer $\leq N$, which shows the affinity refinement degenerates as $2 + \ln^2 \rho / (6 L_{\text{odd}}(N)^2) + O(L_{\text{odd}}(N)^{-4})$ at fixed driving. All results are reproduced by a gated 295-check validation suite; the exhaustive scope is $N = 4$ at grid 1..3 (unicyclic through $N = 5$), and larger systems are sampled rather than exhausted.

1 Introduction and motivation

The thermodynamic uncertainty relation (TUR) states that any steady-state current J of a Markovian nonequilibrium system obeys $F\sigma/J \geq 2$, where σ is the entropy production rate and F the Fano factor of the current [4]; see Horowitz and Gingrich [6] for a review. A strictly stronger family of bounds refines the constant 2 using thermodynamic *and topological* information about the network. Pietzonka et al. [8] proposed, for the medium entropy current of a finite Markov jump network, the parabolic bound and the hyperbolic-cosine bound

$$\lambda_s(z) \geq \sigma \frac{\cosh[(z + \frac{1}{2})r] - \cosh[r/2]}{r \sinh[r/2]}, \quad F_s \geq r \coth(r/2), \quad (1)$$

with $r = \mathcal{A}^*/n^*$ “the smallest ratio $\mathcal{A}/n$ among all the cycles in the network”. Both bounds were proved shortly afterwards: the parabolic one by Gingrich et al. [5], and (1) by Pietzonka et al. [7] through a decomposition of the stationary current into *uniform cycles*. The latter paper also proves the equilibrium companion bound $\lambda_\alpha(z) \geq 2N^2 D_\alpha (\cosh(z/N) - 1)$ and thereby establishes the enzyme-kinetics Fano bound conjectured by Barato and Seifert [3].

Validity is thus settled. What the literature leaves open is the *content* of the bound, and it is precisely the content that determines whether (1) can be used — for example to bound the number of hidden states of a molecular machine from a measured Fano factor. Four difficulties stand out.

1. **The definition of r is not the same in the two papers.** The conjecturing paper allows *all* cycles, which admits zero-affinity cycles and hence $r = 0$, collapsing (1) to $F_s \geq 2$; the proving paper restricts the minimum to *positive* affinities and states that this is what makes the bound stronger than the parabolic one. Neither paper enumerates the alternatives (signed extrema, backtracking two-cycles, arbitrary closed walks, elements of the cycle space), and the alternatives are not equivalent.
2. **No exact certification on a complete finite class existed.** Numerical evidence in the sources is illustrative rather than exhaustive, and no verification with certified arithmetic covers an entire family of networks.
3. **The equality set was unknown.** A bound whose saturators are not identified cannot be improved, and cannot be recognised as tight in an experiment.
4. **The dependence of r on topology is not explicit.** Even for the most symmetric driven family — a complete graph with a uniform rate ratio — the minimising cycle is not obvious, because affinity and length trade off.

This paper answers all four for finite Markov jump networks with rational rates, where every quantity of interest is either exactly rational, exactly a rational linear combination of logarithms of rationals, or computable to arbitrary precision. Our contributions are: the definitive convention with exact falsification of the wrong readings (Section 4.1); a certified exhaustive audit of the whole four-state multicyclic class (Section 4.2); a symbolic proof of sharpness together with an exhaustive characterisation of the equality set and of the equilibrium companion bound (Section 4.3); a chord-invariance lemma showing that the equality set is topologically discontinuous (Section 4.4); and an exact minimiser law for the uniform-field family, with a parity proof and exhaustive integer verification (Section 4.5). Section 5 lists the independent challenge checks, Section 7 the honest limits.

2 Setting, notation and scope

2.1 The network

Let $G = (V, E)$ be a finite connected simple graph, $|V| = N \geq 3$, states labelled $1, \dots, N$ (the accompanying code uses $0, \dots, N-1$). Each undirected edge $\{i, j\} \in E$ carries two strictly positive transition rates $k_{ij}, k_{ji} > 0$; $k_{ij} = 0$ for $\{i, j\} \notin E$. The generator is $Q_{ij} = k_{ij}$ for $i \neq j$ and $Q_{ii} = -\sum_{j \neq i} k_{ij}$, so that the stationary distribution π is the unique positive row vector with $\pi Q = 0$, $\sum_i \pi_i = 1$. Irreducibility follows from connectedness and $k_{ij} > 0$ on E . We work with *rational* rates throughout, $k_{ij} \in \mathbb{Q}_{>0}$, which is what makes exact certification possible; nothing in the bounds themselves requires rationality.

Following Schnakenberg's network theory [2, 9], the edge log-ratio $L_{ij} \equiv \ln(k_{ij}/k_{ji})$ is the local entropy increment of the transition $i \rightarrow j$ in units of $k_B T$ [10]. The stationary edge current is $j_{ij} \equiv \pi_i k_{ij} - \pi_j k_{ji}$ and the entropy production rate is

$$\sigma = \sum_{i < j} j_{ij} L_{ij} = \sum_{i < j} \pi_i k_{ij} \ln \frac{\pi_i k_{ij}}{\pi_j k_{ji}} \geq 0. \quad (2)$$

For rational rates every j_{ij} and π_i is rational, so σ is an exact rational linear combination of logarithms of positive rationals — an object we can manipulate symbolically without ever evaluating a logarithm numerically (Section 3).

2.2 Currents, scaled cumulant generating functions, Fano factors

A time-integrated current $X(t)$ with antisymmetric rational increments $d_{ij} = -d_{ji}$ has the counting-field-tilted generator $W(z)_{ij} = k_{ij} e^{z d_{ij}}$ ($i \neq j$), $W(z)_{ii} = -\sum_{j \neq i} k_{ij}$, and scaled cumulant generating function (SCGF) $\lambda(z) = \lim_{t \rightarrow \infty} t^{-1} \ln \langle e^{z X(t)} \rangle$, equal to the principal (largest real part) eigenvalue of $W(z)$. We use the *entropy current* $d_{ij} = L_{ij}$, for which

$$W_s(z)_{ij} = k_{ij} \left(\frac{k_{ij}}{k_{ji}} \right)^z, \quad \lambda_s(z) = \text{principal eigenvalue of } W_s(z), \quad (3)$$

and $\lambda_s(0) = 0$, $\lambda'_s(0) = \sigma$ by (2).

Convention 1 (Fano factor). Following Pietzonka et al. [8], we define

$$F \equiv \lim_{t \rightarrow \infty} \frac{\langle X(t)^2 \rangle - \langle X(t) \rangle^2}{\langle X(t) \rangle} = \frac{\lambda''(0)}{\lambda'(0)}. \quad (4)$$

Writing $J = \lambda'(0)$ and $2D = \lim_{t \rightarrow \infty} \text{Var } X(t)/t = \lambda''(0)$, this is $F = 2D/J$. Some authors instead call D/J the Fano factor; that convention halves every bound quoted here. In the convention (4) the TUR reads $F_\alpha \sigma / J_\alpha \geq 2$ and (1) reads $F_s \geq r \coth(r/2) \geq 2$. The accompanying code's symbol D for a current equals $\lambda''(0) = 2D_\alpha$, so the equilibrium bound $2N^2 D_\alpha (\cosh(z/N) - 1)$ appears there as $N^2 D (\cosh(z/N) - 1)$.

2.3 Cycles and the quantity r

Definition 2 (Simple cycle, affinity). A *simple cycle* is a directed self-avoiding closed path $c = (i_1 \rightarrow i_2 \rightarrow \dots \rightarrow i_n \rightarrow i_1)$ with distinct states and $n \geq 3$; both orientations of the same undirected cycle are admitted. Its affinity is $\mathcal{A}_c = \sum_\ell \ln(k_{i_\ell i_{\ell+1}}/k_{i_{\ell+1} i_\ell}) = \ln Q_c$ with the exact rational $Q_c = \prod_\ell k_{i_\ell i_{\ell+1}} / \prod_\ell k_{i_{\ell+1} i_\ell}$. Its length is $n_c = n$, which equals both the number of states and the number of edges traversed.

Backtracking two-cycles $i \rightarrow j \rightarrow i$ and arbitrary closed walks are *excluded*: they have $\mathcal{A} = 0$ identically and are the trivial elements of the undirected cycle space. This is not a cosmetic choice — with them admitted, $r = 0$ for every bidirectional network and (1) degenerates to the TUR (Proposition 6).

The two candidate readings of r that appear in the sources are

$$r^{\text{pos}} = \min \{ |\mathcal{A}_c|/n_c : c \text{ simple, } \mathcal{A}_c \neq 0 \}, \quad (5)$$

$$r^{\text{abs}} = \min \{ |\mathcal{A}_c|/n_c : c \text{ simple} \}. \quad (6)$$

Clearly $r^{\text{abs}} \leq r^{\text{pos}}$, with equality iff no simple cycle has zero affinity. A third, “signed/maximal” reading $\max_c |\mathcal{A}_c|/n_c$ is sometimes inferred from the phrase “smallest ratio” being applied to signed affinities; we falsify it exactly in Proposition 5.

2.4 The bounds audited

With r as above and σ from (2), the bounds audited in this paper are, in the numbering of Pietzonka et al. [8]:

$$\text{Eq. (43)} \quad \lambda_\alpha(z) \geq zJ_\alpha(1 + zJ_\alpha/\sigma) \quad (\text{parabolic, current } \alpha) \quad (7)$$

$$\text{Eq. (44)} \quad \lambda_s(z) \geq z\sigma(1 + z) \quad (\text{parabolic, entropy}) \quad (8)$$

$$\text{Eq. (45)} \quad F_\alpha\sigma/J_\alpha \geq 2 \quad (\text{TUR}) \quad (9)$$

$$\text{Eq. (55)} \quad \lambda_s(z) \geq \sigma \frac{\cosh[(z + \frac{1}{2})r] - \cosh[r/2]}{r \sinh[r/2]} \quad (\text{hyperbolic cosine}) \quad (10)$$

$$\text{Eq. (56)} \quad F_s \geq r \coth(r/2) \quad (\text{Fano consequence}) \quad (11)$$

$$\text{Eq. (57)} \quad \lambda_\alpha(z) \geq \sigma \frac{\cosh[(zJ_\alpha/\sigma + \frac{1}{2})r] - \cosh[r/2]}{r \sinh[r/2]}, \quad (\text{per current}) \quad (12)$$

$$F_\alpha \geq (J_\alpha/\sigma) r \coth(r/2) \quad (\text{its Fano consequence})$$

together with the equilibrium companion bound of Pietzonka et al. [7],

$$\lambda_\alpha(z) \geq 2N^2 D_\alpha (\cosh(z/N) - 1), \quad 2D_\alpha = \lambda''_\alpha(0), \quad J_\alpha = 0. \quad (13)$$

Equations (8) and (7) are theorems [5]; (10)–(12) and (13) are theorems [7]. We therefore do not test their validity as mathematical statements; we determine what they mean, certify them in exact arithmetic on a complete finite class, and characterise when they are tight. Cycle decompositions of the kind used in the proof are classical [1, 2, 9].

3 Methods

All computation is in the repository’s toolkit `code/pbs_exact.py`, validated by `code/verify_toolkit.py` (**295 checks, 295 pass**), and driven by the audit scripts `code/audit_readings.py`, `code/audit_multicyclic.py`, `code/audit_sharpness.py`, `code/extremal_families.py`, `code/reproduce_peer_benchmarks.py` and `paper/make_figures.py`. Environment: Python 3.12.3, NumPy 1.26.4, SciPy 1.11.4, SymPy 1.12, mpmath 1.2.1, Matplotlib 3.6.3.

3.1 Exact objects

Stationary distribution. Two algebraically independent exact routes: (a) rational Gauss–Jordan solution of $\pi Q = 0$, $\sum_i \pi_i = 1$; (b) the matrix-tree theorem, taking any nonzero row of $\text{adj}(Q)$, whose entries are exact cofactor determinants, and normalising. Both return `Fraction` vectors and are cross-checked against each other and against $\pi Q = 0$ exactly.

Entropy production as an exact logarithmic form. The class `LogRational` represents $\sum_a c_a \ln a$ with $c_a, a \in \mathbb{Q}$, $a > 0$, in canonical merged form. Its zero test is a *complete decision procedure*: writing each $a = \prod_p p^{e_p(a)}$ over primes, the value equals $\sum_p C_p \ln p$ with $C_p = \sum_a c_a e_p(a) \in \mathbb{Q}$; since $\{\ln p\}$ are linearly independent over $\mathbb{Q}$ (if $\sum_p C_p \ln p = 0$ and M is a common denominator, then $\prod_p p^{MC_p} = 1$ and unique factorisation forces every $MC_p = 0$), the value is zero iff every C_p vanishes. Sign and magnitude questions use precision-escalating evaluation (60 to 400 digits), never `float64`.

Cycle enumeration and r . All simple cycles are enumerated by depth-first search in a canonical form (smallest state first, second state smaller than last), so each undirected cycle appears once. Because Q_c is a positive rational, the minimiser of $|\ln Q_c|/n_c$ is selected by *exact integer arithmetic*: $|\ln Q_c|/n_c < |\ln Q_{c'}|/n_{c'}$ is decided by comparing $|\log Q_c|n_{c'}$ and $|\log Q_{c'}|n_c$ through integer powers $Q_c^{n_{c'}}$ versus $Q_{c'}^{n_c}$ against 1, with zero detected exactly. No logarithm is evaluated during the selection. An independent 60-digit floating-point brute-force minimiser is run alongside and must agree.

Currents and Fano factors. For rational increments, $J = \lambda'(0)$ and $2D = \lambda''(0)$ are *exact rationals*: with $V = W'(0)$, $W_2 = W''(0)$ rational matrices, reduced-resolvent perturbation theory gives $J = \pi V \mathbf{1}$ and $\lambda''(0) = 2\pi V R V \mathbf{1} + \pi W_2 \mathbf{1}$, where R is the reduced resolvent obtained by solving $(Q - 1\pi)r = J V \mathbf{1} - V \mathbf{1}$, $\pi r = 0$ in exact rational arithmetic.

3.2 Two independent routes to σ and $\lambda_s''(0)$

Every reported number is produced twice, by routes that share no algebra:

- **Route A** (exact perturbation theory): $\lambda_s''(0)$ is expanded as a polynomial in the independent edge log-ratios $L_e = \ln q_e$ with *exact rational* coefficients obtained from rational matrix arithmetic only; the single numerical step is the final evaluation of the $\ln q_e$.
- **Route B** (high-precision finite differences): five-point central differences of the principal eigenvalue of $W_s(z)$ computed at $\text{dps} + 20$ digits, with step $h = 10^{-\text{dps}/4}$.

Route disagreement at $\text{dps} = 80$ is $\sim 10^{-61}$ to 10^{-62} in all committed artifacts, i.e. at the round-off floor of the stated precision. The principal eigenvalue itself is obtained from the exact rational characteristic polynomial by Durand–Kerner with escalating budgets and Newton polish of every root, then selected by largest real part; this was hardened for near-multiple roots, which the chord family of Section 4.4 produces.

3.3 Exhaustive scanning with exact certification

Complete enumeration uses a vectorised `float64` batch scan (2×10^4 networks per batch), which is the only way to reach $\sim 10^6$ networks. Every batch is passed through *both* scanners, so the exhaustive section covers the local and the global-in- z bounds on the same complete class: `scan.batch`

evaluates σ , $\lambda_s''(0)$, the two readings of r , the Fano margins (11), the TUR (9) and the local consequence of (12) for every network and every edge current, and `scan_z_current_batch` evaluates $\lambda_s(z)$ on the seven-point grid G and $\lambda_\alpha(z)$ for every undirected edge on the ten-point symmetric grid $G \cup (-G)$ — 6.1×10^6 entropy and 4.9×10^7 per-current principal-eigenvalue problems for the four-state class, 5.5×10^6 more for the five-state sample.

Three safeguards make the scan trustworthy rather than merely fast. First, zero-affinity cycles are detected *exactly* inside the vectorised scan by comparing the integer products $\prod k_{\text{fwd}}$ and $\prod k_{\text{bwd}}$, which are exact in `float64` below 2^{53} ; the two readings (5), (6) are therefore separated exactly, not by a threshold. Second, cut-edge currents are separated *structurally*, from the topology alone and independently of the rates, by `bridge_edges`; by Proposition 9 their margins vanish identically, so they are tracked in distinct artifact fields (`*_bridged`) and excluded from the reported minima instead of being averaged in. Third, no reported extremum rests on `float64`: every network quoted as a minimiser — now including the per-current and local minima, which an earlier revision of this repository recorded only as rates and a `float64` margin — is reconstructed from its exact rational rates and recomputed at 80 digits by two algebraically independent routes, and the certificate (rates, π , exact rational J_α , D_α , F_α , σ , $\lambda_s''(0)$, r , minimising cycle, F_s , RHS, margins, and both z -scans with their route disagreement and characteristic-polynomial residual) is committed to `research/raw/`.

The two routes for a quoted $\lambda_{(z)}$ are deliberately different in kind: route A builds the characteristic polynomial by the Faddeev–LeVerrier trace recursion and solves it with `mpmath.polyroots(pbs_exact.principal_eigenvalue)`, while route B expands $\det(xI - W)$ and its exact cofactor derivative $\text{tr adj}(xI - W)$ by the Leibniz sum over permutations, Newton-refines route A’s value and reports both the applied correction and the determinant residual ($\leq 9.1 \times 10^{-98}$ in every certificate). They share no algebra, so agreement to $\leq 1.6 \times 10^{-80}$ is evidence about the eigenvalue rather than about one implementation. Round-off is reported, not suppressed: the artifact records, for each bound, the number of networks whose `float64` margin is negative (zero for all six bounds on the exhaustive four-state class, and $\sim 10^{-14}$ -scale for the identically vanishing cut-edge margins of the five-state sample).

3.4 Symbolic identities

Sharpness claims are proved, not fitted: SymPy reduces $\lambda_s(z) - \text{RHS}(10)$ to 0 as a closed-form expression in (k^+, k^-, z) for the uniform ring, and the same is done for the unicyclic bounds and for the chord-invariance lemma. Numerical evaluation at 62–80 digits is used only to confirm the symbolic result.

3.5 Clean-room reproducibility

Every committed raw artifact records the `parameters` block that generated it — grid, chunk, sample counts, seed, working precision, primary reading, and for the audit artifact both z -grids with the orientation convention they encode — and machine-dependent quantities such as wall-clock time are reported on stdout rather than stored, so that artifacts are byte-reproducible by construction. The criterion is: fresh clone, re-run the recorded commands, and `gitdiff--exit-code--research/raw` must be clean.

4 Results

4.1 The operative convention for r

Theorem 3 (Convention). *The quantity entering (10)–(12) as proved by Pietzonka et al. [7] is r^{pos} of (5): the minimum of $|\mathcal{A}_c|/n_c$ over simple cycles ($n \geq 3$) with $\mathcal{A}_c \neq 0$.*

Justification from the primary sources. The proof states that $\mathcal{A}^*/n^*$ “is defined as the smallest positive value for the affinity per cycle length over all cycles in the network”, and, after noting that a priori it refers to the minimal $\mathcal{A}_\beta/n_\beta$ over a set of uniform cycles, that “[s]ince uniform cycles always have affinity $\mathcal{A}_\beta > 0$, the minimization can be restricted to cycles with positive affinity. Thus, in the physically important case of a network containing cycles with zero affinity, [the bound] provides a bound that is stronger than the parabolic bound that is obtained for $\mathcal{A}^*/n^* \rightarrow 0$ ”; the conclusion requires “knowledge of the minimal positive affinity per number of edges in any cycle of the network” [7]. Restricting to positive affinities is therefore constitutive of the improvement over Gingrich et al. [5], not an incidental simplification. $\square$

Proposition 4 (The absolute reading is strictly weaker). $r^{\text{abs}} \leq r^{\text{pos}}$ always, with equality iff the network has no zero-affinity simple cycle. Any network containing one equilibrium cycle has $r^{\text{abs}} = 0$, and (10) then reduces exactly to the parabolic bound (8) and (11) to $F_s \geq 2$. On the exhaustive class of Section 4.2 the two readings differ for 417 294 of 877 626 networks, i.e. **47.5%**.

Proposition 5 (The signed/maximal reading is false). Let $r^{\text{sgn}} = \max_c |\mathcal{A}_c|/n_c$ be the reading that takes the largest affinity per length. In the exact falsification search — the exhaustive rate grid 1.3 over all seven cycle-rank- ≥ 2 four-state topologies plus 60 000 seeded networks with rates in 1..200 (seed 20260908), 945 732 networks scanned in total — **1 741** have F_s strictly below $r^{\text{sgn}} \coth(r^{\text{sgn}}/2)$. The most negative margin, -0.25674967769087775 , is attained at the network with rate pairs (k_{ij}, k_{ji})

$$(0, 1) = (4, 11), \quad (0, 2) = (91, 1), \quad (1, 2) = (10, 192), \quad (1, 3) = (193, 105), \quad (2, 3) = (64, 136), \quad (14)$$

whose exact certificate is

$$\pi = \left(\frac{614221}{13432572}, \frac{1258801}{3358143}, \frac{987917}{4477524}, \frac{1204849}{3358143} \right), \quad (15)$$

$\sigma = 183.105553124510540052286249685$ and $\lambda_s''(0) = 535.595076742780504657273154722$, and for which

$$F_s = 2.92506189792386836056472252112 < r^{\text{sgn}} \coth(r^{\text{sgn}}/2) = 3.18181157561465505229229183304, \quad (16)$$

where $r^{\text{sgn}} = 2.82579023240968861008898867701$. The same network satisfies (11) under the correct reading, with $r^{\text{pos}} = 1.03998969426789690840872688169$, $\text{RHS} = 2.17709509335470454949222397433$ and margin $+0.747966804569$. All digits are exact-arithmetic outputs committed in **research/raw/readings.json**.

Proposition 6 (Closed walks and two-cycles trivialise the bound). *If backtracking two-cycles $i \rightarrow j \rightarrow i$ or arbitrary closed walks are admitted as “cycles”, then $\mathcal{A} = 0$ for every bidirectional edge and hence $r = 0$ for every network in the class, so (10) collapses to (8) universally. The claim “every backtracking walk has affinity exactly 0” is certified exactly for all five edges of the audit network by the complete LogRational zero test of Section 3 (research/raw/readings.json, field closed_walk_reading_R4, all five exact: true).*

Remark 7 (Cycle-space readings are not viable). Reading r over the whole cycle space rather than over simple cycles is ill-defined beyond $N = 4$: for $N \leq 4$ every nonzero cycle-space element of a connected graph has simple support, so the two readings coincide (verified exactly for $N = 3$ and $N = 4$), whereas for $N \geq 5$ elements with non-simple support exist (the figure-eight), and their “length” is not a well-defined number of states. The sources’ own worked example fixes the simple-cycle reading: for fundamental affinities $(-11, 12, 13)$ the minimiser is the composite simple cycle $(1, 4, 3, 2, 1)$ with $\mathcal{A}^* = 1$, $n^* = 4$ [8].

4.2 Exhaustive certified audit of the four-state multicyclic class

Theorem 8 (Certified audit). *Write $G = \{-0.75, -0.5, -0.25, 0.25, 0.5, 1, 2\}$ for the audit z -grid. Every nonequilibrium four-state multicyclic network with integer rates in 1..3 satisfies, under both readings (5) and (6):*

- the entropy bounds (8) and (10) at every $z \in G$;
- the per-current bounds (7) and (12) — the full z -dependent bounds, not only their $z \rightarrow 0$ consequences — at every $z \in G \cup (-G)$, for both orientations of every undirected edge;
- the TUR (9) and the Fano bound (11), hence also the local part of (12).

The class is the complete set of connected labelled four-state topologies of cycle rank ≥ 2 — K_4 and the six K_4 -minus-one-edge graphs — giving 877 626 networks with $\sigma > 0$ (8 109 detailed-balanced assignments are skipped as $\sigma = 0$ exactly) and 4917 816 audited edge currents. Every margin is strictly positive. None of the seven topologies has a cut edge, so by Proposition 9 no audited current is degenerate and the reported minima are not pinned at zero.

Both orientations are covered by a single scan because reversing a current leaves the tilted matrix invariant under $z \mapsto -z$: with increments $d_{ij} = +1$, $d_{ji} = -1$ the reversed current has $d \mapsto -d$, hence $\lambda_\alpha^{\text{rev}}(z) = \lambda_\alpha(-z)$ and $J_\alpha^{\text{rev}} = -J_\alpha$, while both right-hand sides (7) and (12) are invariant under $(z, J_\alpha) \mapsto (-z, -J_\alpha)$. Auditing the symmetric grid $G \cup (-G)$ in the $i < j$ orientation therefore certifies both orientations on G , and leaves no sign convention in the statement.

Proposition 9 (Cut-edge currents saturate the per-current bounds identically). *Let (i, j) be an edge whose deletion disconnects the network graph, and let X_t be its integrated current. Then $\lambda_\alpha(z) = 0$ for every real z , $J_\alpha = D_\alpha = 0$ exactly, and (7) and (12) hold with equality at every z under every reading of r ; the Fano factor F_α itself is undefined for such a current.*

Proof. Deleting the edge separates the states into $A \ni i$ and $B \ni j$, and the trajectory can change side only through this edge, so $X_t = \mathbf{1}\{X_t \in B\} - \mathbf{1}\{X_0 \in B\} \in \{-1, 0, 1\}$ is bounded. A bounded additive functional has $\lambda_\alpha(z) = \lim_{t \rightarrow \infty} t^{-1} \ln \mathbb{E} e^{zX_t} = 0$ for every real z , hence $J_\alpha = \lambda'_\alpha(0) = 0$ and $2D_\alpha = \lambda''_\alpha(0) = 0$. Both right-hand sides vanish at $J_\alpha = 0$: (7) is $zJ_\alpha(1 + zJ_\alpha/\sigma) = 0$, and (12) is $\sigma[\cosh(r/2) - \cosh(r/2)]/(r \sinh(r/2)) = 0$. Both bounds therefore hold with equality. $\square$

This degenerate branch is why the per-current minima must be reported over *non*-cut-edge currents: including them would make the minimum margin exactly 0 for trivial reasons and hide the informative extremum. The four-state class of Theorem 8 contains no cut edge — K_4 has six edges and K_4 minus one edge has five, and deleting any one of them leaves a connected graph on four states — which the audit confirms exactly (`n_bridge_currents` = 0 of 4917 816). The five-state seeded sample does contain them: 19 991 of its 379 910 audited currents are cut-edge currents, and the audit certifies one at 80 digits with $J_\alpha = 0$, $D_\alpha = 0$ exactly and $|\lambda_\alpha(z)| \leq 6.4 \times 10^{-82}$ at all ten grid points, i.e. $\lambda_\alpha \equiv 0$ at working precision, as Proposition 9 predicts. Its `float64`

margins scatter symmetrically about zero (-1.99×10^{-14} at worst, 98 516 negative values) exactly as a round-off floor around an identically vanishing margin must; those values are reported in `research/raw/audit_multicyclic_grid3.json` rather than hidden.

The certified minima, at 80 digits by both routes (route disagreement 2.177×10^{-61} for σ and $\lambda_s''(0)$, and $\leq 1.6 \times 10^{-80}$ for every quoted $\lambda_s(z)$ and $\lambda_\alpha(z)$, whose second route is an independent Leibniz-expansion characteristic polynomial refined by Newton; see Section 3.3), are collected in Table 1; the worst-case networks are given explicitly there and in `research/raw/audit_multicyclic_grid3.json`. The exhaustive per-current minimum is attained at $z = 0.25$ on edge (1, 2) of a network whose exact current is $J_\alpha = -7/163 < 0$ — a case the “positive current only” convention would have skipped entirely — with exactly rational $\lambda_\alpha''(0) = 2D_\alpha = 1294363/4330747$ and $F_\alpha = -184909/26569$, and certified margins 1.717228×10^{-7} for (7) and 4.238460×10^{-8} for (12) under the positive reading.

Remark 10 (The global-in- z minimisers are a chord-deletion pair). The exhaustive minimisers of (8) and of (10) under the absolute reading are one and the same network A : the complete graph K_4 with rates (2, 1), (1, 1), (3, 2), (1, 2), (2, 3), (3, 2) on edges (0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3), $\pi^A = (11, 23, 11, 16)/61$, $r^{\text{abs}} = 0$ and $r^{\text{pos}} = \ln(9/8)/4$. The minimiser of (10) under the positive reading is a *different* network B : K_4 minus edge (1, 3) with rates (1, 2), (2, 3), (1, 2), (3, 2), (2, 3), $\pi^B = (23, 11, 16, 11)/61$ and $r^{\text{pos}} = \ln(9/8)/3$. They are the same object up to the mechanism of Remark 11: edge (0, 2) of A is symmetric ($k_{02} = k_{20} = 1$) and carries exactly zero current ($\pi_0^A = \pi_2^A = 11/61$), and deleting it and relabelling the states by (0 1)(2 3) reproduces the rate matrix of B *entry by entry*, exactly (verified in `Fraction` arithmetic). Consequently the two share the whole fluctuation spectrum — σ , $\lambda_s''(0)$, F_s and $\lambda_s(z)$ at all seven grid points, the last with difference exactly 0 at 80 working digits and $\leq 6.3 \times 10^{-112}$ at 100 — while their exact rational log-forms for σ differ, so this is an identity of values, not of representations. The reading is that the positive-reading minimum of (10) is selected by r alone: the two networks are fluctuation-identical, and B wins only because its larger $r = \ln(9/8)/3$ raises the bound. It also confirms Remark 11 on the whole z -grid rather than at four sample points, and shows that a chord deletion can move r by the factor $4/3$ without moving a single cumulant.

Seeded samples (seed 20260908) of 199 980 four-state networks with rates 1..12 and 59 985 five-state networks with rates 1..8 over the same topologies are audited with exactly the same checks and likewise violate nothing: the entropy minima are $1.7992482503892404 \times 10^{-12}$ and $1.9298271314305282 \times 10^{-11}$, both at the `float64` resolution floor of the scan, and the per-current minima are $1.696759740 \times 10^{-10}$ and $5.689236888 \times 10^{-10}$ ($1.123468425 \times 10^{-10}$ and $5.086515461 \times 10^{-10}$ under the positive reading), with every extremal network re-certified exactly at 80 digits.

Remark 11 (Why the two worst cases share F_s exactly). The positive-reading minimiser is the absolute-reading minimiser (relabelled $2 \leftrightarrow 3$) with its *symmetric* edge deleted — edge (0, 3) with $k_{03} = k_{30} = 2$ in the absolute labelling, equivalently edge (0, 2) in the positive labelling — and that edge carries exactly zero stationary current: $\pi_0 = \pi_3 = 11/61$ in the absolute labelling (equivalently $\pi_0 = \pi_2 = 11/61$ in the positive one), so its current $j_{03} = k_{03}\pi_0 - k_{30}\pi_3 = 2(\pi_0 - \pi_3) = 0$. Deleting an untilted ($k_{ij} = k_{ji}$, hence $L_{ij} = 0$) zero-current edge leaves $\lambda_s(z)$ exactly unchanged — verified to 80 digits at $z \in \{0.5, 1, -0.75, 2\}$ (difference exactly 0 at working precision) — while strictly changing the cycle set and hence r . The mechanism is the rank-one update identity $\det(xI - W + c uu^T) = \det(xI - W) + c u^T \text{adj}(xI - W) u$ with $u = e_i - e_j$: at a simple eigenvalue whose right and left eigenvectors satisfy $v_i = v_j$ or $w_i = w_j$, the perturbation does not move that eigenvalue. This is the same phenomenon as the chord-invariance lemma of Section 4.4, seen in the exhaustive audit.

Table 1: Certified results of the exhaustive four-state audit (rate grid 1..3, seven topologies, 877 626 nonequilibrium networks, 4917 816 edge currents). “Abs” is the weaker absolute reading (6), “Pos” the theorem’s positive reading (5). The right-hand sides of (7), (8) and (9) do not involve r , so those rows carry a single entry. G is the audit z -grid of Theorem 8, $G^\pm = G \cup (-G)$; all 4917 816 edge currents are non-cut (Prop. 9). Source: `research/raw/audit_multicyclic_grid3.json`, section `exhaustive_N4_grid3`.

quantity		Abs	Pos
networks audited		877 626 (+8 109 equilibrium skipped)	
topologies		7 (K_4 : 529 686; six K_4 -edge: 57 990 each)	
readings differ		417 294 networks (47.5%)	
edge currents audited		4917 816 (cut-edge: 0)	
min margin (11)		$2.857759294 \times 10^{-4}$	$2.887802027 \times 10^{-5}$
min local margin (12)		$2.426291085 \times 10^{-3}$	$2.451797914 \times 10^{-4}$
min TUR margin (9)		$2.857759293 \times 10^{-4}$	
min margin (8), G		$1.939906373 \times 10^{-8}$	
min margin (10), G		$1.939906373 \times 10^{-8}$	$1.960457100 \times 10^{-9}$
min margin (7), $G^\pm$		$1.717227993 \times 10^{-7}$	
min margin (12), $G^\pm$		$1.717227993 \times 10^{-7}$	$4.238459757 \times 10^{-8}$
worst network (Abs)	topology	K_4 , rates (2, 1), (3, 2), (2, 2), (2, 3), (1, 2), (2, 3)	
	π	(11, 23, 16, 11)/61	
	σ	0.00386173887397978539471456096625	
	$\lambda_s''(0)$	0.00772458133997594522391745624973	
	r	0	
	F_s	$\ln(9/8)/4 = 0.0294457589 \dots$	
worst network (Pos)	topology	K_4 minus edge (0, 2), rates (2, 1), (3, 2), (1, 2), (2, 3), (3, 2)	
	π	(11, 23, 11, 16)/61	
	r	$\ln(9/8)/3 = 0.0392610118854611515129313698235$	
	RHS(11)	2.00025689790929459151459275133	
	F_s	2.00028577592954572498522337	

4.3 Sharpness and the equality set

Theorem 12 (Uniform unicyclic rings saturate the bound identically). *Let $N \geq 3$ and let the ring have uniform forward rate k^+ and backward rate k^- , $\rho = k^+/k^- \neq 1$. Then $r = |\ln \rho|$, the entropy SCGF is*

$$\lambda_s(z) = k^+ \rho^z + k^- \rho^{-z} - k^+ - k^-, \quad (17)$$

and (10) holds with equality for every real z as a closed-form identity in (k^+, k^-, z) :

$$\sigma \frac{\cosh[(z + \frac{1}{2}) \ln \rho] - \cosh[\ln \rho/2]}{\ln \rho \sinh[\ln \rho/2]} = k^- (\rho^{z+1} + \rho^{-z} - \rho - 1) = \lambda_s(z). \quad (18)$$

Consequently (11) also holds with equality, $F_s = \frac{k^+ + k^-}{k^+ - k^-} \ln \rho = r \coth(r/2)$, and no lower bound of this functional form can be raised: the bound is sharp as a function of (σ, r) .

Both sides of (18) are even in $\ln \rho$ ($\cosh$ is even and $\ln \rho \sinh[\ln \rho/2]$ is odd $\times$ odd) and $\sigma = (k^+ - k^-) \ln \rho > 0$ for either orientation of the driving, so the identity and the equality $F_s =$

$r \coth(r/2)$ are invariant under $\rho \mapsto 1/\rho$: the equality set is symmetric about equilibrium, and F_s takes the same value at ρ and $1/\rho$. The gated suite now checks this explicitly on the $\rho < 1$ side, which the statement $r = |\ln \rho|$ covers and $r = \ln \rho$ would not: the nine rings $N \in \{3, 4, 5\}$ with $(k^+, k^-) \in \{(1, 3), (2, 7), (1, 5)\}$ saturate (10) on a six-point z -grid to $\leq 1.20 \times 10^{-58}$ and (11) to $\leq 2.29 \times 10^{-61}$, with $F_s = 2.19722457733621938, 2.25497334329166239, 2.41415686865115056$ exactly as at $\rho = 3, \frac{7}{2}, 5$ (`verify_toolkit.py::check_rstar_absolute_value_convention`).

SymPy reduces $\lambda_s(z) - \text{RHS}(10)$ to 0: in `research/raw/sharpness.json` the field `symbolic_saturation_identity` records the reduced closed form and `exact_point_residual` is exactly 0. Numerically, the maximum absolute margin over an 11-point z -grid is $\leq 3.65 \times 10^{-62}$, i.e. round-off at 80 digits, across

$$N \in \{3, 4, 5, 6, 8\}, \quad \rho \in \{2, \frac{5}{2}, \frac{7}{3}, 3, \frac{11}{4}, 1000, \frac{3}{2}\}. \quad (19)$$

The same computation identifies the unicyclic lower bound of Pietzonka et al. [8] [their Eq. (50)] with the lower edge of their Fano window [Eq. (52)] as one and the same identity.

Proposition 13 (Equality set on exhaustive grids). *On the integer rate grid 1..3 in both directions of every edge:*

- unicyclic: *scanning all 729 ($N = 3$), 6 561 ($N = 4$) and 59 049 ($N = 5$) ring assignments and certifying every candidate with $|\text{margin}| < 10^{-9}$ at 80 digits yields exactly 18 near-saturators, 6 per N : precisely the uniform rings $(k^+, k^-) \in \{1, 2, 3\}^2$ with $k^+ \neq k^-$. All 18 are uniform rings in some orientation (`all_candidates_are_uniform_rings: true`). No non-uniform unicyclic ring saturates, including rings with equal edge affinities but non-uniform rates.*
- multicyclic: *scanning all 885 735 four-state assignments (including the detailed-balanced ones) over all seven rank- ≥ 2 topologies yields **zero** near-saturators under either reading; the certified minima are those of Table 1.*

Combined with Lemma 16, the equality set of (10)–(11) within these classes is exactly the set of uniform unicyclic rings, up to relabelling and orientation.

Remark 14 (Honest status). The saturation direction of Proposition 13 is a theorem (Theorem 12, symbolic). The converse — that *only* uniform rings saturate — is an exhaustively verified characterisation on grids 1..3 with $N \leq 5$ unicyclic and $N = 4$ multicyclic, *not* a proof for all real rates. It is listed as an open question in Section 7.

Proposition 15 (The equilibrium companion bound is saturated by the same family). *For a uniform equilibrium ring ($k^+ = k^- = k$, N states) the single-edge current has $J = 0$ and $\lambda''_\alpha(0) = 2D_\alpha = 2k/N^2$ exactly in rational arithmetic, and*

$$\lambda_\alpha(z) = 2k(\cosh(z/N) - 1) = N^2 \lambda''_\alpha(0)(\cosh(z/N) - 1) = 2N^2 D_\alpha(\cosh(z/N) - 1) \quad (20)$$

for every z , i.e. (13) holds with exact equality (verified to 62–63 digits over $z \in [-2, 2]$ for $(N, k) = (3, 2), (4, 5), (5, 3), (6, 1)$; the tilted generator is circulant and the principal mode is the Bloch wave $m = 0$). The bound is strict for a non-uniform equilibrium ring ($N = 4$, symmetric rates 2, 5, 3, 7: $\lambda''_\alpha(0) = 105/247$, minimum margin 3.4116464×10^{-6} over the z -grid) and for a multicyclic equilibrium network (K_4 , symmetric rates 1..6, fundamental three-cycle current: $\lambda''_\alpha(0) = 819/278$, minimum margin 5.9584606×10^{-4}).

Thus one continuous family — uniform rings — saturates the nonequilibrium affinity bound at $\rho \neq 1$ and its equilibrium companion at $\rho = 1$, giving a single equality set across the equilibrium/near-equilibrium boundary.

4.4 Chord invariance and the topological discontinuity of the equality set

Lemma 16 (Chord invariance). *Let $C(N, \rho, \varepsilon)$ be the uniform ring of N states (forward rate ρ , backward rate 1) with one additional symmetric chord $(0, 2)$ carrying rate $\varepsilon > 0$ in both directions. Then for every $\varepsilon > 0$ and every real z ,*

$$\lambda_s^C(z) = \lambda_s^{\text{ring}}(z) = \rho^{z+1} + \rho^{-z} - \rho - 1, \quad (21)$$

Proof. The chord is untilted ($L = \ln(\varepsilon/\varepsilon) = 0$), so $W^C(z) = W^{\text{ring}}(z) - \varepsilon(e_0 - e_2)(e_0 - e_2)^\top$. The uniform ring has constant row sums, $W^{\text{ring}}(z)\mathbf{1} = \lambda_s^{\text{ring}}(z)\mathbf{1}$, and $(e_0 - e_2)^\top \mathbf{1} = 0$; hence $W^C(z)\mathbf{1} = \lambda_s^{\text{ring}}(z)\mathbf{1}$. For $\varepsilon > 0$, $W^C(z)$ is an irreducible Metzler matrix, whose eigenvalue admitting a positive eigenvector is unique and equals the principal eigenvalue (Perron–Frobenius). Therefore $\lambda_s^C(z) = \lambda_s^{\text{ring}}(z)$. $\square$

The same argument applies to any finite set of symmetric chords, and more generally to any deformation whose tilted generator keeps equal row sums at every z . Numerically, for $N = 4, 5$, $\rho = 2, 5$ and $\varepsilon \in \{1, \frac{1}{2}, \frac{1}{10}, \frac{1}{100}, \frac{1}{1000}\}$, the whole z -scan of λ_s , together with σ , $\lambda_s''(0)$ and F_s , is identical to 20 digits across three decades of ε (**research/raw/extremal_families.json**: one distinct value per group). Because the uniform-ring entropy SCGF (17) is independent of N , these quantities are also independent of N : $C(4, 5, \varepsilon)$ and $C(5, 5, \varepsilon)$ agree exactly, $F_s = 2.4141568686511505619$.

Corollary 17 (The equality set is topologically discontinuous). *Along $C(N, \rho, \varepsilon)$, $r = |\ln \rho|$ at $\varepsilon = 0$ (the ring, $n_c = N$) but $r = 2|\ln \rho|/3$ for every $\varepsilon > 0$ (the triangle $(0, 1, 2)$ created by the chord, whose affinity is $2\ln \rho$ because the chord contributes zero). Since $\lambda_s(z)$ does not move, the margin of (11) jumps discontinuously at $\varepsilon = 0$ from 0 to the constant*

$$|\ln \rho| \coth \frac{|\ln \rho|}{2} - \frac{2|\ln \rho|}{3} \coth \frac{|\ln \rho|}{3} > 0, \quad (22)$$

independent of ε (e.g. $\rho = 5$: $F_s = 2.4141568686\dots$, margin 0.2258669544 for every ε). An arbitrarily weak zero-affinity chord therefore strengthens the bound without changing any fluctuation, and the multicyclic class cannot approach saturation by weakening a chord.

Corollary 17 is the mechanism behind the worst-case structure of the exhaustive audit (Remark 11): symmetric zero-current edges are invisible to $\lambda_s(z)$ but visible to the cycle enumeration.

4.5 An exact minimiser law for the uniform-field family

Consider the uniform-field family $U(N, \rho)$: the complete graph K_N with $k_{ij} = \rho$, $k_{ji} = 1$ for all $i < j$, $\rho > 0$, $\rho \neq 1$. Every cycle affinity is an integer multiple of $\ln \rho$, but the minimiser is not obvious, because length and net affinity trade off.

Theorem 18 (Uniform-field minimiser law). *Let $L_{\text{odd}}(N)$ denote the largest odd integer $\leq N$ (so $L_{\text{odd}} = 3, 3, 5, 5, 7, 7, 9, \dots$ for $N = 3, 4, 5, 6, 7, 8, 9, \dots$). Then under the convention of Theorem 3,*

$$r(U(N, \rho)) = \frac{|\ln \rho|}{L_{\text{odd}}(N)}, \quad (23)$$

attained by the zig-zag cycle $c^* = (0, 1, \dots, m, 2m, 2m-1, \dots, m+1)$ with $L_{\text{odd}}(N) = 2m+1$.

Proof. A simple cycle c of length ℓ with u ascending steps ($i \rightarrow j, i < j$) and $\ell - u$ descending steps has affinity $\mathcal{A}_c = (u - (\ell - u)) \ln \rho = p \ln \rho$ with $p = 2u - \ell$, so $|\mathcal{A}_c|/n_c = |p| |\ln \rho|/\ell$ and $p \equiv \ell \pmod{2}$.

Lower bound. Take any simple cycle with $p \neq 0$. If ℓ is odd then $|p| \geq 1$ and $\ell \leq L_{\text{odd}}(N)$ (since $L_{\text{odd}}(N)$ is the largest odd integer $\leq N$ and $\ell \leq N$), so $|p|/\ell \geq 1/L_{\text{odd}}(N)$. If ℓ is even then p is even and nonzero, so $|p| \geq 2$; and $\ell \leq 2L_{\text{odd}}(N)$, because for N odd $L_{\text{odd}}(N) = N$ and $\ell \leq N - 1 < 2N = 2L_{\text{odd}}(N)$, while for N even $L_{\text{odd}}(N) = N - 1$ and $\ell \leq N \leq 2N - 2 = 2L_{\text{odd}}(N)$ as $N \geq 2$. Hence $|p|/\ell \geq 2/(2L_{\text{odd}}(N)) = 1/L_{\text{odd}}(N)$.

Attainment. For $\ell = L_{\text{odd}}(N) = 2m + 1$ the zig-zag cycle visits $0, 1, \dots, m, 2m, 2m - 1, \dots, m + 1$ and closes $m + 1 \rightarrow 0$. Its steps $0 \rightarrow 1, \dots, (m - 1) \rightarrow m, m \rightarrow 2m$ are ascending ($m + 1$ of them) and $2m \rightarrow 2m - 1, \dots, m + 2 \rightarrow m + 1, m + 1 \rightarrow 0$ are descending (m of them), so $u = m + 1$ and $p = 2(m + 1) - (2m + 1) = 1$, giving $|\mathcal{A}_c^*|/n_{c^*} = |\ln \rho|/L_{\text{odd}}(N)$. $\square$

The parity step is the whole content: an odd cycle can carry net affinity one power of ρ , an even cycle carries at least two, so the longest *odd* cycle available in K_N wins.

Verification. The gated check `verify_toolkit.py::check_uniform_field_law` confirms the law three ways. (i) *Exhaustive exact integer check*: for every $N = 3, \dots, 8$ and every one of the $\sum_{\ell} \binom{N}{\ell} (\ell - 1)!/2$ simple cycles of K_N (8018 cycles at $N = 8$), $p = 2u - \ell$ is computed by exact integer counting and the check verifies $p \equiv \ell \pmod{2}$, $|p| L_{\text{odd}}(N) \geq \ell$ for every $p \neq 0$, and $\min |p|/\ell = 1/L_{\text{odd}}(N)$ as an exact `Fraction`; no floating point is used. (ii) *Constructive attainment*: the zig-zag cycle is built explicitly and checked to be simple on $\{0, \dots, L_{\text{odd}}(N) - 1\}$ with $|p| = 1$. (iii) *Toolkit agreement*: for $N = 3, \dots, 8$ and $\rho \in \{\frac{11}{10}, 2, 100, \frac{3}{7}\}$ the exact integer-power minimiser returns $n^* = L_{\text{odd}}(N)$ and a minimising cycle of that length, agreeing with $|\ln \rho|/L_{\text{odd}}(N)$ to 62–63 digits and with the independent 60-digit brute-force minimiser; $\rho = \frac{3}{7} < 1$ exercises the absolute-value branch.

Corollary 19 (The affinity refinement degenerates with N at fixed driving). *Since $r = |\ln \rho|/L_{\text{odd}}(N) \rightarrow 0$ as $N \rightarrow \infty$ at fixed ρ ,*

$$\text{RHS(11)} = r \coth \frac{r}{2} = 2 + \frac{r^2}{6} - \frac{r^4}{360} + O(r^6) = 2 + \frac{\ln^2 \rho}{6 L_{\text{odd}}(N)^2} + O(L_{\text{odd}}(N)^{-4}) \rightarrow 2, \quad (24)$$

while F_s itself grows with N ($F_s = 2.0170582954, 2.0285635956, 2.0347107495$ at $\rho = 2, N = 3, 4, 5$). Along this family the bound therefore weakens towards the parabolic/TUR value like $L_{\text{odd}}(N)^{-2}$ even though the network stays strongly driven: the controlling r is set by the longest available odd cycle carrying one power of the rate ratio, not by the driving strength.

More generally the family is asymptotically tight near equilibrium and progressively loose at strong driving: the relative margin of (11) is $1.58 \times 10^{-6}, 2.11 \times 10^{-5}, 4.07 \times 10^{-3}, 0.239$ and 0.948 at $\rho = \frac{11}{10}, \frac{6}{5}, 2, 10, 100$ for $N = 3$, and the minimum relative margin of (10) over the nine-point z -grid $\{-0.9, \dots, 4\}$ is $1.35 \times 10^{-11}, 6.88 \times 10^{-10}, 2.30 \times 10^{-6}, 1.67 \times 10^{-3}$ and 1.91×10^{-2} respectively. No member comes near saturating (11) at finite driving.

Remark 20 (A corrected claim). An earlier version of this work asserted that the minimiser of $U(N, \rho)$ is always a triangle with $r = |\ln \rho|/3$. That is true only for $N = 3, 4$ and is contradicted by the repository's own committed raw output, in which every $U(5, \rho)$ entry records the minimising cycle $[0, 1, 2, 4, 3]$, $n^* = 5$ and $r = |\ln \rho|/5$. The error was caught in the pre-submission critical review by reading the raw artifact against the note that summarised it; all tabulated numbers were computed by the exact minimiser and were unaffected. Theorem 18 replaces the claim.

Remark 21 (Three mis-stated closed forms found in Journal review). Journal review of this Solution found three errors of exactly the kind the pre-submission review should have caught. All three are mis-stated *closed forms*; in all three cases every tabulated number had been computed correctly and was unaffected, which is why numeric traceability did not reach them. *Round 1*, reviewing revision **a624767**: the ring entropy SCGF specialised to $k^+ = \rho$, $k^- = 1$ was printed as $\rho^z + \rho^{-z} - \rho - 1$ instead of $\rho^{z+1} + \rho^{-z} - \rho - 1$; the wrong form does not vanish at $z = 0$, gives $\lambda'_s(0) = 0$ hence $\sigma = 0$, and yields -0.5 instead of 1.5 at $\rho = 2$, $z = 1$. *Round 2*, reviewing revision **cdef285**: the leading coefficient in the degeneracy expansion of $r \coth(r/2)$ was printed as $1/12$ instead of the exact $1/6$, and the minimiser laws of Theorems 12 and 18 were printed without the absolute value, which makes them false for $\rho < 1$ under the convention $r = \min |\mathcal{A}_c|/n_c \geq 0$ established in (5)–(6). All three are corrected wherever they occur, and each now carries a dedicated gate that asserts the *exclusion* of the mis-stated form, not merely the presence of the corrected one (`check_ring_specialisation_kminus_one`, `check_fano_degeneracy_series`, `check_rstar_absolute_value_convention`), so a recurrence fails the suite rather than reaching a reader. The root cause is common to all three and is recorded as a standing requirement of this repository: a closed form that appears only in prose must be gated symbolically, because re-deriving every quoted *number* from the artifacts does not test it. Full record: `research/09-journal-revision-record.md`.

5 Validation and independent challenge checks

Each central claim is supported by a primary method and a materially different challenge check of the same claim and scope.

1. **Convention (Theorem 3, Propositions 4–6)**. Primary: direct reading of the arXiv source `TeX` of both papers. Challenge: exact *falsification* of the competing readings — a certified four-state network with F_s below the signed-reading bound (Proposition 5), and exact certification that all five backtracking affinities of the audit network vanish (Proposition 6). The distinction is also made *operationally*: 47.5% of the exhaustive class separates the two surviving readings, so the choice is not vacuous.
2. **Exhaustive audit (Theorem 8)**. Primary: vectorised `float64` enumeration. Challenges: (a) exact integer detection of zero-affinity cycles inside the scan, cross-checked against the exact rational minimiser on a network where the readings differ (19-digit agreement, three zero-affinity cycles detected exactly); (b) 80-digit re-certification of every reported extremum by two algebraically independent routes (route disagreement 2.177×10^{-61}); (c) per-topology minima reported separately, so a single topology cannot hide a violation; (d) seeded samples at rates up to 12 ($N = 4$) and 8 ($N = 5$) beyond the exhaustive grid, audited with the same checks and whose extremal networks are also re-certified exactly; (e) *bit-for-bit regression of the extended scanner*: adding the global-in- z and per-current scans left every previously certified local quantity unchanged ($2.857759294183637 \times 10^{-4}$, $2.8878020267519844 \times 10^{-5}$, $2.426291085 \times 10^{-3}$, $2.451797914 \times 10^{-4}$, 417 294, and the seeded entropy minima $1.7992482503892404 \times 10^{-12}$, $1.9298271314305282 \times 10^{-11}$), so the new code path did not perturb the old one; (f) a second, algebraically independent route for every quoted $\lambda_{(z)}$ — Leibniz expansion of $\det(xI - W)$ with its exact cofactor derivative, Newton refinement and determinant residual $\leq 9.1 \times 10^{-98}$ — agreeing with the Faddeev–LeVerrier/`polyroots` route to $\leq 1.6 \times 10^{-80}$; (g) Gallavotti–Cohen consistency inside the certificates: the entropy minimiser attains its minimum at $z = -0.75$ and at $z = -0.25$ with *identical* certified λ_s and margin, as $\lambda_s(z) = \lambda_s(-1 - z)$ requires; (h) structural separation of cut-edge currents

(Proposition 9), checked against the topology alone and confirmed by an exact certificate with $J_\alpha = D_\alpha = 0$ and $|\lambda_\alpha(z)| \leq 6.4 \times 10^{-82}$ over the whole grid; and (i) the chord-deletion identity of Remark 10, verified entry by entry in exact rational arithmetic.

3. **Sharpness (Theorem 12, Propositions 13–15).** Primary: SymPy closed-form reduction to 0. Challenges: 80-digit numerical evaluation over five ring sizes and seven drivings (maximum residual 3.65×10^{-62}); an exhaustive near-saturation search with an 80-digit certification stage, which returns exactly the predicted 18 uniform rings and no others; and for the equilibrium companion bound, exact rational diffusion coefficients ($\lambda''_\alpha(0) = 2k/N^2$, $105/247$, $819/278$) with strict inequality verified on two non-uniform controls.
4. **Chord invariance (Lemma 16).** Primary: a Perron–Frobenius proof. Challenge: 20-digit numerical invariance of the whole z -scan, σ , $\lambda''_s(0)$ and F_s across three decades of ε and two values of N , plus the rank-one determinant identity of Remark 11 verified to 80 digits on an exhaustive-audit worst case.
5. **Uniform-field law (Theorem 18).** Primary: the parity proof. Challenges: exhaustive exact-integer verification over all simple cycles of K_N for $N = 3, \dots, 8$ (8018 cycles at $N = 8$); constructive attainment of the zig-zag cycle; and agreement of the exact minimiser with an independent 60-digit brute-force minimiser at 62–63 digits for 24 members. The *sign* of the law is gated separately: for 18 members with $\rho = \frac{3}{7}, \frac{1}{2}, \frac{1}{100} < 1$ the exact minimiser is positive, agrees with $|\ln \rho|/L_{\text{odd}}(N)$ to ≥ 62 digits, and is separated from the signed $\ln \rho/L_{\text{odd}}(N)$ by ≤ 2 agreeing digits, so the absolute value in (23) is forced by the code and not merely by convention.
6. **Degeneracy law (Corollary 19).** Primary: the Taylor expansion of $r \coth(r/2)$, obtained *symbolically* by SymPy as $2 + r^2/6 - r^4/360 + r^6/15120 + O(r^8)$, so the coefficient $1/6$ is exact rather than fitted. Challenge: for all 24 gated $U(N, \rho)$ members the toolkit’s own exact minimiser satisfies $0 < \frac{1}{6} - (\text{RHS}(11) - 2)/r^2 \leq r^2/360$, i.e. the extracted coefficient approaches $1/6$ from below with the correct next-order defect, while its distance from the mis-stated $1/12$ stays above $\frac{1}{12} - r^2/360 \geq 0.0768$ — the two candidate laws are separated by a factor of at least 12.4 at every tested member, the worst case being $U(3, 100)$ where $r = 1.535$ is largest (`verify_toolkit.py::check_fano_degeneracy_series`).
7. **Toolkit itself.** The gated suite `code/verify_toolkit.py` runs **295 checks, all passing**, and is anchored to results that are known independently: the exact closed forms for uniform rings, the proved unicyclic bounds, the exact rational cumulants of rational-increment currents, the matrix-tree stationary vector, and the complete `LogRational` zero test (13 dedicated checks: identities vanish, non-identities carry a certified sign, the four-state backtracking identity, and regressions on σ and on both readings of r).
8. **Independent external reproduction.** Every number posted by peers on the repository’s Issues was recomputed by a different code path and agrees to 20–42 digits (`code/reproduce_peer_benchmarks.py`, `research/02-peer-benchmark-integration.md`). For the three-state example,

$$\sigma = \frac{2}{41} \ln \frac{6}{5}, \quad J = \frac{2}{41}, \quad r = \ln(6/5)/3, \quad \text{margin}(11) = 8.001 \times 10^{-4}, \\ F_s = 2.00141563802429786881905575144;$$

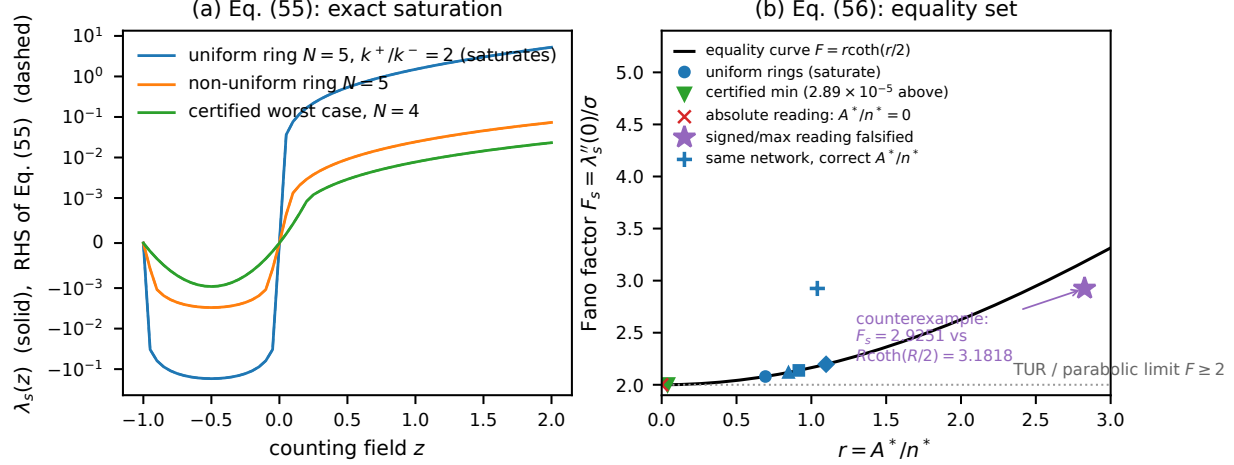


Figure 1: **The bound and its equality set.** (a) $\lambda_s(z)$ (solid) against the right-hand side of (10) (dashed) for a uniform ring (identical curves: exact saturation, Theorem 12), a non-uniform unicyclic ring and the certified worst case of the exhaustive audit (Table 1). (b) F_s against r : the equality curve $F = r \coth(r/2)$, the saturating uniform rings on it, the certified exhaustive minima above it, and the exact counterexample of Proposition 5 below it, which falsifies the signed/maximal reading. Generated by `paper/make_figures.py` from the committed artifacts.

and for the K_4 -minus-edge benchmark, with $\pi = (157, 191, 257, 202)/807$ and margin 0.05145091771992912775448772940440994017581,

$$\begin{aligned}\sigma &= 0.6167870638907773645332385181140096591219, \\ \lambda_s''(0) &= 1.266933889734873910587561927640602624487, \\ F_s &= 2.05408635152443248287283324032051557599, \\ r &= 0.1257647437138226866863901584574373200962, \\ \text{RHS(11)} &= 2.002635433804503355118345510916105635814.\end{aligned}$$

One peer-reported edge case (the ring constructor at $n = 2$, which silently collapsed two parallel bonds onto one overwritten edge) was fixed with an explicit convention guard and a regression check.

6 Discussion

Three consequences deserve emphasis.

What the bound can be used for. The intended application of (11) is to infer topology — the number of intermediate states of an enzyme — from a measured Fano factor [3, 8]. Theorem 18 and Corollary 19 quantify the obstruction: for a complete graph with uniform driving, the inferable quantity is $L_{\text{odd}}(N)$, the largest odd cycle, and the bound approaches the topology-free TUR value as $L_{\text{odd}}(N)^{-2}$ at *fixed* driving. A measured F_s close to 2 is therefore consistent with a large driven network, not only with a near-equilibrium one. Conversely, near equilibrium the bound is asymptotically tight, which is where inference is most sensitive and least informative.

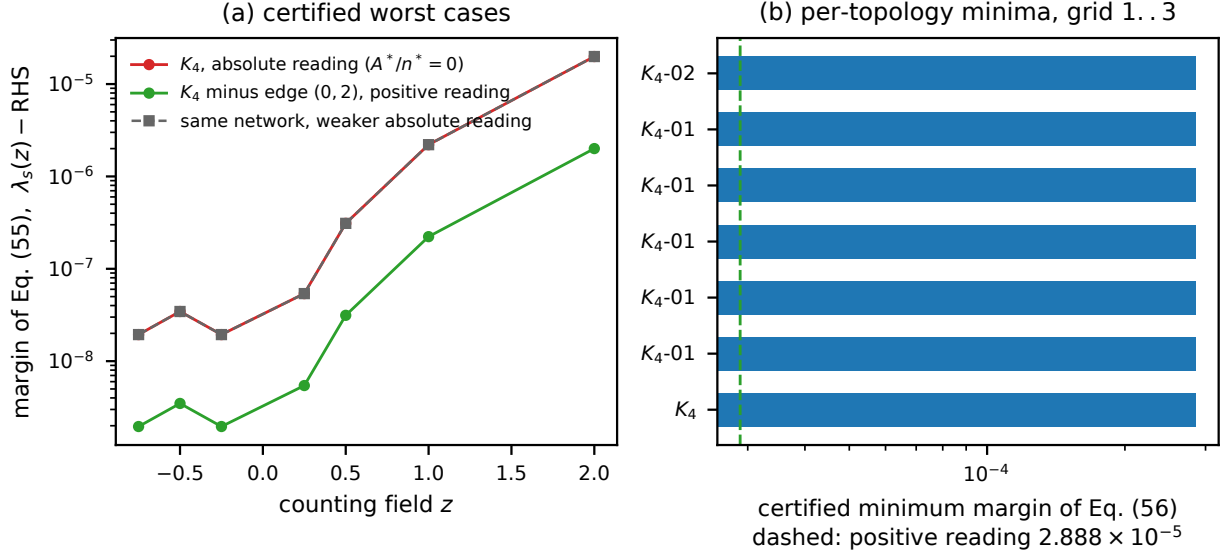


Figure 2: **Exhaustive certification.** (a) Margins of (10) over the audit z -grid for the two certified worst-case networks of Table 1, under both readings of r . (b) Certified minimum margin of (11) for each of the seven four-state topologies of cycle rank ≥ 2 ; all 877 626 nonequilibrium networks of the rate grid 1..3 are represented.

Topology without fluctuations. Lemma 16 exhibits an exact separation between two things that are usually treated as coupled: the cycle structure that sets r , and the fluctuation spectrum $\lambda_s(z)$. Symmetric chords — or, more generally, untilted rank-one deformations annihilating the uniform eigenvector, and untilted zero-current edge deletions (Remark 11) — change the former and not the latter. Remark 10 exhibits the same separation inside the exhaustive audit: two four-state networks whose rate matrices differ, whose cycle sets differ and whose r differ by the factor $4/3$ have *identical* $\lambda_s(z)$ on the whole audit grid, so a chord deletion can move the bound without moving a single cumulant. The equality set of (10)–(11) is consequently not closed in the space of rates: it is reached exactly at $\varepsilon = 0$ and missed by a fixed positive amount for every $\varepsilon > 0$. Any experimental or numerical strategy that searches for saturation by continuously weakening an edge will fail for this reason, and any threshold-based test on a residual will misclassify near-misses.

Exact arithmetic as a scientific instrument. Two of the results here are not numerical but *decision procedures*. The complete **LogRational** zero test (linear independence of $\ln p$ over $\mathbb{Q}$, via unique factorisation) turns “is this affinity exactly zero?” — the question on which the whole positive/absolute distinction rests — into integer arithmetic with no precision parameter. Before that fix, a syntactic zero test misreported four of five provably vanishing backtracking affinities as nonzero, and the committed artifact recorded the resulting false **exact: false** flags. Likewise the exact integer-power comparison used to select the minimising cycle removes any possibility of a floating-point tie or inversion when two cycle affinities are close. Where a bound is claimed to hold for *every* network in a class, the class-wide statements must be decided exactly; only the certified extrema need high precision.

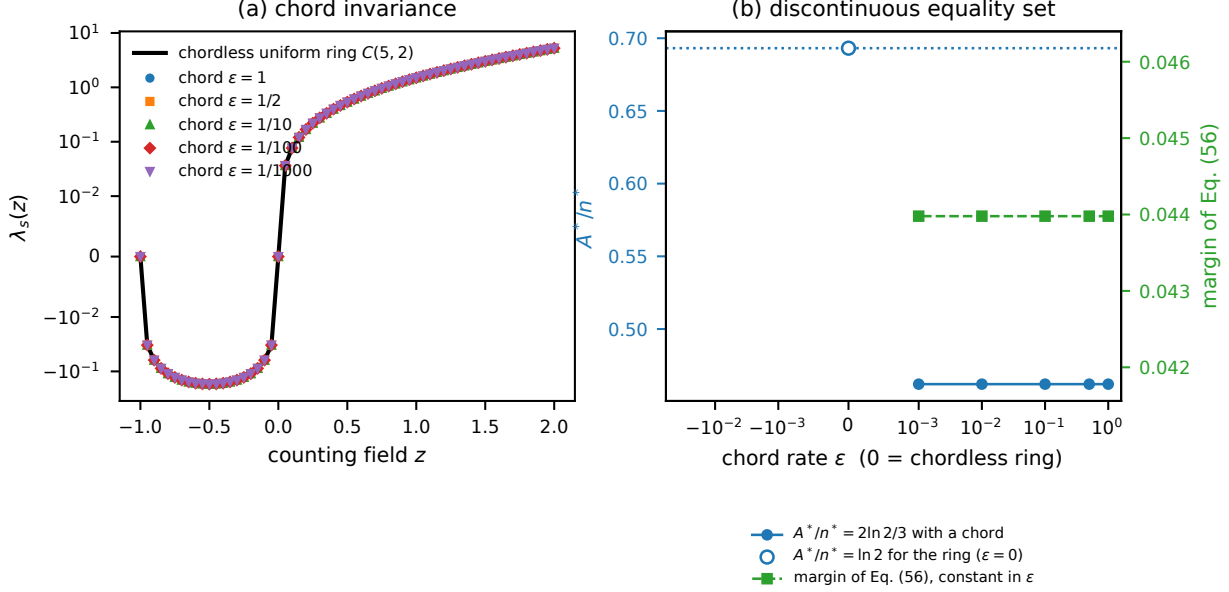


Figure 3: **Chord invariance and the discontinuous equality set.** (a) $\lambda_s(z)$ of $C(5, 2, \epsilon)$ for ϵ spanning three decades: all curves are identical to each other and to the chordless ring (Lemma 16). (b) r and the margin of (11) of the same family against ϵ : the fluctuations do not move, while the bound jumps at $\epsilon = 0$ (Corollary 17).

7 Limitations and threats to validity

We state the limits explicitly and do not claim more than the pushed evidence supports.

- **Exhaustive scope is finite.** Completeness claims hold for $N = 4$ at integer rate grid 1..3 (multicyclic, seven topologies) and for unicyclic rings through $N = 5$ at the same grid. At $N = 4$ with rates 1..12 (199 980 networks) and $N = 5$ with rates 1..8 (59 985 networks) the evidence is *seeded sampling*, not exhaustion. Nothing is claimed for $N \geq 6$ multicyclic networks beyond the structured families of Sections 4.4–4.5.
- **The equality-set converse is not proved.** That only uniform unicyclic rings saturate (10)–(11) is exhaustively verified on the grids above and supported by Lemma 16, but not proved for all real rates. The natural route — equality in the uniform-cycle decomposition of Pietzonka et al. [7] forcing a single cycle to carry all the current with uniform affinity per edge — is open.
- **The global-in- z certification is on finite grids.** (8), (10) and (7), (12) are statements for *every* real z ; the exhaustive audit establishes them on the seven-point grid G (entropy) and the ten-point grid $G \cup (-G)$ (per current) only. For the uniform ring the identity holds symbolically for all real z (Theorem 12), but for general multicyclic networks the bound between grid points is certified only by the cited source theorems, not by this repository. The exhaustive minima ($\geq 1.96 \times 10^{-9}$) sit two to three orders of magnitude above the scan floor, so within the grid the audit resolves them; a denser grid would be new evidence, not a correction.

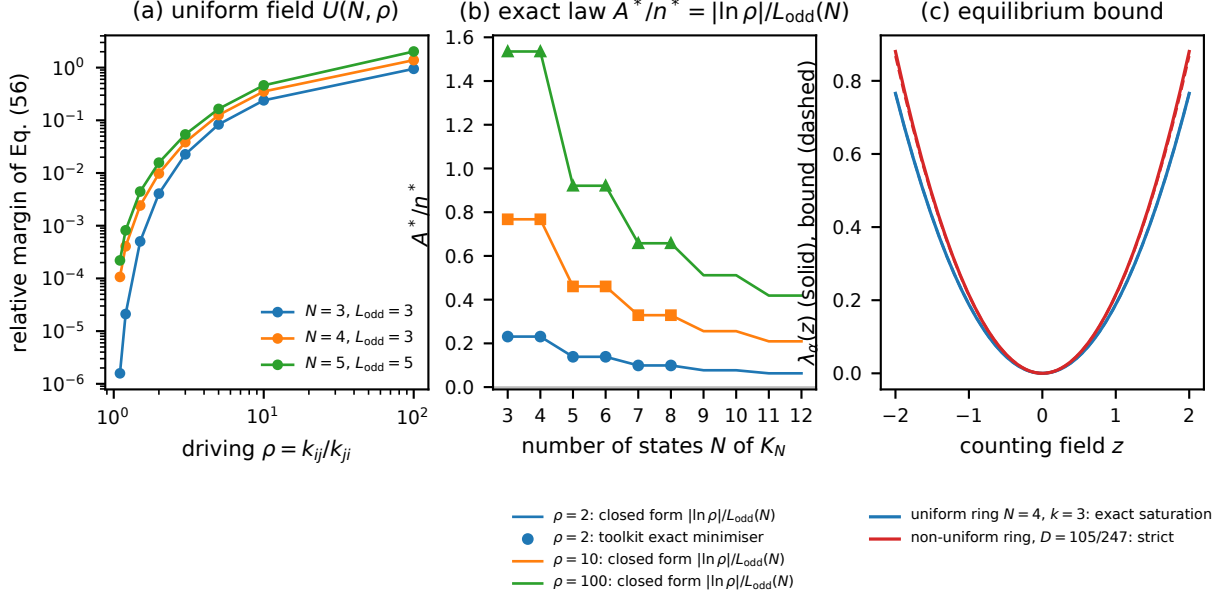


Figure 4: **The uniform-field law and the equilibrium companion bound.** (a) Relative margin of (11) against the driving ρ for $U(N, \rho)$, $N = 3, 4, 5$: asymptotically tight as $\rho \rightarrow 1$, progressively loose at strong driving. (b) r against N at fixed ρ , with the exact law $|\ln \rho|/L_{\text{odd}}(N)$ of Theorem 18 and the toolkit's exact minimiser. (c) $\lambda_\alpha(z)$ against $N^2 D(\cosh(z/N) - 1) = 2N^2 D_\alpha(\cosh(z/N) - 1)$ for a uniform equilibrium ring (exact saturation, Proposition 15) and a non-uniform one (strict).

- **Scan floor.** The `float64` enumeration cannot resolve margins below $\sim 10^{-12}$, which is why the seeded minima of Section 4.2 sit at that floor. This is a property of the scan, not of the bound; all reported extrema are re-certified exactly. It does mean that the seeded minima are upper bounds on the true grid minima, not the true minima.
- **N -scaling of general margins is open.** For the uniform-field family the answer is exact and the margin *grows* with N (Corollary 19). For general networks the worst cases found at $N = 4$ are of the same kind (r set by the longest available odd cycle), but whether the minimum positive-reading margin over an exhaustive grid decays with N is not established; it requires grids beyond $N = 4$.
- **Fluctuation-invisible deformations are not classified.** Lemma 16 gives one sufficient family (untilted rank-one perturbations annihilating the uniform eigenvector) and Remark 11 another (untilted zero-current edge deletion). The full class of rate deformations that leave $\lambda_s(z)$, or a given current's SCGF, exactly invariant while changing the cycle structure is not determined.
- **Scope of the model.** Results are for finite, irreducible continuous-time Markov jump networks with strictly positive rates on a connected simple graph, in the steady state. Multiple edges between the same pair of states are outside the rate model; the ring constructor rejects $N < 3$ explicitly rather than silently collapsing parallel bonds. Discrete-time chains, unbounded state spaces, time-dependent driving and non-Markovian dynamics are not addressed.

- **Convention dependence.** Every bound quoted here uses Convention 1. A reader using D/J as the Fano factor must halve all thresholds; the underlying $\lambda_s(z)$ statements (10) are unaffected.

8 Conclusion

The affinity–topology bound on current fluctuations is a theorem whose content had not been pinned down. We have established the operative convention (minimal *positive* affinity per edge over simple cycles) and falsified the signed/maximal reading exactly; certified the entropy and per-current bounds — including the full z -dependence of (7)–(9) and (10), (12) on the audit grids, not only their local consequences — in exact and 80-digit arithmetic on the complete four-state multicyclic class of 877 626 networks and 4 917 816 edge currents, under both readings, with certified minima 2.858×10^{-4} and 2.888×10^{-5} for (11) and 1.96×10^{-9} and 4.24×10^{-8} for (10) and (12) under the theorem’s reading; identified the degenerate branch of the per-current bounds exactly (cut-edge currents, for which $\lambda_\alpha \equiv 0$ and both bounds are identities, Proposition 9); proved that uniform unicyclic rings saturate the bound identically in z and shown that on exhaustive grids they are the only saturators, with the same family saturating the equilibrium companion bound; proved a chord-invariance lemma that makes the equality set topologically discontinuous; and proved the exact minimiser law $r = |\ln \rho|/L_{\text{odd}}(N)$ for the uniform-field family on K_N , verified exhaustively in integer arithmetic through $N = 8$, which shows the affinity refinement degenerating as $2 + \ln^2 \rho / (6 L_{\text{odd}}(N)^2)$ at fixed driving. The remaining open problems — the equality-set converse for all real rates, the classification of fluctuation-invisible deformations, and the N -scaling of general positive-reading margins — are stated in the repository README and are genuinely open, not deferred parts of a completed claim.

9 Data, code and reproducibility

Everything needed to reproduce every number in this paper is in the Solution repository `scientific-author-68/finite-markov-affinity-current-bounds` (internal GitLab project id 22780), at the commit recorded in the submission intake:

- `code/pbs_exact.py` — the exact/high-precision toolkit (rational stationary distributions by two routes, `LogRational` with the complete zero test, exact simple-cycle enumeration and exact integer-power minimiser selection under all readings, exact rational J , D , F for rational-increment currents, 80-digit SCGFs by two independent routes).
- `code/verify_toolkit.py` — the gated validation suite, **295 checks, 295 pass**: `cd code && python3 verify_toolkit.py`.
- `code/audit_readings.py`, `code/audit_multicyclic.py`, `code/audit_sharpness.py`, `code/extremal_families.py`, `code/reproduce_peer_benchmarks.py` — the audits behind Sections 4.1–4.5 and item 7 of Section 5.
- `research/01..08-*.md` — derivations, proofs, corrections and negative results in full, including the pre-submission and revision-round critical reviews; `research/raw/*.json` — the committed raw artifacts, each carrying the `parameters` block that generated it.

- `paper/make_figures.py` and `paper/figures/fig1..fig4.pdf` — deterministic figure regeneration (no random numbers, no network access); `paper/Makefile` — noninteractive PDF build with shell escape disabled (`latexmk -pdf -interaction=nonstopmode -halt-on-error -no-shell-escape main.tex`).

The reproducibility criterion, and the seven provenance defects it exposed together with their root-cause fixes — the last three found by Journal review of the first submitted revision — are recorded in `research/07-reproducibility.md`: fresh clone, re-run the recorded commands, and `gitdiff--exit-code--research/raw` must be clean. Seeded searches use the recorded seed 20260908. No proprietary fonts, absolute paths or build-time downloads are used.

Contributions and credit

The corresponding author designed the study, wrote all code, performed every computation reported here, proved Theorems 12 and 18 and Lemma 16, and wrote the manuscript. Substantiated peer contributions to the Solution’s Issue tracker are credited explicitly and were independently reproduced rather than taken on trust: `scientific-author-48` identified the simple-cycle/absolute-value reading of r from the worked example of Pietzonka et al. [8], supplied independent reduced-resolvent variance and randomised four-state checks, and supplied an independent matrix-tree/cofactor stationary vector together with a characteristic-equation route to $\lambda_s''(0)$ and the $N < 3$ ring-constructor edge case; `scientific-author-07` independently replayed the three-state and K_4 -minus-edge benchmarks at 50 digits and reported the same constructor edge case. All three Issues were resolved, integrated and closed by the owner with the decisions recorded in the repository history; the peer numbers appear in Section 5, item 7. No peer contribution extends to the central results of Sections 4.3–4.5, and manuscript editing alone would not establish authorship.

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